2.4 The Quotient Rule

Theorem

Given two differentiable functions $f(x)$ and $g(x)$, **then** the quotient function

$$H(x)=\frac{f(x)}{g(x)}, \quad g(x)\neq 0$$

is also differentiable, and

$$H'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}.$$

Proof: By defn

$$H'(x)=\lim_{h\to 0}\left(\frac{H(x+h)-H(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\frac{f(x+h)g(x)-f(x)\cdot g(x+h)}{g(x+h)g(x)}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(x+h)\cdot g(x)-f(x)\cdot g(x+h)}{h\left(g(x+h)\cdot g(x)\right)}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(x+h)\cdot g(x)-g(x)\cdot f(x)+g(x)\cdot f(x)-f(x)g(x+h)}{h\left(g(x+h)\cdot g(x)\right)}\right)$$

$$=\lim_{h\to 0}\left(\frac{g(x)\cdot\overbrace{\left(f(x+h)-f(x)\right)}^{f'(x)}}{h\left(g(x+h)\cdot g(x)\right)}-\frac{f(x)\overbrace{\left(g(x+h)-g(x)\right)}^{g'(x)}}{h\left(g(x+h)\cdot g(x)\right)}\right)$$

$$=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{(g(x))^2}$$

Example 2.4.1

Differentiate using the Quotient Rule, simplifying as much as possible.

$$f(x)=\frac{5x-2}{x+1}$$

$$F(x)=\frac{T}{B}$$

$$F'(x)=\frac{T'\cdot B-T\cdot B'}{B^2}$$

$$f'(x)=\frac{(5)(x+1)-(5x-2)(1)}{(x+1)^2}$$

$$=\frac{7}{(x+1)^2}$$

Example 2.4.2

Differentiate, and simplify.

$$g(x)=\frac{(2x+1)^2}{(x-1)^2}$$

$$g'(x)=\frac{2(2x+1)(2)(x-1)^2-(2x+1)^2(2(x-1))}{(x-1)^4}$$

$$=\frac{2(2x+1)\cancel{(x-1)}\left(2(x-1)-(2x+1)\right)}{(x-1)^{\cancel{4}\,3}}, \quad x\neq 1$$

$$=\frac{2(2x+1)(-3)}{(x-1)^3}=\frac{-6(2x+1)}{(x-1)^3}$$

Example 2.4.3

Differentiate and simplify.

$$g(t)=\frac{\sqrt{t}}{2t^2+3}$$

Note:

$f(x)=\sqrt{x}$

$f(x)=(x)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}(x)^{-\frac{1}{2}}$

$=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$=\frac{1}{2\sqrt{x}}$

$$g'(t)=\frac{\frac{1}{2\sqrt{t}}(2t^2+3)-\sqrt{t}\,(4t)}{(2t^2+3)^2}$$

$$=\frac{\frac{2t^2+3}{2\sqrt{t}}-4t^{3/2}}{(2t^2+3)^2}$$

$\left(4t^{3/2}\cdot 2\sqrt{t} = 8t^2\right)$

$$=\frac{\frac{2t^2+3-8t^2}{2\sqrt{t}}}{(2t^2+3)^2}=\frac{-6t^2+3}{2\sqrt{t}\,(2t^2+3)^2}$$

Example 2.4.4

Differentiate and simplify.

$$(\sqrt{x})^2 = x$$

$$h(x) = \frac{(4x^2-5)^3}{\sqrt{x}}$$

$$h'(x) = \frac{3(4x^2-5)^2(8x)(x^{\frac{1}{2}}) - (4x^2-5)^3\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{(4x^2-5)^2\left(24x^{3/2} - \frac{4x^2-5}{2\sqrt{x}}\right)}{x}$$

$$= \frac{\frac{(4x^2-5)^2}{1}\left(\frac{48x^2 - 4x^2 + 5}{2\sqrt{x}}\right)}{x} = \frac{(4x^2-5)^2(44x^2+5)}{2x^{3/2}}$$

"squish bam"

Example 2.4.5

From your text, Pg. 97 #7

Determine the points on the graph of $f(x) = \frac{3x}{x-4}$ where the slope of the tangent is $-\frac{12}{25}$

$$f'(x) = \frac{3(x-4) - (3x)(1)}{(x-4)^2}$$

$$= \frac{-12}{(x-4)^2}$$

we want the slope to be $-\frac{12}{25}$

$\Rightarrow |x-4| = 5$

$$(m_{tan} =) \quad -\frac{12}{25} = \frac{-12}{(x-4)^2}$$

$$\Rightarrow (x-4)^2 = 25$$

$\therefore x - 4 = \pm 5$

$\therefore x = 5+4 = 9$ or $x = -5+4 = -1$

$\therefore$ The points are $(9, f(9))$, $(-1, f(-1))$

$= (9, \frac{27}{5})$, $(-1, \frac{3}{5})$

Example 2.4.6

From your text, Pg. 97 #5d

Determine $\frac{dy}{dx}$ at $x=4$ for $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

To simplify we will expand the numerator & denom

$$y = \frac{x^2+3x+2}{x^2-3x+2}$$

$$y' = \frac{(2x+3)(x^2-3x+2) - (x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$$

no need to simplify since we are just calculating a #

$$y'\Big|_{x=4} = \frac{(2(4)+3)(4^2-3(4)+2) - (4^2+3(4)+2)(2(4)-3)}{(4^2-3(4)+2)^2}$$

$$= \frac{-84}{36} = -\frac{21}{9} = -\frac{7}{3}$$

Class/Homework for Section 2.4

Pg. 97 – 98 #2, 4 – 6, 8 – 13, 15