CALCULUS

Chapter 1 –Introduction to the Calculus

(*Material adapted from Chapter 1 of your text*)

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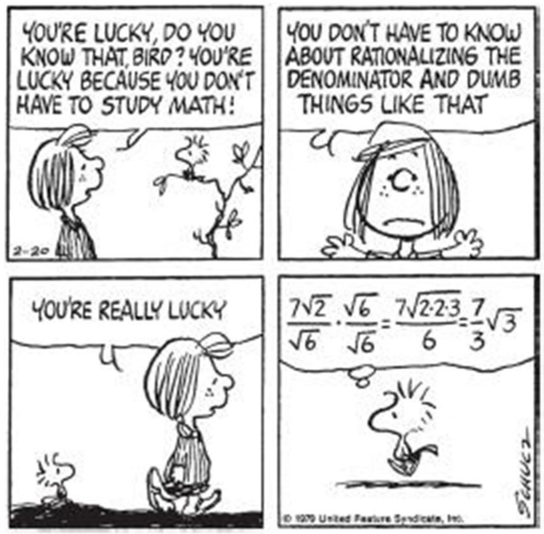
MATH@TD

1.1 Radical Expressions: Rationalizing



In the above cartoon Peppermint Patty calls the bird lucky for not having to know how to rationalize radicals (square roots, really). As it turns out, the Woodstock is actually lucky because he *can* rationalize. In Calculus being able to rationalize a denominator (or a numerator) is a necessary skill, and so we'll spend a little time honing that skill.

Definition 1.1.1

The **conjugate** of a binomial expression $a+b$ is $a-b$.

Example 1.1.1

$(\sqrt{a})^2 = a$

Determine the conjugate of

a) $5-\sqrt{3x}$ b) $\sqrt{2x^2}+\sqrt{10}$

$5+\sqrt{3x}$ $x\sqrt{2}-\sqrt{10}$

We can use the conjugate to **rationalize** a binomial expression which contains square roots. That is to say, we can **eliminate the square roots** (the irrational part) of a binomial expression (sort of).

multiply by the conjugate

Note: Only "conjugate" the part of the expression indicated. So, in this example, **leave the numerator alone**. Also remember **FACTORED FORM IS YOUR FRIEND**.

↳ with an exception

Example 1.1.2

Rationalize the denominator of $\dfrac{3+2x}{\sqrt{8+h}-\sqrt{h}}$

$$= \frac{3+2x}{\sqrt{8+h}-\sqrt{h}} \cdot \frac{\sqrt{8+h}+\sqrt{h}}{\sqrt{8+h}+\sqrt{h}}$$

$$= \frac{(3+2x)(\sqrt{8+h}+\sqrt{h})}{\underbrace{(8+h)-(h)}_{\text{skipped a step}}} = \frac{(3+2x)(\sqrt{8+h}+\sqrt{h})}{8}$$

rational!

Example 1.1.3

Rationalize the numerator of $\dfrac{\sqrt{7}+5}{3\sqrt{7}-6}$.

$$\frac{\sqrt{7}+5}{3\sqrt{7}-6} \cdot \frac{\sqrt{7}-5}{\sqrt{7}-5}$$

$$= \frac{7-25}{\underbrace{(3\sqrt{7}-6)(\sqrt{7}-5)}} = \frac{-18}{21-15\sqrt{7}-6\sqrt{7}+30}$$

$$= \frac{-18}{51-21\sqrt{7}} = \frac{-(\cancel{3})(6)}{\cancel{3}(17-7\sqrt{7})} = \frac{-6}{17-7\sqrt{7}}$$

no variables ⇒ expand using FOIL & simplify.

Class/Homework for Section 1.1

Pg. 9 #2, 3, 5 – 7

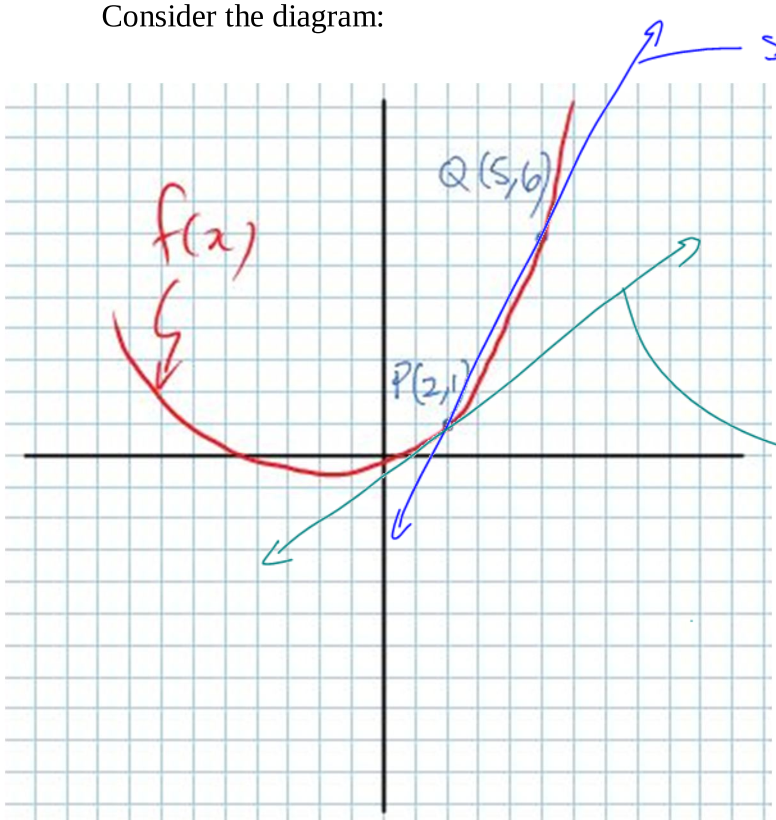
1.2 The Slope of a Tangent

This concept is a key to unlocking the tool box of Differential Calculus.

We'll begin by looking at a couple of examples.

Example 1.2.1

Consider the diagram:



secant has a calculable slope

$$m_{sec} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

tangent to f(x) at P(2,1) has a non-calculable slope

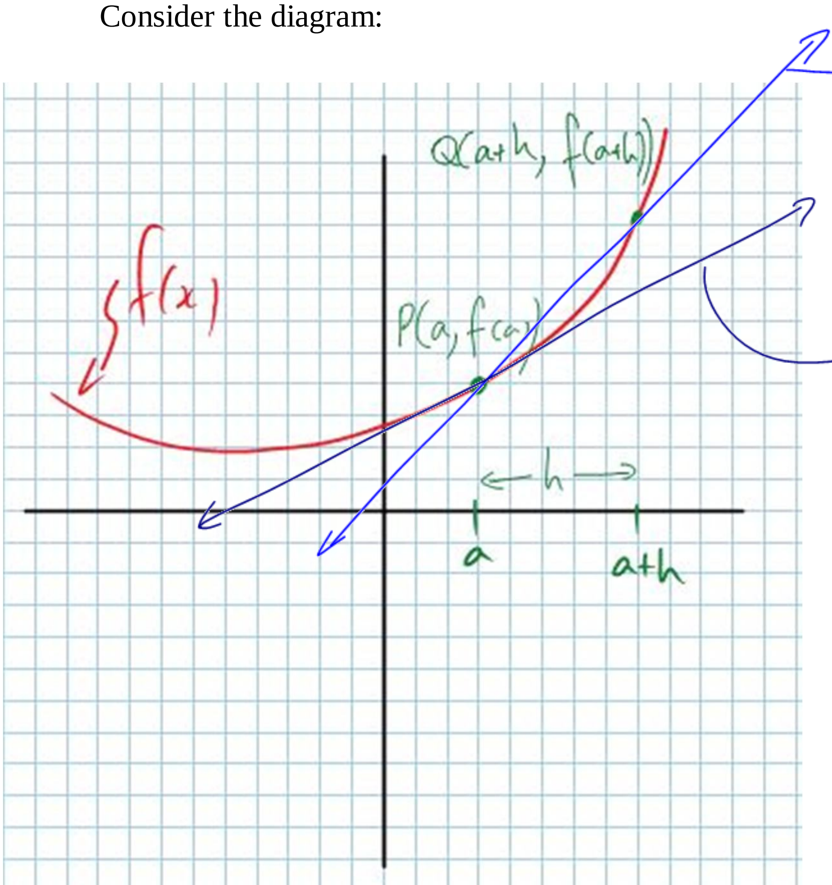
We know from Adv. Fns we can use m_{sec} to approximate m_{tan}

$\Rightarrow m_{tan} \sim m_{sec}$ $\boxed{\text{IF}}$ 'P' & 'Q' are close together

Question: Why can we **always** calculate the slope of a secant?

Example 1.2.2

Consider the diagram:



$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$

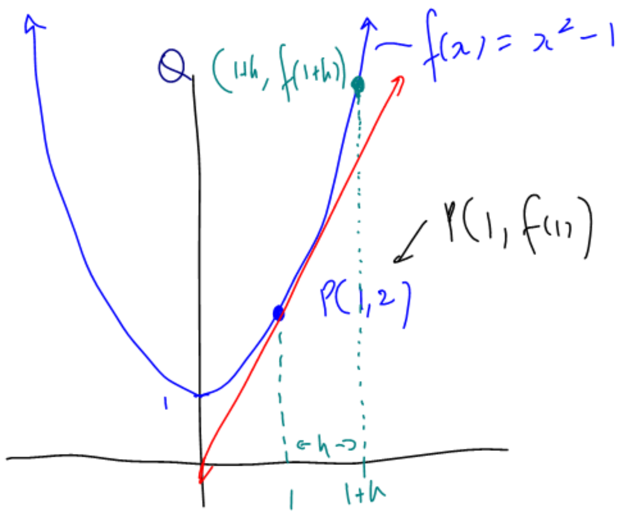
difference quotient

$m_{tan} \sim m_{sec}$ for small h

$$m_{tan} \sim \frac{f(a+h) - f(a)}{h}, \quad \text{for } h \text{ is small}$$

Example 1.2.3

Given $f(x) = x^2 + 1$ ***numerically*** approximate the slope of the tangent to the function at the point $P(1,2)$



Pictures are as much your **friends** as Factors are

$f(x) = x^2 + 1$

$f(☺) = (☺)^2 + 1$

$f(\square) = (\square)^2 + 1$ Simplifying a D.Q.

$f(1+h) = (1+h)^2 + 1$

Consider the difference quotient

$$m_{tan} \sim \frac{f(1+h) - f(1)}{h}$$

$$= \frac{((1+h)^2 + 1) - (2)}{h}$$

$$= \frac{1 + 2h + h^2 + 1 - 2}{h}$$

$$= \frac{2h + h^2}{h}$$

$$= \frac{\cancel{h}(2+h)}{\cancel{h}}, \ h \neq 0$$

$$= 2 + h$$

$(1+h, f(1+h))$

$= (1+h, (1+h)^2 + 1)$

h	P	Q	m_{sec}
1	(1,2)	(2,5)	3
0.5	(1,2)	(1.5, 3.25)	2.5
0.25	(1,2)	(1.25, f(1.25))	2.25
0.1	(1,2)	(1.1, 2.1)	2.1
0.01	(1,2)	(1.01, 2.02)	2.02

$\Rightarrow \quad m_{tan} \sim 2$

h tends to zero

but never gets there.

Algebraic Technique

In making h smaller and smaller (that is, as we let $h \to 0$), we are actually using what we call a **limit technique**.

If we write for the slope of a secant to a function

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

then, $m_{\text{tan}} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$

Example 1.2.4

Determine the slope of the tangent to $f(x) = 3x^2 + 1$ at $x = 2.$ — $a = 2$

$$m_{\tan} = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\left(3(2+h)^2 + 1\right) - 13}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{3(4 + 4h + h^2) + 1 - 13}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{12h + 3h^2}{h} \right) = \lim_{h \to 0} \left(\frac{\cancel{h}(12 + 3h)}{\cancel{h}} \right) = 12$$

Example 1.2.5

Calculate the slope of the tangent to $g(x)=\sqrt{x+1}$ at $x=3$.

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

$$m_{tan} = \lim_{h\to 0}\left(\frac{g(3+h)-g(3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\right)$$

when simplifying a diff. quot, if you see a radical CONJUGATE!

$$= \lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$

$$= \lim_{h\to 0}\left(\frac{4+h-4}{h(\sqrt{4+h}+2)}\right)$$

$$\frac{1}{\sqrt{4}+2} = \frac{1}{2+2}$$

$$= \lim_{h\to 0}\left(\frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{4+h}+2)}\right) = \frac{1}{4}$$

Class/Homework for Section 1.2

Pg. 18 – 21 #4,6 – 9, 11, 16, 20 – 22.