Hwk Check

m) $\lim_{x \to 8} \left(\frac{x^{\frac{1}{3}} - 2}{x - 8} \right)$ "$\frac{0}{0}$"

Let $u = x^{\frac{1}{3}}$ as $x \to 8$

$\Rightarrow x = u^3$ $\Rightarrow u \to 2$

$= \lim_{u \to 2} \left(\frac{u - 2}{u^3 - 8} \right)$ "$\frac{0}{0}$"

$= \lim_{u \to 2} \left(\frac{\cancel{u-2}\ 1}{(\cancel{u-2})(u^2 + 2u + 4)} \right)$

$= \frac{1}{12}$

1.6 Continuity

Before embarking on the wonder filled road that is "Continuity", we should take another quick look at a couple of examples in Limit Evaluation (Section 1.5). Before looking at the examples, however, let's consider the definition of the Absolute Value.

Definition 1.5.1

The Absolute Value of x, written $|x|$ is defined as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 1.5.5

Determine the limit, if it exists:

$$\lim_{x \to \frac{5}{2}} \left(\frac{2x-5}{|2x-5|} \right) \qquad \text{"}\frac{0}{0}\text{"}$$

Note: do not cancel absolute value

Try two one sided limits

$$\lim_{x \to \frac{5}{2}^-} \left(\frac{2x-5}{|2x-5|} \right) = \lim_{x \to \frac{5}{2}^-} \left(\frac{\cancel{2x-5}}{-(\cancel{2x-5})} \right) = -1$$

Note:

as $x \to \frac{5}{2}^-$

$\Rightarrow x < \frac{5}{2}$

$\Rightarrow 2x-5 < 0$

$\Rightarrow |2x-5| = -(2x-5)$

$$\lim_{x \to \frac{5}{2}^+} \left(\frac{2x-5}{|2x-5|} \right) = \lim_{x \to \frac{5}{2}^+} \left(\frac{\cancel{2x-5}}{+(\cancel{2x-5})} \right) = 1$$

$$\therefore \lim_{x \to \frac{5}{2}} \left(\frac{2x-5}{|2x-5|} \right) \text{ d.n.e.}$$

Example 1.5.6

Determine the limit, if it exists:

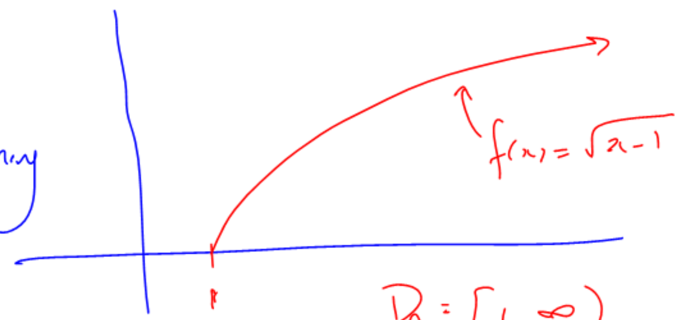
$$\lim_{x\to 1}\left(\sqrt{x-1}\right)$$

Consider the sketch

Since $\therefore$ $x<1$ is not in D_f

$\Rightarrow \lim_{x\to 1^-}\left(\sqrt{x-1}\right)$ has no meaning

$\Rightarrow \lim_{x\to 1}\left(\sqrt{x-1}\right)$ d.n.e.



$D_f = [1,\infty)$

$x<1$ is not in the domain

However: we can say $\lim_{x\to 1^+}\left(\sqrt{x-1}\right) = 0$

And now on to **Continuity**

A Geometric View

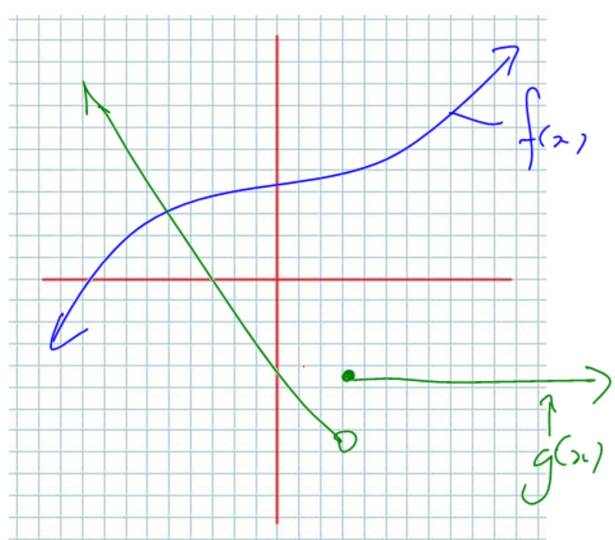
Lame defn

A function, $f(x)$, is continuous (cts) if its sketch can be drawn without lifting your pen/pencil from the page.

Example 1.6.1

$f(x)$ is cts

$g(x)$ is not cts



An Algebraic Definition (*Memorize!*)

Definition 1.6.1

A function $f(x)$ is **continuous** at (the domain value) $x = a$ if:

(Single value continuity)

① $f(a)$ exists

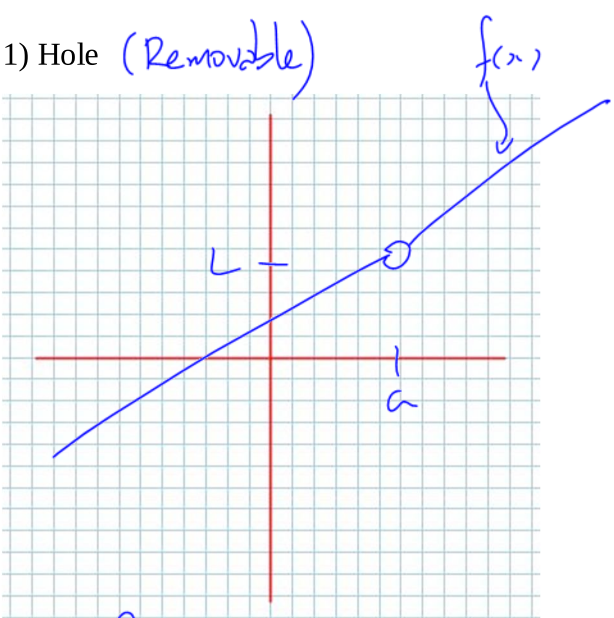
② $\lim_{x \to a} \left(f(x) \right)$ exist (i.e. $\lim_{x \to a^-} \left(f(x) \right) = L = \lim_{x \to a^+} \left(f(x) \right)$, L is a finite #)

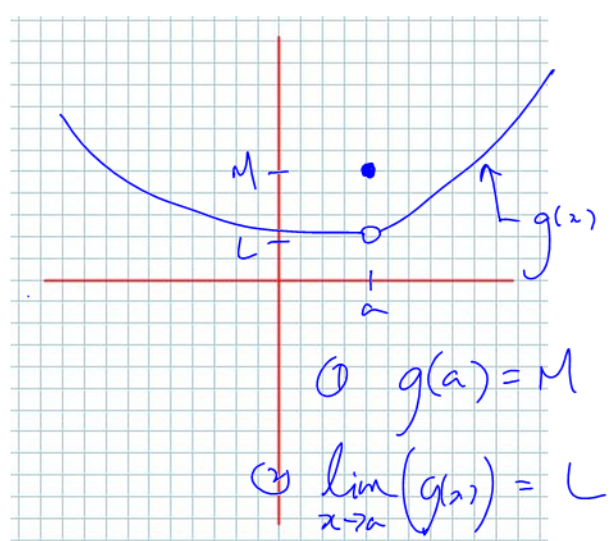
③ $\lim_{x \to a} \left(f(x) \right) = f(a)$

> Note: If **any** of these conditions is/are **not met**, we say the function is **discontinuous** at $x = a$

Recall the three types of discontinuities:

1) Hole (Removable)





① $g(a) = M$

② $\lim_{x \to a} \left(g(x) \right) = L$

③ but $M \neq L$

Note: $f(a)$ dne.

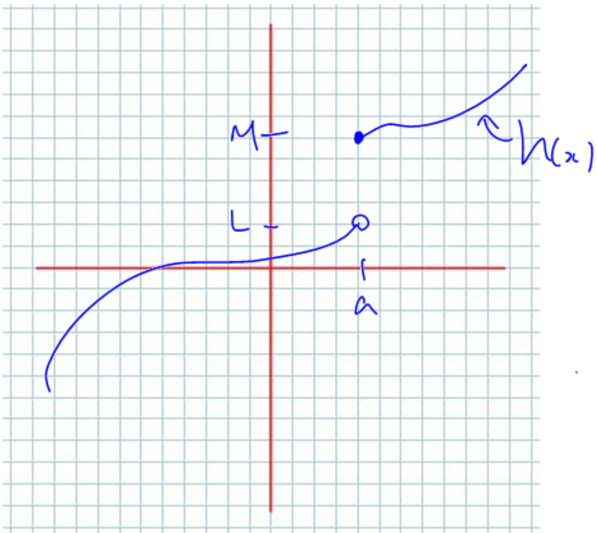
$\Rightarrow$ Condition ① is broken

$\therefore f(x)$ is not cts at $x = a$

$\therefore$ Condition ③ is broken

$\Rightarrow g(x)$ is NOT cts at $x = a$

2) Jump



① $h(a) = M$

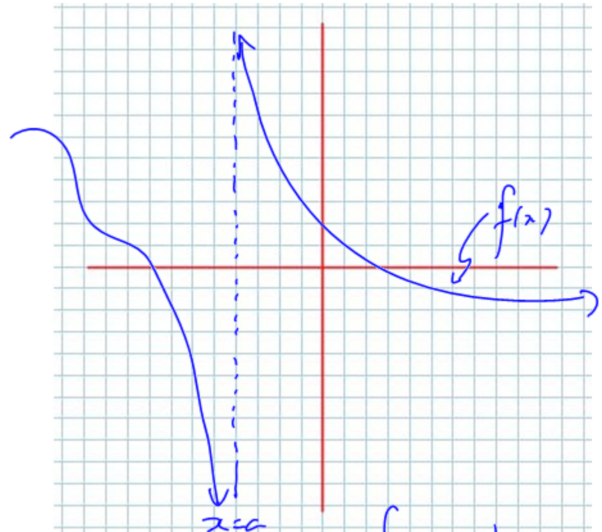
② $\lim_{x\to a^-}(h(x)) = L$

$\lim_{x\to a^+}(h(x)) = M \neq L$

$\therefore \lim_{x\to a}(h(x))$ dne $\quad \therefore$ condition ② is broken

$\therefore h(x)$ is NOT cts at $x=a$

3) Infinite (or Asymptotic)



$f(a)$ dne

$\lim_{x\to a}(f(x))$ dne

$\therefore f(x)$ is not cts at $x=c$

Continuity at a **single point** is vital for Differential (and Integral) Calculus, **BUT** functions are defined over domains (intervals of the "real line") ∞'ly many values

We CANNOT check ∞'ly many values to determine whether a function is continuous (or not) over its entire domain!

Q. How do we "talk about" a fn being cts over its domain?

Thankfully we have the following results:

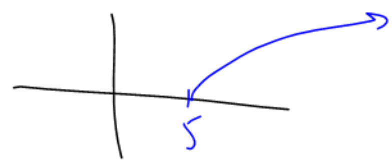
1) Polynomial Functions are cts everywhere (ie cts on $(-\infty, \infty)$)

2) Rational Functions (*recall the definition of a rational function*)

$$\left(R(x) = \frac{P(x)}{Q(x)},\ Q(x) \neq 0,\ P(x) \text{ and } Q(x) \text{ both polynomials} \right)$$

cts everywhere on their natural domains (all domain values which are not restrictions)

3) Radical Functions eg $f(x) = \sqrt{x-5}$



are cts everywhere on their domains

Pickiness!! technically speaking, a radical fn is "not" cts at its start pt. – we say the fn is cts "from one side"

4) Exponential, Logarithmic and Trigonometric Functions

are cts everywhere on their domains

5) Piecewise Defined Functions

have to be checked for continuity

Check ① "around" break points using the above results

② check cty at the single valued break pts.

Example 1.6.2

Determine where the function is continuous:

$$f(x)=\begin{cases}3x^2-1, & x\geq 0\\ x-1, & x<0\end{cases}$$

break pt $x=0$

On $(-\infty,0)$ $f(x)=x-1$ which is a polynomial

$\Rightarrow f(x)$ is $\boxed{\text{cts on } (-\infty,0)}$

On $(0,\infty)$ $f(x)=3x^2-1$ which is a polynomial

$\Rightarrow f(x)$ is $\boxed{\text{cts on } (0,\infty)}$

for $x=0$ (check the 3 conditions!)

$f(0)=-1$

$\lim_{x\to 0}(f(x))$ requires 2 one sided limits

$$\lim_{x\to 0^-}(f(x)) = \lim_{x\to 0^-}(x-1) = -1$$

$$\lim_{x\to 0^+}(f(x)) = \lim_{x\to 0^+}(3x^2-1) = -1$$

$\therefore \lim_{x\to 0}(f(x)) = -1 = f(0) \longrightarrow \therefore f(x)$ is cts $\boxed{\cancel{\text{at}}\ x=0}$

$\therefore f(x)$ is cts on $(-\infty,\infty)$

Example 1.6.3

Determine if $g(x)$ is cts at $x=3$:

$$g(x)=\begin{cases}\dfrac{2x^2-5x-3}{x-3}, & x\neq 3\\ 7\ 6 & x=3\end{cases}$$

break pt.

for $x \neq 3$ $g(x)=\dfrac{2x^2-5x-3}{x-3}$ which is rational

$\Rightarrow g(x)$ is cts on $(-\infty,3)\cup(3,\infty)$

for $x=3$ check single value cty (continuity)

$g(3)=6$

$$\lim_{x\to 3}\left(g(x)\right) = \lim_{x\to 3}\left(\frac{2x^2-5x-3}{x-3}\right) \quad \text{“}\frac{0}{0}\text{”}$$

$$= \lim_{x\to 3}\left(\frac{(2x+1)\cancel{(x-3)}}{\cancel{x-3}}\right)$$

$$= 7$$

$\because g(3)=6\neq 7=\lim_{x\to 3}(g(x))$, $\therefore g(x)$ is NOT cts at $x=3$

$\therefore g(x)$ is cts on $(-\infty,3)\cup(3,\infty)$

Class/Homework for Section 1.6

Pg. 52 – 53 #3 – 5,7 – 8, 10, 12 – 15