CALCULUS

Chapter 2 –The Derivative

(*Material adapted from Chapter 2 of your text*)

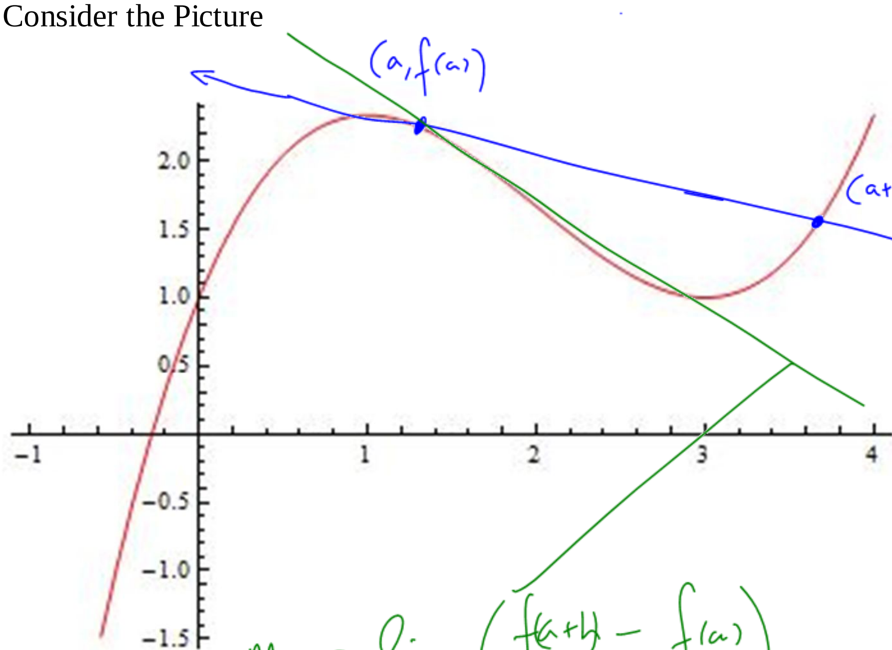
A∞Ω

MATH@TD

2.1 The Derivative as a Function

Recall the concepts of **AROC (AVERAGE RATE OF CHANGE)** and **IROC (INSTANTANEOUS RATE OF CHANGE)**

Consider the Picture



$m_{sec} = \dfrac{f(a+h) - f(a)}{h}$

This slope (NUMBER) represents the AROC of f(x) over the (domain) interval [a, a+h]

$m_{tan} = \lim_{h \to 0} \left(\dfrac{f(a+h) - f(a)}{h} \right)$

m_{tan} (NUMBER) represents the INSTANTANEOUS rate of change of f(x) AT (the single domain value) $x = a$

Clearly the AROC and the IROC are measuring **change** in a function. In a sense, **Calculus is the study of change using functions**. Because **the IROC is fundamental** to studying change through functions, it was given a name: **The Derivative**.

The Derivative of a Function at a Point

Definition 2.1.1

Given a function $f(x)$, and a point on the function $P(a, f(a))$, then the **derivative** of $f(x)$ at $x = a$ is

$$f'(a) = \frac{df}{dx}(a) = \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

prime notation

Leibniz' notation

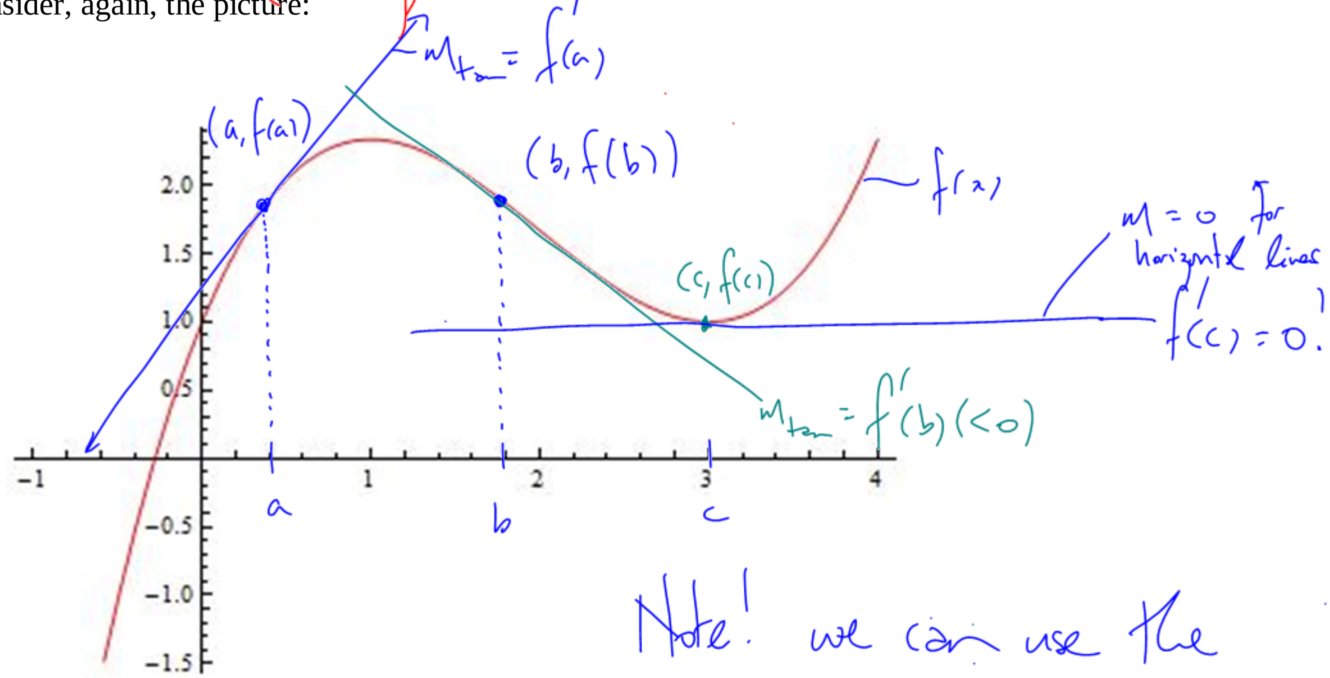
Another notation

$$\left.\frac{df}{dx}\right|_{x=a}$$

The Derivative is a mathematical tool for measuring the IROC of a fn (which may be used to describe some aspect of reality)

Note: $f'(a)$ is a NUMBER representing the IROC of $f(x)$ at $x = a$

Consider, again, the picture:



Note! we can use the derivative to find local max's & min's $(!!!!!)^3$

Example 2.1.1

Determine the derivative of $f(x)=\dfrac{1}{x+2}$ at $x=-1$. using the definition $(a=-1)$ $x=0, 1, 2, 3, 4$

$$f'(-1) = \lim_{h\to 0}\left(\frac{f(-1+h)-f(-1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{1}{-1+h+2}-\frac{1}{-1+2}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{1}{h+1}-\frac{1}{1}}{h}\right)$$

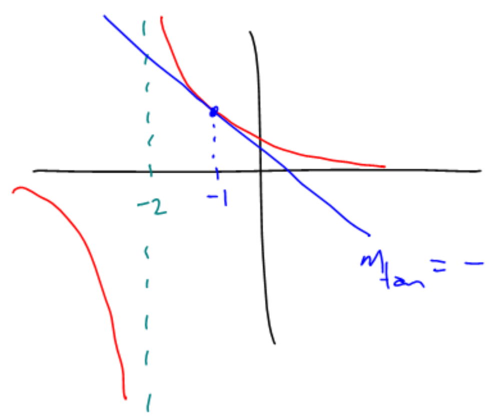
numerator $\Rightarrow$ common denom.

$$= \lim_{h\to 0}\left(\frac{\frac{1}{h+1}-\frac{h+1}{h+1}}{\frac{h}{1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{-\cancel{h}}{(h+1)}\cdot\frac{1}{\cancel{h}}\right)$$

$$= -1$$

picture $f(x)=\dfrac{1}{x+2}$



Calculating the derivative at a single point is useful, but if the calculation needs to be done at multiple points, tedium may set in. It is much more useful to have the derivative as a "number generator. That is, it will be useful to consider the derivative as a FUNCTION.

The Derivative as a Function

MEMORIZE

Definition 2.1.2

The derivative of a function, $f(x)$, is given by

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

x varies!

Example 2.1.2

Determine the derivative of $g(x)=\sqrt{x+1}$ at and point $(x, g(x))$. Use the derivative function to determine the **numbers** $g'(3)$, and $\frac{dg}{dx}(0)$.

$$g'(x) = \frac{1}{2\sqrt{x+1}}$$

$$g'(x) = \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\right)$$

use the conjugate!

$$= \lim_{h\to 0}\left(\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x+h+1)-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{h}\ 1}{\cancel{h}\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}\right) = \frac{1}{2\sqrt{x+1}}$$

$$g'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}$$

$$\frac{dg}{dx}(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

Example 2.1.3

Determine $\frac{df}{dx}(x)$ for $f(x) = x^3$.

$$\frac{df}{dx}(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Two techniques
① Expand & collect like terms
② Factor the diff of cubes

$$= \lim_{h \to 0}\left(\frac{(x+h)^3 - x^3}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\cancel{(x+h-x)}\left((x+h)^2 + x(x+h) + x^2\right)}{\cancel{h}}\right)$$

$$= 3x^2$$

Differentiability

Definition 2.1.3

A function $f(x)$ is said to be **differentiable** at $x = a$ if

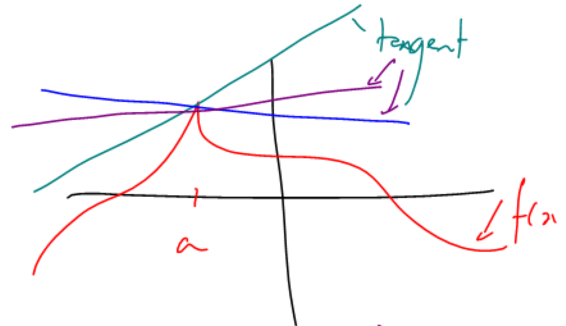
$f'(a)$ exists

ie, the limit $\lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$ must exist

There are three types of non-differentiability.

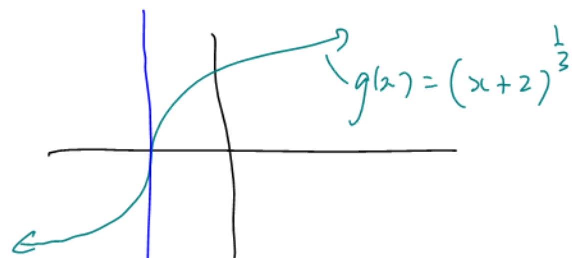
Pictures

① At a 'cusp' of a fn



at $x = a$, there are ∞'ly many tangents through $(a, f(a))$ $\Rightarrow$ No derivative

② At a tangent which is vertical



m_{tan} is undefined

$\Rightarrow$ No derivative

③ At any discontinuity of the f

Class/Homework for Section 2.1

Pg. 73 – 75 #5 – 7, 10, 11, 14, 19