

1.5 Evaluating Limits

It will be helpful for you to remember that a **Limit of a Function** is a **potential functional value**. Because functional values are **simply numbers**, then **limits are just numbers** too. And so, to **evaluate** a limit **means** to **calculate the number that the limit is**.

On Page 40 of your text we see seven Properties of Limits listed. These seven properties can be thought of as the algebra of limits (where we think of “algebra” as a “set of rules for calculating”).

The first two properties are important enough that we should look at them in some small detail.

$$1) \lim_{x \rightarrow a} (k) = k \quad \text{where } k \text{ is a constant}$$

(k is unaffected by a
varying domain value)

$$2) \lim_{x \rightarrow a} (x) = a$$

The five other properties allow us to “use” the above two.

We will see that **using the Properties of Limits is a mathematical practice** very much like what you’ve been doing with algebra over the last few years. The **context** is just a little different. Instead of solving an equation, we are calculating potential functional values.

Section 1.5—Properties of Limits

The statement $\lim_{x \rightarrow a} f(x) = L$ says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a), such that $x \neq a$. This means that when finding the limit of $f(x)$ as x approaches a , there is no need to consider $x = a$. In fact, $f(a)$ need not even be defined. The only thing that matters is the behaviour of $f(x)$ near $x = a$.

EXAMPLE 1

Reasoning about the limit of a polynomial function

Find $\lim_{x \rightarrow 2} (3x^2 + 4x - 1)$.

Solution

It seems clear that when x is close to 2, $3x^2$ is close to 12, and $4x$ is close to 8. Therefore, it appears that $\lim_{x \rightarrow 2} (3x^2 + 4x - 1) = 12 + 8 - 1 = 19$.

In Example 1, the limit was arrived at intuitively. It is possible to evaluate limits using the following properties of limits, which can be proved using the formal definition of limits. This is left for more advanced courses.

Properties of Limits

For any real number a , suppose that f and g both have limits that exist at $x = a$.

1. $\lim_{x \rightarrow a} k = k$, for any constant k
2. $\lim_{x \rightarrow a} x = a$
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} [cf(x)] = c[\lim_{x \rightarrow a} f(x)]$, for any constant c
5. $\lim_{x \rightarrow a} [f(x)g(x)] = [\lim_{x \rightarrow a} f(x)][\lim_{x \rightarrow a} g(x)]$
6. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, provided that $\lim_{x \rightarrow a} g(x) \neq 0$
7. $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$, for any rational number n

Example 1.5.1

Using the Properties of Limits, determine $\lim_{x \rightarrow 2} \left(\frac{3x^2 - 5x}{x + 3} \right)$.

$$= \frac{\lim_{x \rightarrow 2} (3x^2 - 5x)}{\lim_{x \rightarrow 2} (x + 3)} \quad (\text{property 6})$$

$$= \frac{\lim_{x \rightarrow 2} (3x^2) - \lim_{x \rightarrow 2} (5x)}{\lim_{x \rightarrow 2} (x) + \lim_{x \rightarrow 2} (3)} \quad (\text{Prop 3})$$

$$= \frac{3 \cdot \lim_{x \rightarrow 2} (x^2) - 5 \cdot \lim_{x \rightarrow 2} (x)}{\lim_{x \rightarrow 2} (x) + \lim_{x \rightarrow 2} (3)} \quad (\text{prop 4})$$

$$= \frac{3 \cdot \left(\lim_{x \rightarrow 2} (x) \right)^2 - 5 \cdot \lim_{x \rightarrow 2} (x)}{\lim_{x \rightarrow 2} (x) + \lim_{x \rightarrow 2} (3)} \quad (\text{prop 7})$$

$$= \frac{3(2)^2 - 5(2)}{(2) + 3}$$

$$= \frac{2}{5}$$

prop (1) & (2)

To be frank, it seems a little silly and certainly is tedious to evaluate limits by stating the various properties as we use them. In fact, using the various Properties of Limits can be boiled down to a single statement (or single Limit Law, if you will allow):

plug it in, and see what happens
 ↓
 the approached domain value

One of three things will happen:

- 1) You calculate a definite (a finite) number.

Hardy you are done

Example 1.5.2

Evaluate $\lim_{x \rightarrow -3} \left(\frac{2x-5}{7x} \right) = \frac{11}{21}$

$$\begin{array}{l} \frac{2(-3)-5}{7(-3)} \\ = \frac{-6-5}{-21} \\ = \frac{11}{21} \end{array}$$

- 2) Your calculation arrives at $\frac{\text{definite non-zero \#}}{0}$

Example 1.5.3

Evaluate $\lim_{x \rightarrow 2} \left(\frac{3x^2-5}{x-2} \right)$

the limit d.n.e.

$$\begin{array}{l} \frac{3(2)^2-5}{2-2} \\ = \frac{7}{0} \end{array}$$



Note: $x=2$ is a V.A

** If * we are concerned w/ final behaviour around the V.A. then 2 one sided limits are necessary*

↙ not quite determinable

3) Your calculation arrives at an **indeterminate** form:

$$\frac{0}{0}, \text{ or } \frac{\infty}{\infty}, \text{ or } 0 \cdot \infty$$

↪ could be anything

If we have an indeterminate form

MORE WORK MUST BE DONE
eg FACTORING, or CONJUGATE

Example 1.5.4

Determine the limits, if they exist:

$$2x^2, 3x-2x-3$$

$$x(2x+3) - (2x+3) \\ (2x+3)(x-1)$$

a) $\lim_{x \rightarrow 1} \left(\frac{2x^2 + x - 3}{x - 1} \right)$ "0/0"

$$= \lim_{x \rightarrow 1} \left(\frac{(2x+3)(\cancel{x-1})}{\cancel{x-1}} \right)$$

$$= \lim_{x \rightarrow 1} (2x+3)$$

$$= 5$$

b) $\lim_{x \rightarrow 3} \left(\frac{\sqrt{x^2-5}-2}{x-3} \right)$ "0/0"

$$= \lim_{x \rightarrow 3} \left(\frac{\sqrt{x^2-5}-2}{x-3} \cdot \frac{\sqrt{x^2-5}+2}{\sqrt{x^2-5}+2} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{(x^2-5)-4}{(x-3)(\sqrt{x^2-5}+2)} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{x^2-9}{(x-3)(\sqrt{x^2-5}+2)} \right)$$
 "0/0"

$$= \lim_{x \rightarrow 3} \left(\frac{(\cancel{x-3})(x+3)}{(\cancel{x-3})(\sqrt{x^2-5}+2)} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{x+3}{\sqrt{x^2-5}+2} \right) = \frac{6}{4} = \frac{3}{2}$$

factoring cube roots is NOT EASY

TRICK - use a change of variable

$$c) \lim_{x \rightarrow 0} \left(\frac{(x+27)^{\frac{1}{3}} - 3}{x} \right) \quad \frac{0}{0}$$

$$= \lim_{u \rightarrow 3} \left(\frac{u - 3}{u^3 - 27} \right) \quad \frac{0}{0}$$

difference of cubes

$$= \lim_{u \rightarrow 3} \left(\frac{\cancel{u-3}}{(\cancel{u-3})(u^2 + 3u + 9)} \right)$$

⇒ Factor!

$$= \lim_{u \rightarrow 3} \left(\frac{1}{u^2 + 3u + 9} \right) = \frac{1}{27}$$

$$(x+27)^{\frac{1}{3}} = u$$

$$\text{if } x \rightarrow 0 \Rightarrow u \rightarrow (0+27)^{\frac{1}{3}} = 3$$

$$\text{Now, since } (x+27)^{\frac{1}{3}} = u$$

$$\Rightarrow x+27 = u^3$$

$$\Rightarrow x = u^3 - 27$$

Class/Homework for Section 1.5

Pg. 45 – 47 #3, 4, 7, 8abc (for #3, **explain** your thinking)