Chapter 1 – Introduction to Calculus - ***Review***

In Chapter 1 we learned some of the **fundamental** concepts of (differential) Calculus. Two of the concepts we considered are **key** to your understanding Calculus, so make sure you understand the meaning of the *Limit of a Function*, and *Continuity of Functions*. We also reviewed how to rationalize expressions with radicals, and saw (finally) how to calculate the Instantaneous Rate of Change (or the slope of a tangent) of a function at a point.

Section 1.1 – Rationalizing

Keep in mind the notion of a conjugate: e.g. Given $a + b$ we call $a - b$ the conjugate.

Section 1.2 – The Slope of a Tangent (or The Instantaneous Rate of Change)

With a small modification to the **difference quotient**, we defined the **slope of a tangent** to a function, $f(x)$ at a domain value $x = a$ to be:

$$m_{\tan} = \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

You must have this definition memorized, and you must know how to use it.

Section 1.4 – The Limit of a Function

In this section we considered the concept of a limit, and learned that a **limit can be thought of** as a "**potential functional value**". The main point of considering limits is to be able to explore the behavior of a function near any problem domain values. We learned two definitions:

Definition:

We take $\lim_{x \to a}\left(f(x)\right) = L$ to mean that as x gets "ridiculously" close to the domain value a, the functional value is getting "ridiculously" close to the finite number L.

Definition:

Given the function $f(x)$ then $\lim_{x \to a}\left(f(x)\right) = L$ exists **if and only if**

$$\lim_{x \to a^-}\left(f(x)\right) = L = \lim_{x \to a^+}\left(f(x)\right)$$

Section 1.5 – Evaluating Limits

Here we saw the "**seven properties of limits**" and that they actually boil down to the statement "**Plug it in and see what happens**". When evaluating limits, there are three possible outcomes at the "plug it in…" stage. Those possibilities are:

- We obtain a definite answer, and so the problem is done.
- We obtain an answer which is undefined (i.e. something like $\frac{2}{0}$) and again, the problem is complete.
- We obtain an **indeterminate** form $\frac{0}{0}, \frac{\infty}{\infty}$, or $0 \cdot \infty$ and so "**more work**" must be done to **determine** an answer.

Section 1.6 – Continuity

In this final section of the chapter we learned the definition of continuity of a function at a point. You must have the definition memorized.

Definition:

A function, $f(x)$, is continuous at the domain value $x = a$ if:

1. $f(a)$ is defined.
2. $\lim_{x \to a}(f(x))$ exists.
3. $\lim_{x \to a}(f(x)) = f(a)$

We also learned that all of the functions we will consider in the Calculus half of the course are "nicely behaved". That is, **the functions we consider are all continuous on their domains**, **with the exception of piecewise defined functions which will require some analysis to determine their continuity.** Make sure you **know the three types of discontinuities**!

Some examples:

1. Rationalize the numerator

$$\frac{\sqrt{x-2}+3}{x-7}$$

$$= \frac{\sqrt{x-2}+3}{x-7} \cdot \frac{\sqrt{x-2}-3}{\sqrt{x-2}-3}$$

$$= \frac{x-2-9}{(x-7)(\sqrt{x-2}-3)}$$

$$= \frac{x-11}{(x-7)(\sqrt{x-2}-3)}$$

it's nice when you can cancel - but that doesn't happen all the time tangents are lines $\therefore$ they have the form $\boxed{y = mx + b}$

slope — y intercept

2. Determine the **equation** of the tangent to the function:

a) $f(x) = 2x^3 + 3x - 1$ at $x = 0$

We need to find the slope of the tangent

$$m_{tan} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right), \quad a = 0$$

$$= \lim_{h \to 0} \left(\frac{(2(0+h)^3 + 3(0+h) - 1) - (2(0)^3 + 3(0) - 1)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{2h^3 + 3h}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(2h^2 + 3)}{\cancel{h}} \right)$$

$$= 3$$

$\therefore$ we have $y = 3x + b$

use $(0, f(0)) = (0, -1)$ to find b

$\Rightarrow -1 = 3(0) + b \Rightarrow b = -1$

$\boxed{\therefore y = 3x - 1 \text{ is the eqn}}$

b) $g(x) = 2\sqrt{x+1} - 4$ at $x = 3$ $\quad (g(3) = 0)$

$$m_{tan} = \lim_{h \to 0} \left(\frac{g(a+h) - g(a)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(2\sqrt{4+h} - 4) - (0)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{2\sqrt{4+h} - 4}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{2\sqrt{4+h} - 4}{h} \cdot \frac{2\sqrt{4+h} + 4}{2\sqrt{4+h} + 4} \right)$$

$$= \lim_{h \to 0} \left(\frac{4(4+h) - 16}{h(2\sqrt{4+h} + 4)} \right)$$

$$= \lim_{h \to 0} \left(\frac{4\cancel{h}}{\cancel{h}(2\sqrt{4+h} + 4)} \right)$$

$$= \frac{4}{8} = \frac{1}{2}$$

$\therefore$ we have

$y = \frac{1}{2}x + b$

use $(3, 0)$ to find b

$0 = \frac{1}{2}(3) + b \Rightarrow b = -3/2$

$\therefore \boxed{y = \frac{1}{2}x - 3/2}$

TELL ME YOU SEE "$\frac{0}{0}$"

3. Evaluate the following limits:

a) $\lim_{x\to 5}\left(\frac{x^2-25}{(x-5)^2}\right)$ "$\frac{0}{0}$"

$$= \lim_{x\to 5}\left(\frac{\cancel{(x-5)}(x+5)}{(x-5)^{\cancel{2}}}\right) \quad \text{"}\frac{10}{0}\text{"}$$

dne

b) $\lim_{x\to 2}\left(\frac{3x+5}{x-3}\right)$

$$= \frac{11}{-1} = -11$$

c) $\lim_{t\to 6}\left(\frac{\sqrt{2t-3}-3}{t^2-36}\right)$ "$\frac{0}{0}$"

$$= \lim_{t\to 6}\left(\frac{\sqrt{2t-3}-3}{(t-6)(t+6)}\cdot\frac{\sqrt{2t-3}+3}{\sqrt{2t-3}+3}\right)$$

$$= \lim_{t\to 6}\left(\frac{(2t-3)-9}{(t-6)(t+6)(\sqrt{2t-3}+3)}\right)$$

$$= \lim_{t\to 6}\left(\frac{2t-12}{(t-6)(t+6)(\sqrt{2t-3}+3)}\right)$$

$$= \lim_{t\to 6}\left(\frac{2\cancel{(t-6)}}{\cancel{(t-6)}(t+6)(\sqrt{2t-3}+3)}\right)$$

$$= \frac{2}{(12)(6)} = \frac{1}{36}$$

d) $\lim_{x\to -1}\left(\frac{\sqrt[3]{x}+1}{x+1}\right)$ "$\frac{0}{0}$"

Let $u = x^{\frac{1}{3}}$

$\Rightarrow x = u^3$

as $x \to -1$

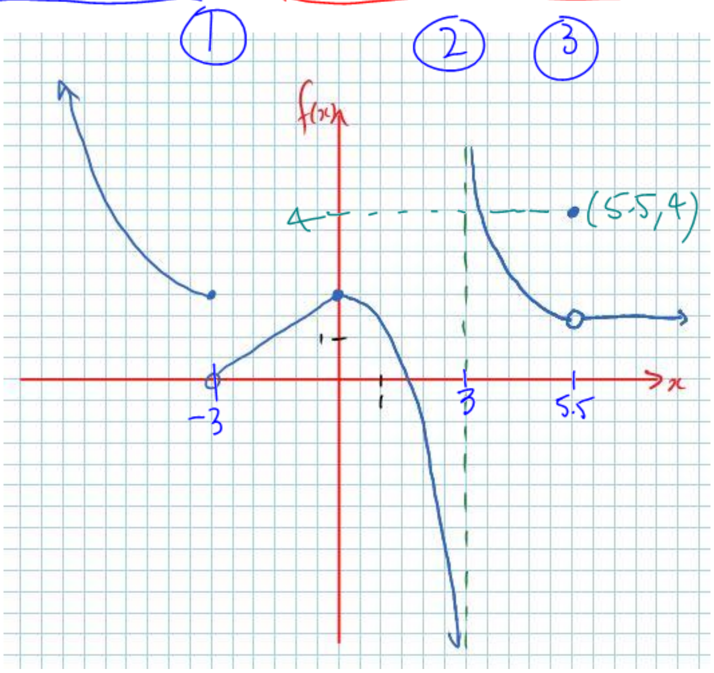
$\Rightarrow u \to (-1)^{\frac{1}{3}} = -1$

$$= \lim_{u\to -1}\left(\frac{u+1}{u^3+1}\right)$$

$$= \lim_{u\to -1}\left(\frac{\cancel{u+1}\ 1}{\cancel{(u+1)}(u^2-u+1)}\right)$$

$$= \frac{1}{3}$$

4. The following sketch of the graph of the function $f(x)$ shows three discontinuities. State where the discontinuity is, what the type of discontinuity it is, and which condition of continuity is being broken.



① at $x = -3$ there is a jump
Condition ② is broken since $\lim_{x \to -3} (f(x))$ dne.

② at $x = +3$ there is an infinite discontinuity (asymptote)
All 3 conditions of continuity are broken

③ at $x = 5.5$ we have a hole
condition ③ is broken since $f(5.5) = 4$
but $\lim_{x \to 5.5} (f(x)) = 1.5$.

5. Show that the given function is continuous for all real numbers.

$$f(x) = \begin{cases} 2x^2 + 1, & x \le 0 \\ \cos(x), & x > 0 \end{cases}$$

$x = 0$ is a "break point" for $f(x)$

We need to test both "sides" of the "break point" and the single value of the "break".

for $x < 0$ $f(x) = 2x^2 + 1$ which is a polynomial
$\Rightarrow$ $\boxed{f(x) \text{ is cts on } (-\infty, 0)}$

for $x > 0$ $f(x) = \cos(x)$ which is cts
$\Rightarrow$ $\boxed{f(x) \text{ is cts on } (0, \infty)}$

for $x = 0$ (single value continuity has 3 conditions)

① $f(0) = 2(0)^2 + 1 = 1$

② $\lim_{x \to 0^-} (f(x)) = \lim_{x \to 0^-} (2x^2 + 1) = 1$ | $\lim_{x \to 0^+} (f(x)) = \lim_{x \to 0^+} (\cos(x)) = 1$

$\therefore \lim_{x \to 0} (f(x)) = 1$

$\therefore f(0) = \lim_{x \to 0} (f(x))$ $\boxed{\therefore f(x) \text{ is cts at } x = 0}$

$\boxed{\therefore f(x) \text{ is cts on } (-\infty, \infty)}$