MCV4U - W14: Some Calculus Exam Practice Problems

SOLUTIONS

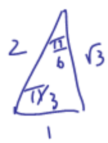
1. Determine the limit for each of the following:

"plug it in and see what happens"

a) $\lim_{x \to 3} \sin\left(\frac{\pi}{x}\right)$ b) $\lim_{x \to -2} \frac{3x^2+x-2}{x^2-4}$ c) $\lim_{x \to \infty} \frac{3x-5x^2-2x^4}{3x^4-7}$ d) $\lim_{x \to 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x^2-3x+2}$

a) $\lim_{x \to 3} \left(\sin\left(\frac{\pi}{x}\right)\right)$

$= \sin\left(\frac{\pi}{3}\right)$



$= \frac{\sqrt{3}}{2}$

b) $\lim_{x \to -2} \left(\frac{(3x-2)(x+1)}{(x-2)(x+2)}\right)$

$= \frac{8}{0}$ = undefined

c) $\lim_{x \to \infty} \left(\frac{3x-5x^2-2x^4}{3x^4-7}\right)$

$\frac{\text{"order 4"}}{\text{order 4}}$

$= -\frac{2}{3}$

d) $\lim_{x \to 2} \left(\frac{\sqrt{x^2+1}-\sqrt{5}}{(x-2)(x-1)}\right)$ "$\frac{0}{0}$"

$= \lim_{x \to 2} \left(\frac{\sqrt{x^2+1}-\sqrt{5}}{(x-2)(x-1)} \cdot \frac{\sqrt{x^2+1}+\sqrt{5}}{\sqrt{x^2+1}+\sqrt{5}}\right)$

$= \lim_{x \to 2} \left(\frac{x^2+1-5}{(x-2)(x-1)(\sqrt{x^2+1}+\sqrt{5})}\right)$

$= \lim_{x \to 2} \left(\frac{x^2-4}{(x-2)(x-1)(\sqrt{x^2+1}+\sqrt{5})}\right)$

$= \lim_{x \to 2} \left(\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-1)(\sqrt{x^2+1}+\sqrt{5})}\right)$

$= \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

2. Let $f(x) = -3(2x+4)^{\frac{2}{3}}$. For what values of x is $f(x)$ concave up? $\rightarrow$ c.c.u.

$$f'(x) = -2(2x+4)^{-\frac{1}{3}}(2)$$
$$= -4(2x+4)^{-\frac{1}{3}}$$

$$f''(x) = \frac{4}{3}(2x+4)^{-4/3}(2) = \frac{8}{3(2x+4)^{4/3}}$$

We note that $f''(x) > 0$ for all $x \Rightarrow f(x)$ is always c.c.u.

3. Determine the derivative of $f(x) = \sqrt{2x-1}$ from first principles.

$$f'(x) = \lim_{h\to 0}\left(\frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2(x+h)-1} + \sqrt{2x-1}}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{2(x+h)-1-(2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}\right)$$

$$= \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$$

4. Differentiate each of the following. State which differentiation rule (or rules) you used in taking the derivative.

a) $f(x) = 3x^3 - 5x + 7x^{-2}$ b) $g(x) = \left(5x^3 - 3x\right)^3\left(2x^2 - x + 5\right)^4$

c) $h(x) = \dfrac{(3x-1)^2}{2x^2+3}$ d) $f(x) = 2^{\sin\left(5x^2\right)}$

e) $y = \cos(3x)\cdot\sin^3(5x)$ f) $g(x) = \ln\left(\dfrac{\sin(x)}{3x^3 - 5x}\right)$

out of order - apologies.

a) $f'(x) = 9x^2 - 5 - 14x^{-3}$ power rule

d) $f'(x) = 2^{\sin(5x^2)} \cdot \cos(5x^2) \cdot 10x \cdot \ln(2)$

exponential, trig, chain

c) $$h'(x) = \frac{2(3x-1)(3)(2x^2+3) - (3x-1)^2(4x)}{(2x^2+3)^2}$$

quotient, chain.

$$= \frac{2(3x-1)\left(3(2x^2+3) - 2x(3x-1)\right)}{(2x^2+3)^2}$$

$$= \frac{2(3x-1)(2x+9)}{(2x^2+3)^2}$$

b) $g(x) = (5x^3-3x)^3(2x^2-x+5)^4$ product, chain, power

$g'(x) = 3(5x^3-3x)^2(15x^2-3)(2x^2-x+5)^4 + (5x^3-3x)^3\left(4(2x^2-x+5)^3(4x-1)\right)$

$$= (5x^3-3x)^2(2x^2-x+5)^3\Big(\underbrace{3(15x^2-3)(2x^2-x+5) + 4(5x^3-3x)(4x-1)}_{\text{you could expand and collect like terms - not necessary here}}\Big)$$

e) $y = \cos(3x)\cdot\sin^3(5x)$ product, trig, chain.

$y' = -3\sin(3x)\cdot\sin^3(5x) + \cos(3x)\cdot 3\sin^2(5x)\cdot\cos(5x)\cdot 5$

$= 3\sin^2(5x)\left(-\sin(3x)\cdot\sin(5x) + 5\cos(3x)\cos(5x)\right)$

f) $g'(x) = \dfrac{1}{\dfrac{\sin(x)}{3x^3-5x}} \cdot \left(\dfrac{\cos(x)(3x^3-5x) - \sin(x)(9x^2-5)}{(3x^3-5x)^2} \right)$

ln, quotient

$= \dfrac{\cancel{3x^3-5x}}{\sin(x)} \left(\dfrac{\cos(x)(3x^3-5x) - \sin(x)(9x^2-5)}{(3x^3-5x)^{\cancel{2}}} \right)$

$= \dfrac{\cos(x)(3x^3-5x) - \sin(x)(9x^2-5)}{\sin(x)(3x^3-5x)}$

5. Find the points on the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ where the tangent to the curve is parallel to the line $y = 2x$. Show that two of these points have the same tangent.

we want slope = 2

$y' = 4x^3 - 18x^2 + 26x - 10$ set = 2

$\Rightarrow 4x^3 - 18x^2 + 26x - 10 = 2$

$\Rightarrow 2x^3 - 9x^2 + 13x - 6 = 0$ Factor theorem. $(x-1)$ is a factor

1	2	-9	13	-6
		2	-7	6
	2	-7	6	0

$\Rightarrow (x-1)(2x^2 - 7x + 6) = 0$

$\Rightarrow (x-1)(2x-3)(x-2) = 0$

$\Rightarrow x = 1, \ 3/2, \ 2$

Consider tangents at points

find values

$(1,3)$ $(2,5)$ with slope 2

at $(1,3)$ $y = 2x + b$

$\Rightarrow 3 = 2(1) + b \Rightarrow b = 1$

$\Rightarrow y = 2x + 1$

at $(2,5)$ $5 = 2(2) + b \Rightarrow b = 1$

$\Rightarrow y = 2x + 1$ same tangent

6. Let $y = \left(\dfrac{x-1}{x+1}\right)^3$. Determine the equation of the tangent to the curve at $\left(2, \dfrac{1}{27}\right)$.

$$y' = 3\left(\frac{x-1}{x+1}\right)^2 \left(\frac{(1)(x+1) - (x-1)(1)}{(x+1)^2}\right)$$

$$= 3\left(\frac{x-1}{x+1}\right)^2 \left(\frac{2}{(x+1)^2}\right)$$

$$m_{tan} = y'\Big|_{x=2} = 3\left(\frac{1}{3}\right)^2 \left(\frac{2}{3^2}\right) = \frac{2}{27}$$

$\therefore$ eq^n is $y = \frac{2}{27}x + b$ using $\left(2, \frac{1}{27}\right)$

$$\Rightarrow \frac{1}{27} = \frac{2}{27}(2) + b \Rightarrow b = -\frac{3}{27} = -\frac{1}{9}$$

$$\therefore y = \frac{2}{27}x - \frac{1}{9}$$

$$\Rightarrow 2x - 27y - 3 = 0$$

7. Find the equation of the tangent line to the curve $f(x) = \dfrac{x^2}{x-6}$ at the point where $x = 3$.

$$f'(x) = \frac{2x(x-6) - x^2(1)}{(x-6)^2} = \frac{x^2 - 12x}{(x-6)^2}$$

$$f(3) = \frac{9}{-3} = -3$$

$\Rightarrow$ point $(3, -3)$

$$m_{tan} = f'(3) = \frac{9-36}{(-3)^2} = -3$$

$\therefore$ eq^n is $y - (-3) = -3(x-3)$

$\Rightarrow y = -3x + 6$ or $3x + y - 6 = 0$

8. Let $f(x)=\dfrac{(1-3x)^2}{(1-4x)^3}$. Determine the intervals of increase or decrease.

$$f'(x)=\frac{2(1-3x)(-3)(1-4x)^3-(1-3x)^2\left(3(1-4x)^2(-4)\right)}{(1-4x)^6}$$

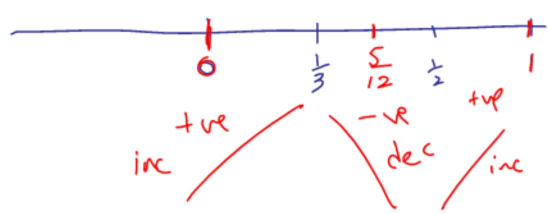
$$=\frac{-6(1-3x)\cancel{(1-4x)^2}\left((1-4x)-2(1-3x)\right)}{(1-4x)^{\cancel{6}\,4}}$$

$$=\frac{-6(1-3x)(2x-1)}{(1-4x)^4}$$ set to zero for C.V.'s

$\Rightarrow -6(1-3x)(2x-1)=0$

$\Rightarrow x=\frac{1}{3}\ ,\quad x=\frac{1}{2}$

Testing



$f'(0)=6>0$

$f'(\frac{5}{12})<0$

$f'(1)>0$

$\therefore$ $f(x)$ is inc on $x\in(-\infty,\frac{1}{3})\cup(\frac{1}{2},\infty)$

$f(x)$ is dec on $x\in(\frac{1}{3},\frac{1}{2})$

→ when $x=e^{-3/2}$ $y=(e^{-3/2})^2(\ln(e^{-3/2}))=(e^{-3})(-\frac{3}{2})$

9. Investigate whether the graph of the curve $y=x^2\ln x$ has a point of inflection. If it does, state the coordinates of the point of inflection.

$y'=2x\ln(x)+x^2\left(\frac{1}{x}\right)$

$=2x\ln(x)+x$

$\Rightarrow y''=2\ln(x)+2x\left(\frac{1}{x}\right)+1$

$=2\ln(x)+3$ set to zero

$2\ln(x)+3=0$

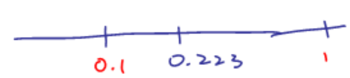
$\ln(x)=-\frac{3}{2}$

$\Rightarrow x=e^{-3/2}=0.223$

since the concavity changed $x=e^{-3/2}$ is a POI

Point $\left(e^{-3/2},-\frac{3e^{-3}}{2}\right)$

Testing



$y''\big|_{x=0.1}=-1.6$ ccd

$y''\big|_{x=1}=3$ ccu

10. 500 g of radioactive material decays exponentially. Amount A of material left after t years is $A(t) = A_0(1.3)^{-t}$. How fast is the material decaying after 5 years?

$$A(t) = 500\,(1.3)^{-t}$$

$$\Rightarrow A'(t) = 500\left((1.3)^{-t}(\ln(1.3))(-1)\right)$$

$$= -131.2\left(1.3^{-t}\right)$$

$$\Rightarrow A'(5) = -131.2\left(1.3^{-5}\right) = -35.3 \text{ g/yr.}$$

11. A beam of length $10m$ is supported at one end. If $5kg$ is the uniform weight per metre of length, the bending moment M at a distance x from the end is given by $M = \frac{1}{2}lx - \frac{1}{2}wx^2$, where l is the beam's length in metres and w is the uniform weight per metre of length. Find the points on the beam at which the bending moment has the maximum value.

↳ derivative

$$M = \tfrac{1}{2}(10)x - \tfrac{1}{2}(5)x^2$$

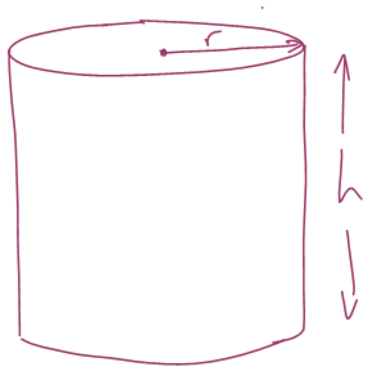
$$= 5x - \tfrac{5}{2}x^2$$ quadratic opening down $\Rightarrow$ c.v. is a max

$$M' = 5 - 5x$$ set to zero $\Rightarrow x = 1$

$\Rightarrow$ 1 m from the end of the beam the bending moment is maximized

12. A closed right circular cylinder is such that the sum of its height and the circumference of its base is 10 m. Find the maximum volume of the cylinder.

(I will assume the smallest r or h can be is 1m)



if $h=1$

$1 \le r \le \frac{9}{2\pi}$

$h + 2\pi r = 10$

$\Rightarrow h = 10 - 2\pi r \quad (*)$

$$V = \pi r^2 h$$

$$V = \pi r^2(10 - 2\pi r)$$

$$\Rightarrow V(r) = 10\pi r^2 - 2\pi^2 r^3$$

$$V'(r) = 20\pi r - 6\pi^2 r^2$$ set to zero

$$\Rightarrow 20\pi r - 6\pi^2 r^2 = 0$$

$$\Rightarrow 2\pi r(10 - 3\pi r) = 0$$

$$\Rightarrow r = 0 \quad \text{or} \quad r = \frac{10}{3\pi} = 1.06\text{ m}$$

Testing

$V(1) = 10\pi - 2\pi^2 = 11.68$

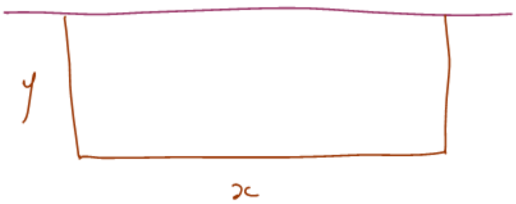
$V\left(\frac{10}{3\pi}\right) = \frac{1000}{9\pi} - \frac{2000}{27\pi} = 11.79$

$V\left(\frac{9}{2\pi}\right) = \frac{810}{4\pi} - \frac{729}{4\pi} = 6.4$

$\therefore$ max volume of cylinder is $11.79\ m^3$

13. Cameron needs to fence off an area of 300 m² for his two sheep. He will use an existing fence on his farm for one side of the enclosed region and will construct the other three sides with cedar posts. Find the dimensions of the sheep pasture that minimize the length of the new fencing.

I will assume the smallest a dimension can be is 1m.



$A = xy$

$\Rightarrow y = \dfrac{300}{x}$ *

'y=1'

$1 \le x \le 300$

$P = x + 2y$

$\Rightarrow P(x) = x + \dfrac{600}{x}$

$\Rightarrow P'(x) = 1 - \dfrac{600}{x^2}$ set to zero

$\Rightarrow 1 - \dfrac{600}{x^2} = 0$

$\Rightarrow 1 = \dfrac{600}{x^2}$

$\Rightarrow x = \pm\sqrt{600}$

$-\sqrt{600}$ inadmissable

Testing:

$P(1) = 601$

$P(\sqrt{600}) \doteq 49$

$P(300) = 302$

$\therefore$ The dimensions which minimize the amount of fencing are

$\sqrt{600}$ m by $\dfrac{\sqrt{600}}{2}$ m (by *)

14. a. Determine the values of a and b for $f(x) = -2x^3 + ax^2 + bx + 6$ so that $f'(-2) = 0$ and $f'(7) = 0$.
b. For what values of x is $f(x)$ increasing?

a) $f'(x) = -6x^2 + 2ax + b$

$f'(-2) = 0 \Rightarrow -24 - 4a + b = 0 \Rightarrow 4a - b = -24$ ①

$f'(7) = 0 \Rightarrow -294 + 14a + b = 0 \Rightarrow 14a + b = 294$ ②

① + ② $\Rightarrow 18a = 270$

$\Rightarrow a = 15$

$\Rightarrow b = 84$ (by ①)

b) $f(x) = -2x^3 + 15x^2 + 84x + 6$

$\Rightarrow f'(x) = -6x^2 + 30x + 84$ set to zero for C.V.'s and $\div -6$

$\Downarrow$ next pg

$\Rightarrow x^2 - 5x - 14 = 0$

$\Rightarrow (x-7)(x+2) = 0$

$\Rightarrow x = 7$ or $x = -2$

intervals	$(-\infty,-2)$	$(-2,7)$	$(7,\infty)$
T.V.	-3	0	8
sign on $f'(x) = -6x^2 + 30x + 84$	$-$ve dec ↘	$+$ve inc ↗	$-$ve dec ↘
		(local min)	(local max)

$\therefore f(x)$ is increasing on $x \in (-2, 7)$

15. Let $f(x) = 2x^3 - 9x^2 - 60x + 1$. Use the second derivative test to determine all local minima of $f(x)$.

$f'(x) = 6x^2 - 18x - 60$ set to zero for C.V.'s and $\div 6$

$\Rightarrow x^2 - 3x - 10 = 0$

$\Rightarrow (x-5)(x+2) = 0$

$\Rightarrow x = 5, -2$ are the C.V.'s

$f''(x) = 12x - 18$

$f''(5) = 12(5) - 18$
$= 42 > 0 \therefore$ local min

$f''(-2) = 12(-2) - 18$
$= -42 < 0 \therefore$ local max

Point $(5, f(5)) = (5, -274)$

Point $(-2, f(-2)) = (-2, 69)$

16. Use the algorithm for curve sketching to sketch the graph of $f(x) = \dfrac{x-2}{x^2 - 3x - 4}$.

$f(x) = \dfrac{x-2}{(x-4)(x+1)}$

INTERCEPTS

y_{int} $(0, \frac{1}{2})$ x_{int}: $(2, 0)$

ASYMPTOTES

H.A. $y = 0$ V.A. $x = 4$, $x = -1$

C.V.'s

$$f(x) = \frac{x-2}{x^2-3x-4}$$

$$\Rightarrow f'(x) = \frac{(1)(x^2-3x-4) - (x-2)(2x-3)}{(x^2-3x-4)^2}$$

$$= \frac{-x^2+4x-10}{(x^2-3x-4)^2}$$

set to zero $\Rightarrow -x^2+4x-10 = 0$

d.n.f. : discriminant b^2-4ac

$= 4^2 - 4(-1)(-10)$

$= 16 - 40 < 0$

$\therefore$ No sol'ns

No C.V.'s.

PPOI

$$f''(x) = \frac{(-2x+4)(x^2-3x-4)^2 - (-x^2+4x-10)(2(x^2-3x-4)(2x-3))}{(x^2-3x-4)^4}$$

$$= \frac{(-2x+4)(x^2-3x-4) - 2(2x-3)(-x^2+4x-10)}{(x^2-3x-4)^3}$$

$$= \frac{(-2x^3+10x^2-4x-16) - (-4x^3+22x^2-64x+60)}{(x^2-3x-4)^3}$$

$$= \frac{2x^3-12x^2+60x-76}{(x^2-3x-4)^2}$$

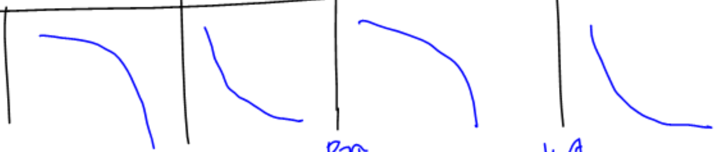
Note: on the exam a sketching problem will be much "nicer".

set to zero $\Rightarrow 2x^3-12x^2+60x-76 = 0$

using graph calc: $x = 1.67$ pt $(1.67, 0.05)$

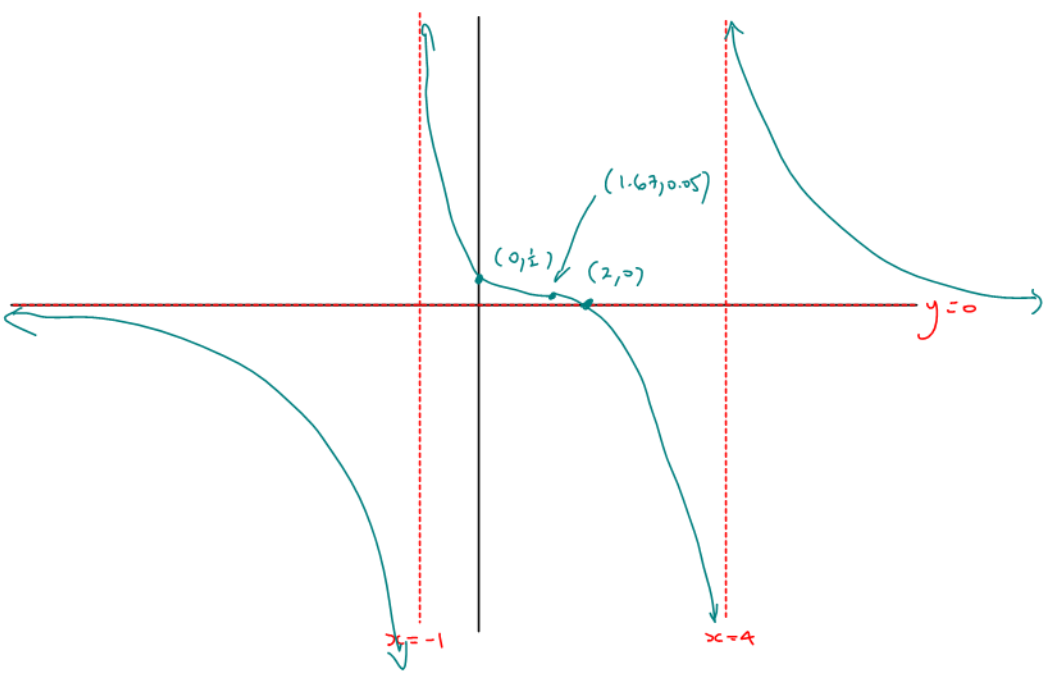
next pg for interval chart and sketch.

INTERVALS	$(-\infty,-1)$	$(-1, 1.67)$	$(1.67, 4)$	$(4,\infty)$
T.V.	-2	0	2	5
SIGN on $f'(x)=\dfrac{-x^2+4x-10}{(x^2-3x-4)^2}$	$-ve$ de	$-ve$ dec	$-ve$ dec	$-ve$ dec
SIGN on $f''(x)=\dfrac{2x^3-12x^2+60x-76}{(x^2-3x-4)^3}$	$-ve$ ccd	$+ve$ ccu	$-ve$ ccd	$+ve$ ccu
SHAPE				



V.A. POI V.A.

SKETCH



again - on an 'exam' any curve you might sketch would have much nicer derivatives.

I hope this helps!