6.8b Linear Combinations and Spanning Sets 2

Yesterday we considered the notion of a **Linear Combination** (***scalar multiplication and vector addition***):

> e.g. Given two vectors $\vec{u}$ and $\vec{v}$, then $\vec{w} = a\vec{u} + b\vec{v}$ is called a linear combination of $\vec{u}$ and $\vec{v}$.

Consider the standard unit vectors for $\mathbb{R}^2$: $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$. We can write any vector in $\mathbb{R}^2$ as a linear combination if the standard unit vectors!!

Consider $\vec{u} = (x, y)$

$$\vec{u} = x\hat{i} + y\hat{j}$$

Thus the set of vectors $\{\hat{i}, \hat{j}\}$ is called a spanning set for $\mathbb{R}^2$

Similarly, the set of vectors $\{(1,0,0),(0,1,0),(0,0,1)\}$ spans $\mathbb{R}^3$

$$\{\hat{i}, \hat{j}, \hat{k}\}$$

Definition 6.8b.1

Given vectors $\vec{u}$ and $\vec{v}$, we call the set of all possible linear combinations of $\vec{u}$ & $\vec{v}$ the **SPAN** of $\{\vec{u}, \vec{v}\}$.

We write $span\{\vec{u}, \vec{v}\} = \{\vec{w} = a\vec{u} + b\vec{v},\ a, b \in \mathbb{R}\}$

Thus we can write $span\{\hat{i}, \hat{j}\} =$ $\mathbb{R}^2$

Now, spanning sets are not unique.

Consider the set $A=\{(1,0),(0,1),(3,2)\}$. Clearly $span\{A\}=\mathbb{R}^2$

BUT we don't need all 3 vectors. In fact

$A=\{(1,0),(0,1),(3,2)\}$ is a linearly dependent set

since $(3,2)=3(1,0)+2(0,1)$

So $(3,2)\in span\{(1,0),(0,1)\}$

means "non-collinear"

Definition 6.8b.2

We call a set which is **NOT linearly dependent** a **linearly independent set**.

Definition 6.8b.3

A **linearly ~~dependent~~ independent set of vectors**, $A=\{\vec{u},\vec{v}\}$, which **spans another set** $B=\{\vec{w}=a\vec{u}+b\vec{v},\ a,b,\in\mathbb{R}\}$ is called a **BASIS** of that set.

So, the set $A=\{\hat{i}=(1,0),\hat{j}=(0,1)\}$ is a basis for $\mathbb{R}^2$

In fact we call $A=\{\hat{i},\hat{j}\}$ the STANDARD BASIS for $\mathbb{R}^2$

Defn: A Basis for a set B is the smallest set of vectors $\{\vec{u},\vec{v}\}$ s.t. $span\{\vec{u},\vec{v}\}=B$

And now for some more difficult stuff…

Example 6.8b.1

Show $span\{(1,2),(-1,3)\} = \mathbb{R}^2$

Note

$(1,2)$ & $(-1,3)$ are <u>not</u> collinear

$\Rightarrow \nexists$ a scalar 'a'

s.t. $(1,2) = a(-1,3)$

We want some general vector

$\vec{\omega} = (x,y)$ along with some

scalars $a,b \in \mathbb{R}$ s.t.

$$\vec{\omega} = a(1,2) + b(-1,3)$$

(we need to find a,b)

$$\Rightarrow (x,y) = a(1,2) + b(-1,3)$$

$$\Rightarrow (x,y) = (a-b,\ 2a+3b)$$

$$\Rightarrow \quad a - b = x \quad (1)$$

$$2a + 3b = y \quad (2)$$

$\Rightarrow$ find a & b!

(1)×3 + (2) $\quad 5a = 3x + y$

$$\therefore \boxed{a = \frac{3}{5}x + \frac{1}{5}y}$$

(1)×2 − (2)

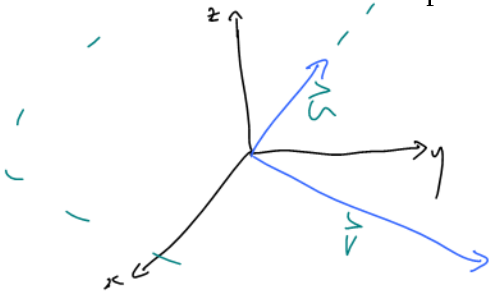
$$-5b = 2x - y$$

$$\Rightarrow \boxed{b = -\frac{2}{5}x + \frac{1}{5}y}$$

quarternions

In $\mathbb{R}^3$, and two non-collinear vectors form a plane.

Picture

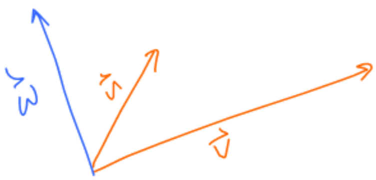


Note: the set $A = \{\vec{u}, \vec{v}\}$ is **NOT**

BASIS for $\mathbb{R}^3$ (span$\{\vec{u}, \vec{v}\} \neq \mathbb{R}^3$, but span$\{\vec{u}, \vec{v}\}$ is a plane in $\mathbb{R}^3$

Three **non-coplanar** vectors **WILL** span $\mathbb{R}^3$.

Picture:



An Important Question:

How can we determine if three vectors (in $\mathbb{R}^3$) are non-coplanar?

Answer: Given $\vec{u}, \vec{v}, \vec{w}$, $\nexists$ scalars a, b s.t.

$$\vec{w} = a\vec{u} + b\vec{v}$$

Because IF $\vec{u}, \vec{v}, \vec{w}$ were coplanar, then we can write $\vec{w}$ as a linear combo of $\vec{u}$ & $\vec{v}$

Example 6.8b.2

From your text: Pg. 41 #13a

Show that the vectors $(-1,2,3)$, $(4,1,-2)$ and $(-14,1,16)$ do not lie in the same plane.

Let $\{(-1,2,3), (4,1,-2)\}$ span a plane in $\mathbb{R}^3$

Assume $(-14,1,16)$ is in the same plane

$\Rightarrow \exists$ scalars a, b st

$$(-14,1,16) = a(-1,2,3) + b(4,1,-2)$$

$$\Rightarrow (-14,1,16) = (-a+4b,\ 2a+b,\ 3a-2b)$$

$$\Rightarrow \quad -a+4b = -14 \quad (1)$$
$$2a+b = 1 \quad (2)$$
$$3a-2b = 16 \quad (3)$$

use 2 eqn to find a & b & test in the 3RD

$(1)\times 2 + (2)$ $\quad 9b = -27 \Rightarrow \boxed{b=-3}$ sub into (1) $\boxed{a=2}$

Sub $b=-3$, $a=2$ into (3)

$$3(2) - 2(-3) = 16$$

$12 = 16$ NO, CONTRADICTION!

Class/Homework for Section 6.8 (pt. 2)

Pg. 340 - 342 #1 – 6, 7, 10, 11, 8, 9, 12b, 13b, 14, 15

$\therefore$ The vectors are not the same plane