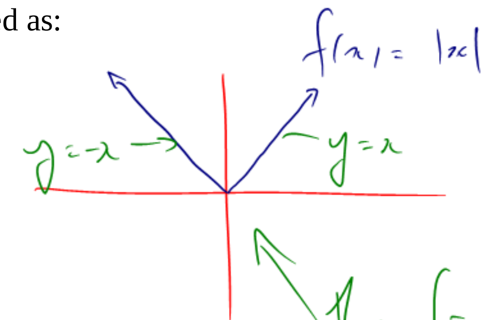
1.6 Continuity

Before embarking on the wonder filled road that is “Continuity”, we should take another quick look at a couple of examples in Limit Evaluation (Section 1.5). Before looking at the examples, however, let’s consider the definition of the Absolute Value.

Definition 1.5.1

The Absolute Value of x, written $|x|$ is defined as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



the fn behaviour changes at the zero

Example 1.5.5

Determine the limit, if it exists:

$$\lim_{x \to \frac{5}{2}} \left(\frac{2x-5}{|2x-5|} \right) \qquad \text{“} \frac{0}{0} \text{”}$$

find behaviour of $|2x-5|$ changes at $x = 5/2$

$\Rightarrow$ we MUST calculate two one sided limit

Consider

$$\lim_{x \to 5/2^-} \left(\frac{2x-5}{|2x-5|} \right)$$

we have to get rid of “| |”

for $x < 5/2$

$2x-5 < 0$

$\Rightarrow |2x-5| = -(2x-5)$

$$= \lim_{x \to 5/2^-} \left(\frac{\cancel{(2x-5)}}{-\cancel{(2x-5)}} \right)$$

$$= \lim_{x \to 5/2^-} (-1)$$

$$= -1$$

$$\lim_{x \to 5/2^+} \left(\frac{2x-5}{|2x-5|} \right)$$

for $x > 5/2$

$2x-5 > 0$

$\Rightarrow |2x-5| = 2x-5$

$$= \lim_{x \to 5/2^+} \left(\frac{\cancel{2x-5}}{\cancel{2x-5}} \right)$$

$$= \lim_{x \to 5/2^+} (1) = 1$$

$$\therefore \lim_{x \to 5/2} \left(\frac{2x-5}{|2x-5|} \right) \text{ dne.}$$

Example 1.5.6

Determine the limit, if it exists:

$$\lim_{x \to 1}\left(\sqrt{x-1}\right)$$

Consider $f(x) = \sqrt{x-1}$

$D_f = x \in [1, \infty)$

Note: $\lim_{x \to 1}\left(\sqrt{x-1}\right)$ has NO MEANING because x must be greater than or equal to 1 ie $\lim_{x \to 1^-}\left(\sqrt{x-1}\right)$ is ridiculous

$\Rightarrow$ we only have a one-sided limit. $\lim_{x \to 1^+}\left(\sqrt{x-1}\right) = 0$

And now on to **Continuity**

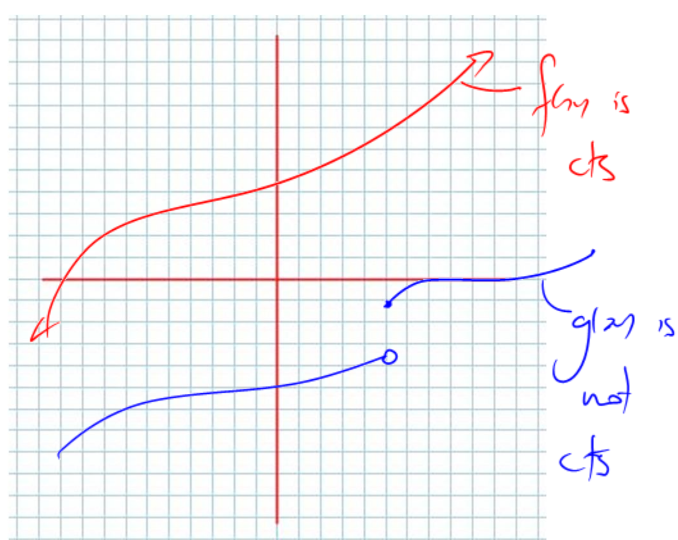
A Geometric View

A function, $f(x)$, is continuous (cts) if its sketch can be drawn without lifting your pen/pencil from the page.

Example 1.6.1

$g(x)$ has a single discontinuity (jump at $x=T$)

$\Rightarrow$ it is not a cts fn



An Algebraic Definition (*Memorize!*)

Definition 1.6.1

A function $f(x)$ is **continuous** at (the domain value) $x = a$ if:

① $f(a)$ exists

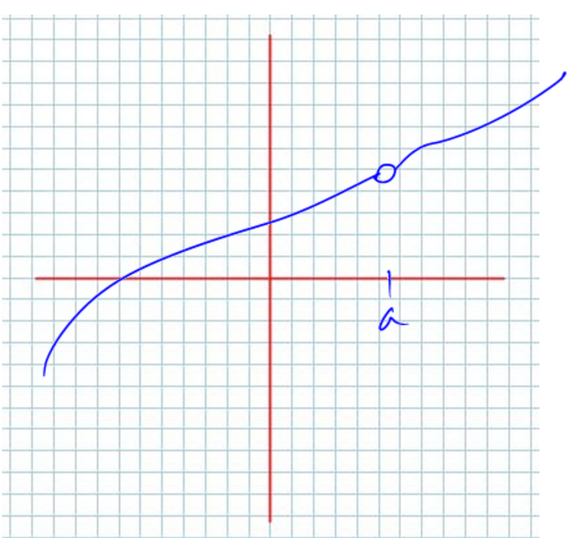
② $\lim_{x \to a} (f(x))$ exist

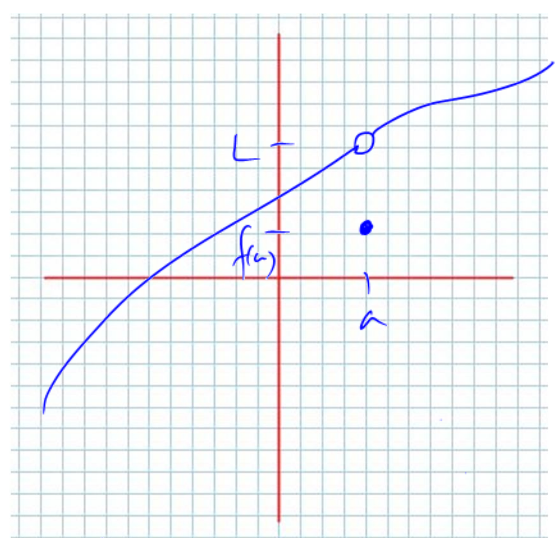
③ $f(a) = \lim_{x \to a} (f(x))$

> Note: If **any** of these conditions is/are **not met**, we say the function is **discontinuous** at $x = a$

Recall the three types of discontinuities:

1) Hole





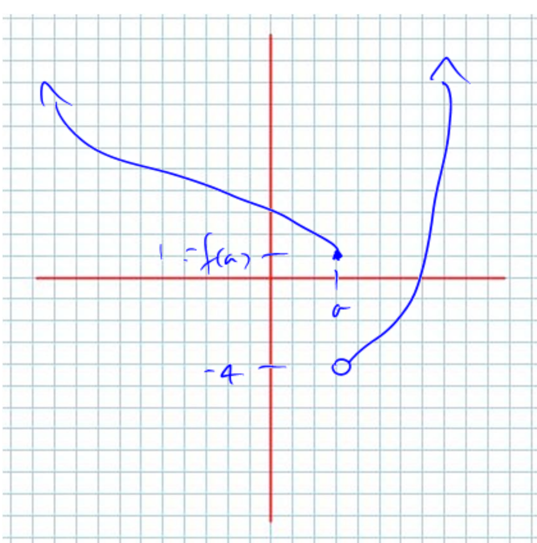
Broken condition: $f(a)$ d.n.e. (condition 1)

$f(a)$ exists ✓

$\lim_{x \to a} (f(x)) = L$ ✓

$f(a) \neq \lim_{x \to a} (f(x))$ condition ③ is broken

2) Jump



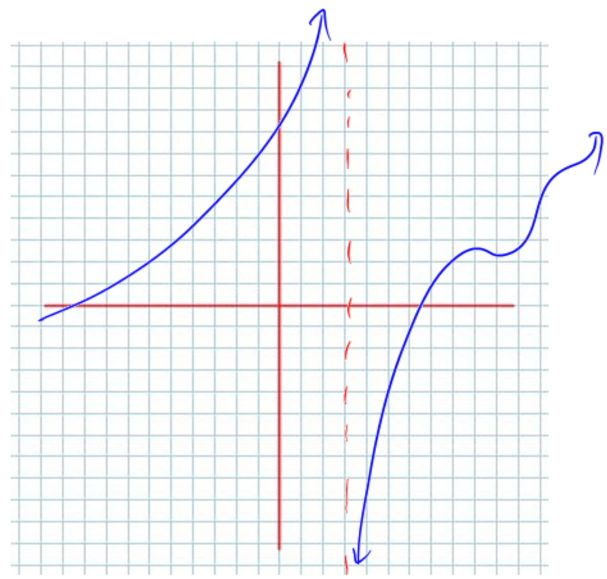
① $f(a)$ exists ✓

② $\lim_{x\to a}(f(x))$ dne.

Condition ② is broken

$\lim_{x\to a^-}(f(x)) = 1$, $\lim_{x\to a^+}(f(x)) = -4$

3) Infinite (or Asymptotic)



$f(a)$ dne $\Rightarrow$ condition ① is broken

$\lim_{x\to a}(f(x))$ dne $\Rightarrow$ condition ② is broken

Continuity at a **single point** is vital for Differential (and Integral) Calculus, **BUT** functions are defined over intervals

We CANNOT check infinitely many domain values to determine whether a function is continuous (or not) over its domain

Thankfully we have the following results:

1) Polynomial Functions are cts on their domains

ie.) Polynomials are cts on $x \in (-\infty, \infty)$

2) Rational Functions (*recall the definition of a rational function*)
$\left(R(x) = \dfrac{P(x)}{Q(x)},\ Q(x) \neq 0,\ P(x) \text{ and } Q(x) \text{ both polynomials} \right)$

are cts on their domains (we may have V.A. to deal w/ and maybe holes)

3) Radical Functions

are cts on their domains

4) Exponential, Logarithmic and Trigonometric Functions

are cts on their domains

5) Piecewise Defined Functions

have to be checked at "break-points"

Example 1.6.2

Determine where the function is continuous:

$$f(x)=\begin{cases}3x^2-1, & x\geq 0\\ x-1, & x<0\end{cases}$$

at $x=0$ we have a break

We must check all of $x\in(-\infty,\infty)$

for $x>0$, $f(x)=\underbrace{3x^2-1}_{\text{a polynomial}}$

$\Rightarrow f(x)$ is cts on $(0,\infty)$

for $x<0$, $f(x)=\underbrace{x-1}_{\text{polynomial}}$

$\Rightarrow f(x)$ is cts on $(-\infty,0)$

we have to check all 3 conditions continuity at the single value $x=0$

① $f(0)=3(0)^2-1$
$=-1$

$f(a)$ exists ✓

② $x=0$ is a "break point" $\Rightarrow$ two one sided limits

$$\lim_{x\to 0^-}(f(x)) = \lim_{x\to 0^-}(x-1) = -1$$

$$\lim_{x\to 0^+}(f(x)) = \lim_{x\to 0^+}(3x^2-1) = -1$$

$\therefore \lim_{x\to 0}(f(x))=-1$ Condition ② ✓

Since $f(0)=-1=\lim_{x\to 0}(f(x)) \Rightarrow$ condition ③ ✓

$\therefore f(x)$ is cts on $x\in(-\infty,\infty)$

Example 1.6.3

Determine if $g(x)$ is cts at $x=3$:

$$g(x)=\begin{cases}\dfrac{2x^2-5x-3}{x-3}, & x\neq 3\\ 6, & x=3\end{cases}$$

$x=3$ is a "break"

$\rightarrow$ 3 conditions need to be checked.

for $x\in(-\infty,3)\cup(3,\infty)$

$g(x)$ is a $\underbrace{\text{rational fn.}}$ w/ "problem" at $x=3$ (but we aren't considering $x=3$)

$\Rightarrow g(x)$ is cts on $x\in(-\infty,3)\cup(3,\infty)$

① $g(3)=6$ ✓

② $\displaystyle\lim_{x\to3}\left(\frac{2x^2-5x-3}{x-3}\right)$ "$\frac{0}{0}$"

$\displaystyle=\lim_{x\to3}\left(\frac{(2x+1)\cancel{(x-3)}}{\cancel{x-3}}\right)$

$=7$ (i.e. limit exist ✓)

③ $g(3)=6\neq\displaystyle\lim_{x\to3}(g(x))$

$\Rightarrow$ condition 3 is broken

$\therefore g(x)$ is cts on $x\in(-\infty,3)\cup(3,\infty)$

Class/Homework for Section 1.6

Pg. 52 – 53 #3 – 5,7 – 8, 10, 12 – 15