Hwk Check. § 2.2

5. Determine the value(s) of x so that the tangent to the function $f(x)=\frac{3}{\sqrt[3]{x}}$ is parallel to the line $x-8y+1=0$.

The line has slope: $x+1=8y \Rightarrow y=\frac{1}{8}x+\frac{1}{8}$

$m=\frac{1}{8}$

derivative

(we want our tangent to have that slope)

$\Rightarrow f'(x)=\frac{1}{8}$

write as a power fn

$$f(x)=3x^{-\frac{1}{3}}$$

$$\Rightarrow f'(x)=-x^{-4/3}$$

$$\Rightarrow \frac{1}{8}=-x^{-4/3}$$

$$\Rightarrow -\frac{1}{8}=\frac{1}{x^{4/3}}$$

$$\Rightarrow x^{4/3}=-8$$

$$\Rightarrow x=(-8)^{3/4}$$

$$x^3=\square$$

$$x=(\square)^{\frac{1}{3}}$$

$\therefore$ no soln

$\therefore$ $\nexists$ an x value where $g(x)$ has a tangent with slope $\frac{1}{8}$

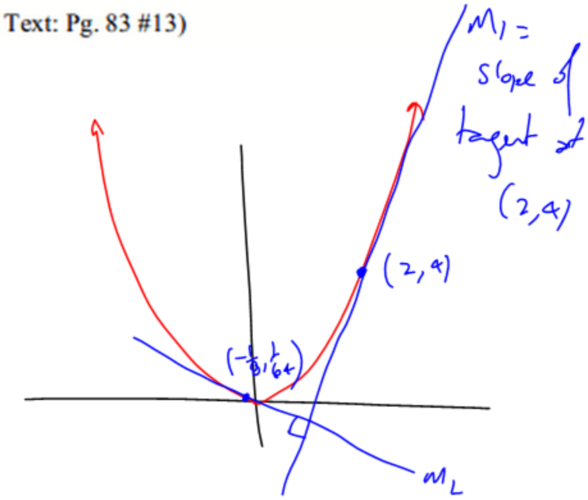
6. Tangents are drawn to the parabola $y = x^2$ at $(2,4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Show that the tangents are perpendicular to each other. (From the Nelson Text: Pg. 83 #13)

derivative

$$y' = 2x$$

$$\therefore m_1 = y'\Big|_{x=2} = 2(2) = 4$$

evaluated at



we expect $m_1 = -\frac{1}{m_2}$

Further

$$m_2 = y'\Big|_{x=-\frac{1}{8}} = 2\left(-\frac{1}{8}\right) = -\frac{1}{4} = -\frac{1}{m_1}$$

$\therefore$ tangent 1 $\perp$ tangent 2

is perpendicular to

7. Show that there are two tangents to the curve $g(x) = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

(From the Nelson Text: Pg. 83#16)

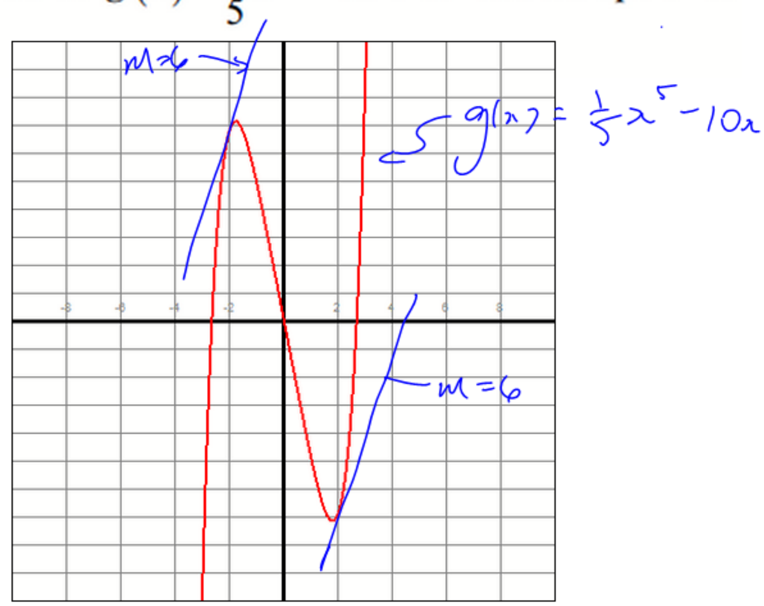
$$g'(x) = x^4 - 10$$

we want $g'(x) = 6$

$$\Rightarrow x^4 - 10 = 6$$

$$\Rightarrow x^4 - 16 = 0$$

$$\Rightarrow (x^2 - 4)(x^2 + 4) = 0$$



$\Rightarrow (x-2)(x+2)(x^2+4)=0$

$\therefore \quad x=2, \quad x=-2$

$\therefore$ There are two points on $g(x)$: $(2, g(2)) = (2, -\frac{68}{5})$

$(-2, g(-2)) = (-2, \frac{68}{5})$

where the slope of the tangents is 6

8. A subway train travels from one station to the next in 2 minutes. Its distance (in *km*) from the first station after *t* minutes is given by $s(t)=t^2-\frac{1}{3}t^3$. At what time with the train have a velocity of $0.5 km/\min$. (From the Nelson Text: Pg. 84 #21)

$\hookrightarrow v = 0.5$

Note: velocity is the $\dfrac{\text{rate of change of } s(t)}{\text{rate of change of } t} = \dfrac{ds}{dt} = s'(t)$

$v(t) = s'(t) = 2t - t^2$

$\Rightarrow 0.5 = 2t - t^2$

$\Rightarrow t^2 - 2t + 0.5 = 0 \quad (\times 2)$ eliminate 0.5

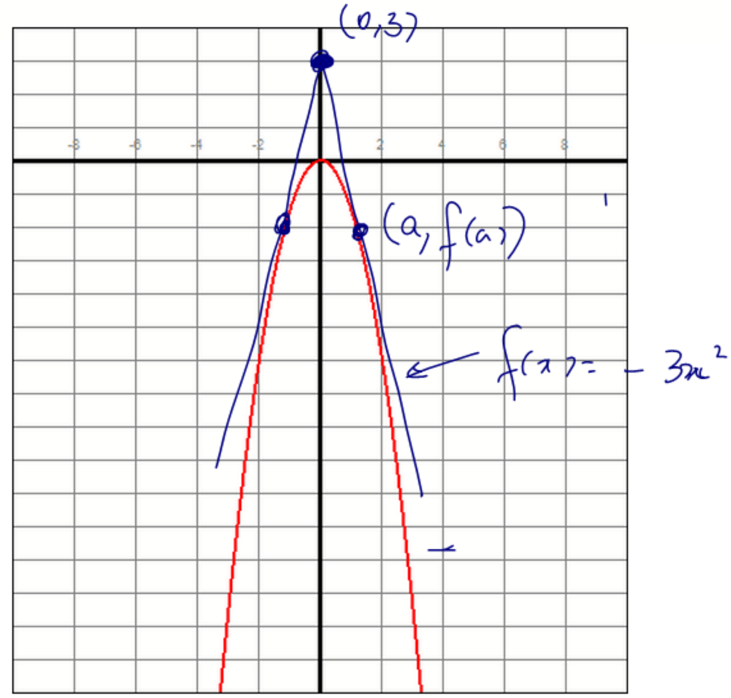
$\Rightarrow 2t^2 - 4t + 1 = 0$ (does not factor)

Q.F. $\rightarrow t = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$t = \frac{4 \pm \sqrt{16-4(2)(1)}}{2(2)} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2}$$

$\therefore t = \frac{2+\sqrt{2}}{2} = 1.71\,\text{min}$ or $t = \frac{2-\sqrt{2}}{2} = 0.29\,\text{min}$.

9. Tangents are drawn from the point $(0,3)$ to the parabola $f(x)=-3x^2$. Find the coordinates of the points where the tangents touch the curve. (Draw a sketch)
(From the Nelson Text: Pg. 84 #23)



$f'(x) = -6x$

$\therefore$ The slope of the tangent at the point $(a, f(a)) = (a, -3a^2)$

is

$$\boxed{m_{tan} = -6a}$$

BUT we can also get slope using

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ using the points $(0,3)$, $(a, -3a^2)$

$$\boxed{\Rightarrow m_{tan} = \frac{-3a^2 - 3}{a - 0}}$$

$\therefore$ we have

$$\frac{-6a}{1} = \frac{-3a^2 - 3}{a}$$ (solve for a)

$\Rightarrow -6a^2 = -3a^2 - 3$

$\Rightarrow 3a^2 - 3 = 0 \rightarrow a^2 - 1 = 0$ $((a-1)(a+1)=0)$

$\therefore a = \pm 1$

$\therefore$ The points of tangency are : $(1, f(1)) = (1, -3)$
$(-1, f(-1)) = (-1, -3)$

4. Determine the equation of the tangent to the curve $f(x) = 2x^3 - 5x^2 + 3$ at the point $P(2, -1)$.

$f'(x) = 6x^2 - 10x$

$f'(2) = 6(2)^2 - 10(2)$

$= 24 - 20$

$= 4$

$y = mx + b$

$-1 = 4(2) + b$

$-1 = 8 + b$

$-9 = b$

$\boxed{y = 4x - 9}$

2.3 The Product Rule

Theorem

Given $f(x)$, and $g(x)$, both differentiable, **then** the **PRODUCT FUNCTION**

$$F(x) = f(x) \cdot g(x)$$

is differentiable, and

$$F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Proof

$$F'(x) = \lim_{h \to 0} \left(\frac{F(x+h) - F(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \right)$$

we added zero (a convenient zero!)

$$= \lim_{h \to 0} \left(\frac{f(x+h) \cdot g(x+h) \underbrace{- f(x) \cdot g(x+h) + f(x) \cdot g(x+h)} - f(x) \cdot g(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{g(x+h)\left(f(x+h) - f(x)\right) + f(x)\left(g(x+h) - g(x)\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(g(x+h) \cdot \frac{f(x+h) - f(x)}{h} \right) + \lim_{h \to 0} \left(f(x) \cdot \frac{g(x+h) - g(x)}{h} \right)$$

$$= g(x) \cdot f'(x) + f(x) \cdot g'(x) \quad \square$$

product of 2 fns

recall $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Example 2.3.1

Differentiate $f(x) = \left(3x^2 - 5x + 2\right)\left(3\sqrt{x} - 5x^{-2}\right)$

Note: The Product Rule says "(deriv. of 1st times 2nd) + (first times deriv. of 2nd)"

$$f'(x) = \left(3x^2 - 5x + 2\right)'\left(3\sqrt{x} - 5x^{-2}\right) + \left(3x^2 - 5x + 2\right)\left(3\sqrt{x} - 5x^{-2}\right)'$$

$$= \left(6x - 5\right)\left(3\sqrt{x} - 5x^{-2}\right) + \left(3x^2 - 5x + 2\right)\left(\frac{3}{2\sqrt{x}} + 10x^{-3}\right)$$

Done

Simplify when you can → "factor" or expand & collect like terms

Triple Product Rule

Given three differentiable functions $f(x)$, $g(x)$, and $h(x)$, **then** the function

$$H(x) = f(x) \cdot g(x) \cdot h(x)$$

is also differentiable, and

$$H'(x) = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

Note: we already *know* the "simple" product rule.

Proof

Let $H(x) = \left(f(x) \cdot g(x)\right) \cdot h(x)$

$$\Rightarrow H'(x) = \left(f(x) \cdot g(x)\right)' \cdot h(x) + \left(f(x) \cdot g(x)\right) \cdot h'(x)$$

$$= \left(f'(x) \cdot g(x) + f(x) \cdot g'(x)\right) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

$$= f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x) \quad \square$$

Example 2.3.2

Differentiate $f(x) = (3x^2 - 2x)^3$

Triple product rule

$$f(x) = (3x^2 - 2x)(3x^2 - 2x)(3x^2 - 2x)$$

$$\Rightarrow f'(x) = (6x-2)(3x^2-2x)(3x^2-2x) + (3x^2-2x)(6x-2)(3x^2-2x) + (3x^2-2x)(3x^2-2x)(6x-2)$$

$$= 3(6x-2)(3x^2-2x)^2$$

The Power of a Function Rule

Given a differentiable function $f(x)$, then the power function

$$F(x) = (f(x))^n$$

is also differentiable and

$$\frac{dF}{dx}(x) = \left[n \cdot (f(x))^{n-1}\right] \cdot \left(\frac{df}{dx}(x)\right)$$

bring the power down (leave inside alone!) reduce power by 1 times (deriv. of inside)

Example 2.3.3

Differentiate $(3x^2 - 2x)^3$

$$\frac{d}{dx}\left((3x^2 - 2x)^3\right) = 3(3x^2 - 2x)^2 \cdot (6x - 2)$$

Example 2.3.4

Differentiate $\left(x^{\frac{2}{3}} - 5x^2 + 3x - 5\right)^{52}$

$$\frac{d}{dx}\left(\left(x^{2/3} - 5x^2 + 3x - 5\right)^{52}\right)$$

$$= 52\left(x^{2/3} - 5x^2 + 3x - 5\right)^{51} \cdot \left(\frac{2}{3}x^{-\frac{1}{3}} - 10x + 3\right)$$

Class/Homework for Section 2.3

Pg. 90 – 91 #2, 3, 5