10. Find constants a, b, and c such that the function $f(x) = ax^3 + bx^2 + c$ will have a local extremum at $(2, 11)$ and a point of inflection at $(1, 5)$. Sketch the graph of $y = f(x)$.

$f'(x) = 3ax^2 + 2bx$

$f'(2) = 0 \Rightarrow 3a(2)^2 + 2b(2) = 0$

$\Rightarrow 12a + 4b = 0$ ①

③ $\times 2$ $\quad 12a + 4b = 0$ ③ No INFO

$f''(x) = 6ax + 2b$

$f''(1) = 0$

$\Rightarrow 6a + 2b = 0$ ②

① $\Rightarrow b = -3a$

using $(2,11)$; $(1,5)$

$\Rightarrow 11 = 8a + 4b + c$ ④ $\quad 5 = a + b + c$ ⑤

④ − ⑤ $\quad 6 = 7a + 3b$ but $b = -3a$

$\Rightarrow 6 = 7a - 9a$

$a = -3 \Rightarrow b = 9 \Rightarrow c = -1$ by ⑤

$\therefore f(x) = -3x^3 + 9x^2 - 1$

11. Find the value of the constant b such that the function $f(x) = \sqrt{x+1} + \frac{b}{x}$ has a point of inflection at $x = 3$.

$f''(3) = 0$

$f'(x) = \dfrac{1}{2\sqrt{x+1}} - \dfrac{b}{x^2}$

$f''(x) = -\dfrac{1}{4(x+1)^{3/2}} + \dfrac{2b}{x^3}$

$0 = \dfrac{1}{-4(4)^{3/2}} + \dfrac{2b}{(3)^3}$

$0 = -\dfrac{1}{32} + \dfrac{2b}{27}$

$\dfrac{2b}{27} = \dfrac{1}{32}$

$\boxed{b = \dfrac{27}{64}}$

4.5 Bringing It All Together: Sketching Curves

What we have learned thus far:

From the First Derivative:

How to find Critical Values (set $f'(x)=0$ and solve for *x*)

How to find intervals of **increase**/**decrease** $\left(f'(x)>0,\ \ f'(x)<0\right)$

How to show if a c.v. is the location of a max or min (First Derivative Test)

From the Second Derivative

How to find **Possible** Points of Inflection $\left(f''(x)=0\right)$

How to find intervals where a function is **concave up**/**concave down**

$\left(f''(x)>0,\ \ f''(x)<0\right)$

How to test c.v.'s for max/min (Second Derivative Test)

Asymptotes

Rational Functions may have:

Vertical Asymptotes: $\lim\limits_{x\to a}\left(f(x)\right)=\infty,\ \Rightarrow x=a$ is a V.A.

Horizontal Asymptotes: $\lim\limits_{x\to\infty}\left(f(x)\right)=b,\ \Rightarrow y=b$ is a H.A.

Oblique Asymptotes: If $f(x)=\dfrac{\text{degree }(n+1)}{\text{degree }n}$

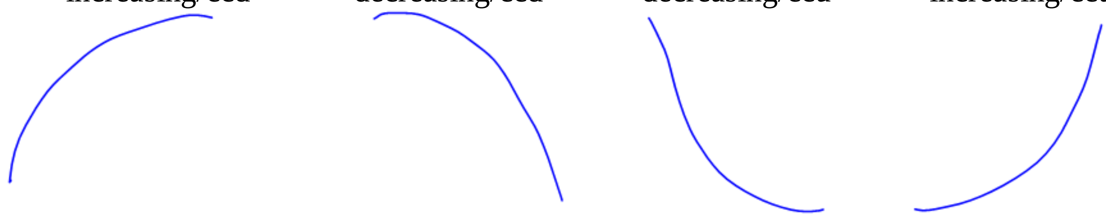
Algorithm for Sketching Curves

We must:

1) Find all intercepts
2) Find all c.v.'s
3) Find all asymptotes
4) Determine all P.P.O.I.
5) Determine all special **points**
6) Analyze all information in an Interval Chart
7) Sketch the curve.

Note: **All** (*infinitely many*) possible functions can be sketched using a combination of **FOUR BASIC SHAPES:**

increasing/ccd decreasing/ccd decreasing/ccu increasing/ccu



Your Interval Chart will look like:

INTERVALS	SPLIT DOMAIN at all C.V., PPOI, V.A.
TEST VAL.	
SIGN OF $f'_{(x)}$ (inc/dec)	
SIGN OF $f''_{(x)}$ (ccu/ccd)	
SHAPE	

Example 4.5.1

Sketch $f(x) = x^4 - 8x^2 + 7$

INTERCEPTS

y int $\rightarrow x = 0$

$(0, 7)$

x int $\Rightarrow f(x) = 0$

$x^4 - 8x^2 + 7 = 0$

$(x^2 - 7)(x^2 - 1) = 0$

$\Rightarrow x = \pm\sqrt{7}$, $x = \pm 1$

Points

$(-\sqrt{7}, 0)$, $(\sqrt{7}, 0)$

$(-1, 0)$ $(1, 0)$

P.P.O.I.

$f''(x) = 12x^2 - 16$

set to zero

$12x^2 - 16 = 0$

$x^2 = \frac{4}{3}$

$\Rightarrow x = -\frac{2}{\sqrt{3}}, +\frac{2}{\sqrt{3}}$

-1.15

$\left(-\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$ $\left(\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$

C.V.

$f'(x) = 4x^3 - 16x$

set to zero

$4x(x^2 - 4) = 0$

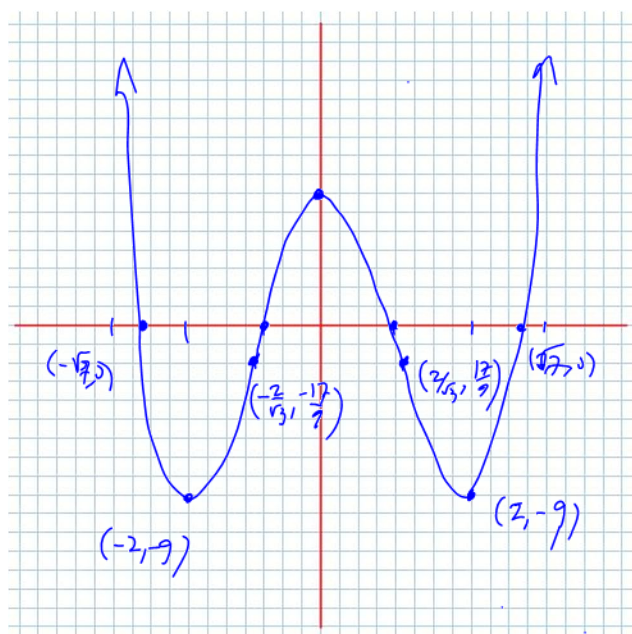
$x = 0$, $x = 2$, $x = -2$

$(0, 7)$ $(2, -9)$ $(-2, -9)$

ASYMPTOTES

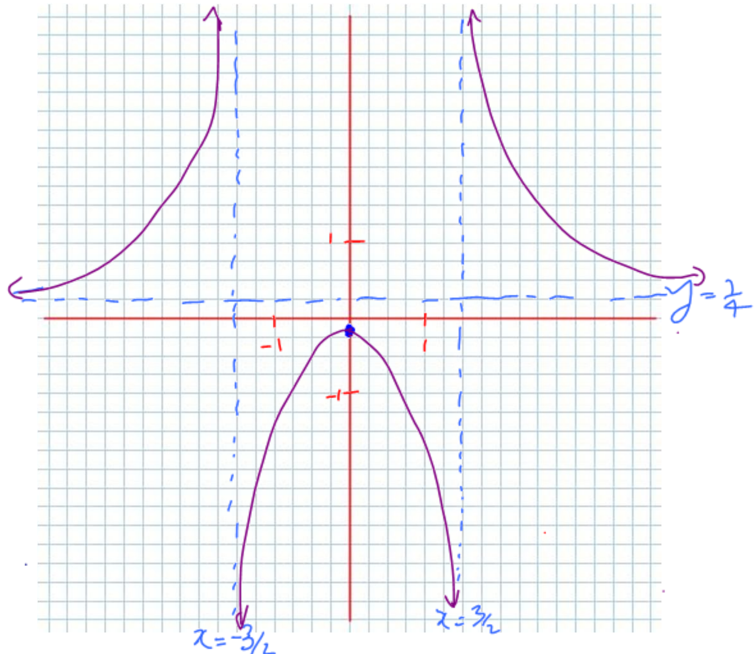
none

INTERVALS	$(-\infty, -2)$	$\left(-2, -\frac{2}{\sqrt{3}}\right)$	$\left(-\frac{2}{\sqrt{3}}, 0\right)$	$\left(0, \frac{2}{\sqrt{3}}\right)$	$\left(\frac{2}{\sqrt{3}}, 2\right)$	$(2, \infty)$
T.V.	-3	-1.5	-1	1	1.5	3
Sign on $f'(x) = 4x^3 - 16x$	-ve dec	+ve inc	+ve inc	-ve dec	-ve dec	+ve inc
Sign on $f''(x) = 12x^2 - 16$	+ve ccu	+ve ccu	-ve ccd	-ve ccd	+ve ccu	+ve ccu
SHAPE	local min		POI	local max	POI	local min



Example 4.5.2

Sketch $g(x)=\dfrac{x^2+1}{4x^2-9}$



INTERCEPTS

y-int $(0,-\frac{1}{9})$ x-int $\rightarrow g(x)=0$

$\Rightarrow x^2+1=0$ NEVER

C.V.'s

$$g'(x)=\frac{2x(4x^2-9)-(x^2+1)(8x)}{(4x^2-9)^2}$$

$$=\frac{-26x}{(4x^2-9)^2}$$

set to zero $\Rightarrow -26x=0$

$\Rightarrow x=0$ $(0,-\frac{1}{9})$

Asymptotes

V.A. $4x^2-9=0$ H.A.

$x=\pm\frac{3}{2}$ $y=\frac{1}{4}$

INTERVALS	$(-\infty,-3/2)$	$(-3/2,0)$	$(0,3/2)$	$(3/2,\infty)$
T.V.	-2	-1	1	2
SIGN on $g'(x)=\frac{-26x}{(4x^2-9)^2}$	+ve inc	+ve inc	-ve dec	-ve dec
SIGN ON $g''(x)=\frac{-26(-12x^2-9)}{(4x^2-9)^3}$	+ve ccu	-ve ccd	-ve ccd	+ve ccu
SHAPE		VA.	local max	V.A

PPOI

$$g''(x)=\frac{-26(4x^2-9)^2-(-26x)(2(4x^2-9)(8x))}{(4x^2-9)^4}$$

$$=\frac{-26\cancel{(4x^2-9)}((4x^2-9)-16x^2)}{(4x^2-9)^{\cancel{4}\,3}}$$

$$=\frac{-26(-12x^2-9)}{(4x^2-9)^3}$$

$\rightarrow$ set to zero

$-26(-12x^2-9)=0$

$\Rightarrow x^2=-\frac{9}{12}$ impossible

$\therefore$ No P.o.I.

Class/Homework for Section 4.5

Pg. 212 – 213 #2 – 6