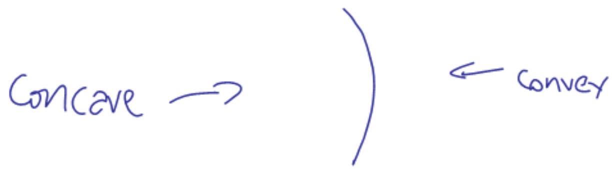
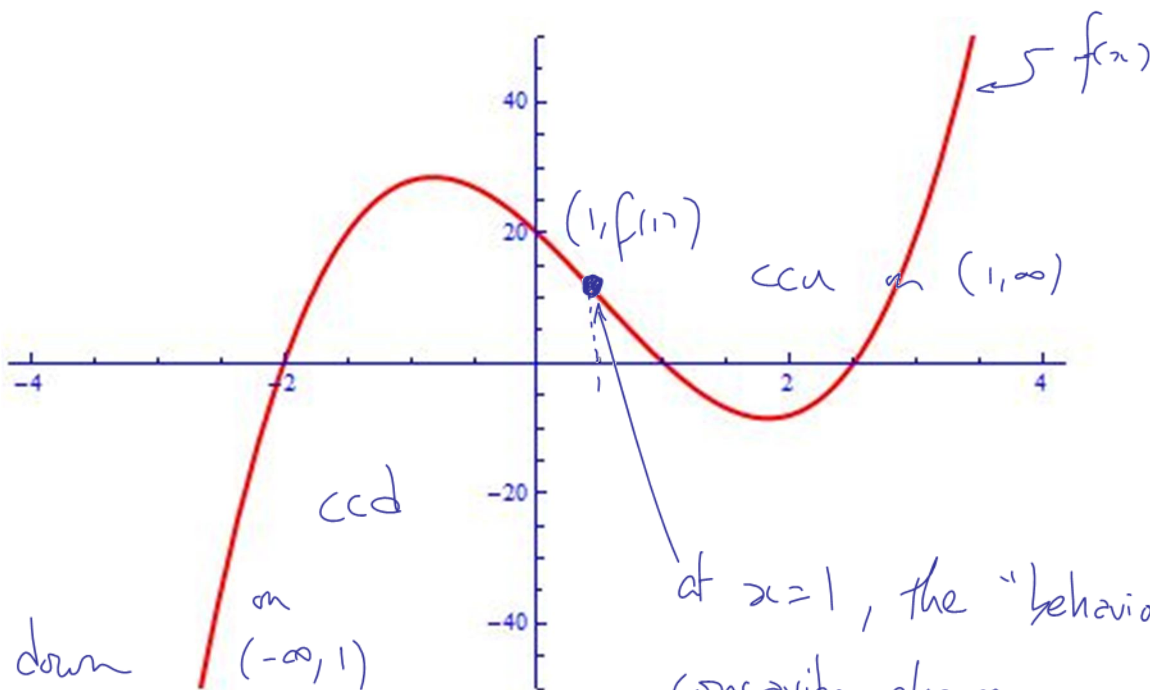
4.4 Concavity and Points of Inflection

Recall that concave means to bend "away" from the perspective of observation.



Consider the sketch of the graph of $f(x)$



ccd

≡ concave down

ccu

≡ concave up

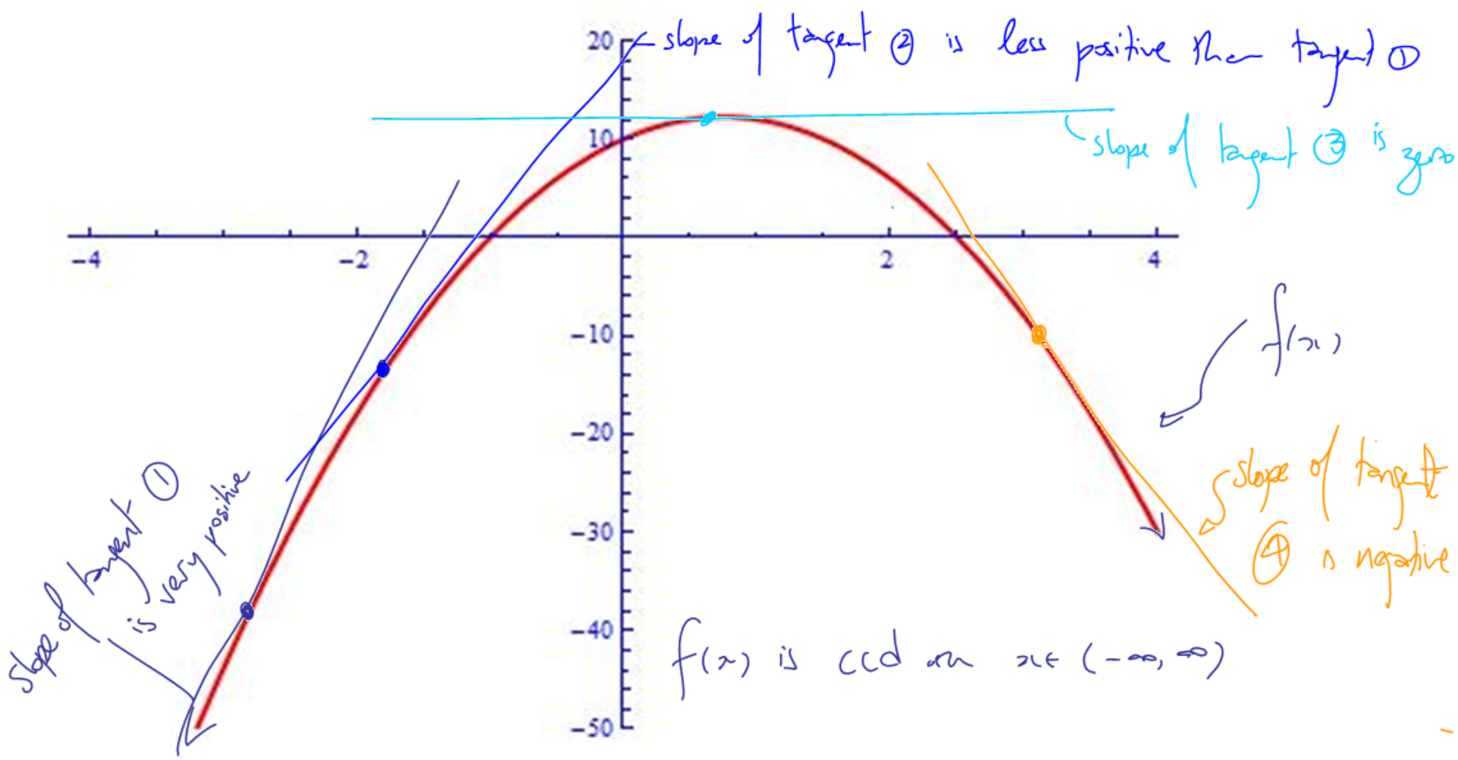
at $x = 1$, the "behavior of" concavity changes

we call such a value a point of inflection

(though $(1, f(1))$ is actually the point of inflection)

Q. How can we determine when an algebraic representation of a function is concave up or concave down?

Consider the Picture



Note: ① The 1st derivative measures how fast the original fn is change

② In the picture above, we see that the 'values' of $f'(x)$ are decreasing as we move left to right on the domain

③ The first derivative is changing, and that change can be measured by the second derivative

It appears that a fn, $f(x)$, is ccd when $f''(x) < 0$

Second Derivative Test

If $x = c$ is a critical value of some **twice differentiable** function $f(x)$ (i.e. $f'(c) = 0$), then

If $f''(c) > 0$, $x = c$ is a critical value where a local min occurs (the fn $f(x)$ is ccu)

If $f''(c) < 0$, $x = c$ is a critical value where a local max. occurs (the fn $f(x)$ is ccd)

Note: **Sometimes** the Second Derivative Test **tells us nothing**!

Consider the following:

Given $f(x) = x^4$, determine and critical values, and use the Second Derivative Test to determine whether the c.v.(s) are locations of local maxima and/or minima.

$f'(x) = 4x^3$ set to zero for C.V.

$\Rightarrow 4x^3 = 0$

$\Rightarrow x = 0$

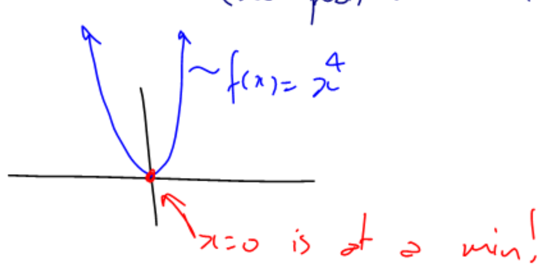
$f''(x) = 12x^2$

Test C.V. $x=0$

$f''(0) = 12(0)^2$

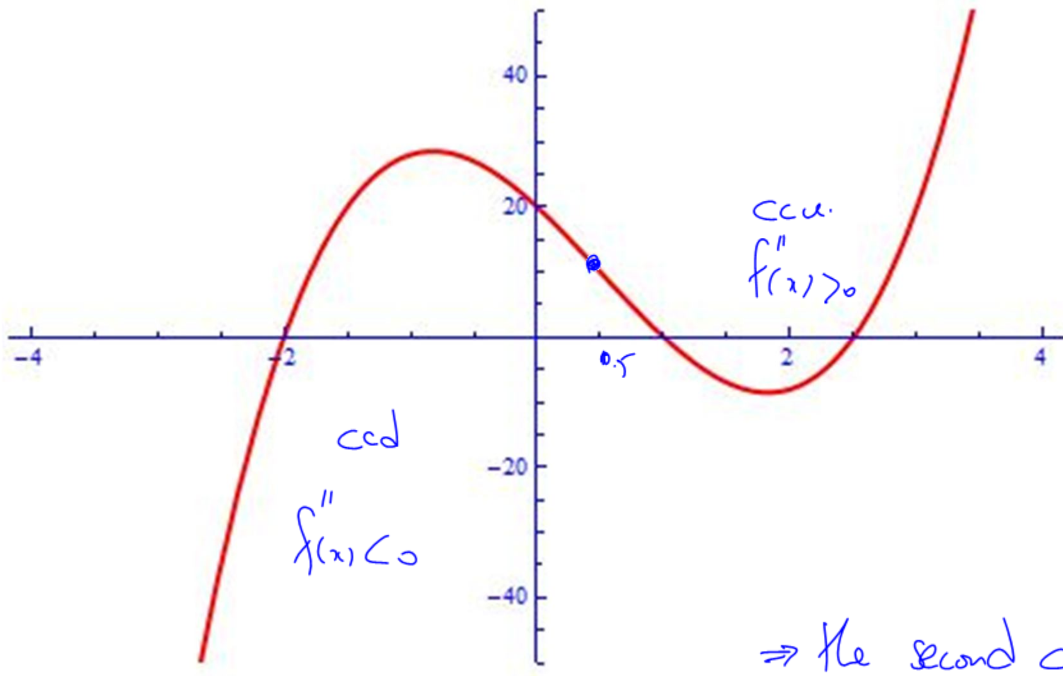
$= 0 \Rightarrow$ we do not have info about max/min.

(not positive nor negative!) $\Rightarrow$ Turn to 1^{ST} derivative test



Q. Well then, what is happening when $f''(x)=0$? where concavity changes

Consider the following Picture:



Note: the concavity of f(x) changes somewhere (x = 0.5)

⇒ the second derivative has to be zero somewhere

$f''(0.5) = 0$

Definition 4.4.1

Given a **twice differentiable** function, $f(x)$, for any value $x=c$ where $f''(c)=0$, we call $x=c$ a **possible point of inflection**.

A **Point of Inflection** is where a function's **concavity changes** (and since functional behaviour changes, P.O.I.'s are also considered "critical"!)

Example 4.4.1

Given $f(x) = x^3 - \frac{5}{2}x^2 - 2x + 1$ determine all c.v.'s. Use the Second Derivative Test for max/min. Determine any P.P.O.I, and state intervals of concavity.

$f'(x) = 3x^2 - 5x - 2$ set to zero for C.V.'s

$\Rightarrow 3x^2 - 5x - 2 = 0$

$(3x+1)(x-2) = 0$

$\therefore x = -\frac{1}{3}$ or $x = 2$

$f''(x) = 6x - 5$ Test $x = -\frac{1}{3}$, $x = 2$ for max/min.

$f''(-\frac{1}{3}) = 6(-\frac{1}{3}) - 5$

$= -7 < 0$

$\therefore$ ccd $\therefore$ at $x = -\frac{1}{3}$ $f(x)$ has a local max with value $f(-\frac{1}{3})$

$f''(2) = 6(2) - 5$

$= +7 > 0$

$\therefore$ ccu

$\therefore$ at $x = 2$, $f(x)$ has a local min (with value $f(2)$)

Class/Homework for Section 4.4

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$P.P.O.I. \Rightarrow f''(x) = 0$

$\Rightarrow 6x - 5 = 0$

$\Rightarrow x = \frac{5}{6}$

$\boxed{\text{A P.P.O.I is a P.O.I } \underline{\text{IF}} \text{ concavity CHANGES } \underline{\text{around}}\ x = \frac{5}{6} \text{ (in this case)}}$

Test around $x = \frac{5}{6}$

Test $x = 0$ & $x = 1$ in $f''(x)$

$f''(0) = 6(0) - 5$
$= -5 < 0 \therefore$ ccd

$f''(1) = 6(1) - 5$
$= 1 > 0$
$\therefore$ ccu

$\therefore$ concavity changes $x = \frac{5}{6}$ $\underline{\text{is}}$ a P.O.I.

and $f(x)$ is ccd on $x \in (-\infty, \frac{5}{6})$

$f(x)$ is ccu on $x \in (\frac{5}{6}, \infty)$