7.4 The Dot Product: An Algebraic View

Definition 7.4.1

Given vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, then

$\vec{a} \cdot \vec{b} =$ $a_1 b_1 + a_2 b_2 + a_3 b_3$ (for a proof see pg. 379 – 380)

(nothing about the angle!)

Example 7.4.1

Given $\vec{a} = (3,2)$ and $\vec{b} = (-5,1)$ determine

a) $\vec{a} \cdot \vec{b}$ (***algebra***)

b) the angle between the vectors (***geometry***)

a) $\vec{a} \cdot \vec{b} = (a_1, a_2) \cdot (b_1, b_2) \quad (= a_1 b_1 + a_2 b_2)$

$= (3,2) \cdot (-5,1)$

$= (3)(-5) + (2)(1)$

$= -13$

Note: since $\vec{a} \cdot \vec{b} < 0$

$\Rightarrow \theta$ (the angle between them) is s.t.

$90 < \theta \leq 180$

b) To get the angle we use the "geometric def."

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$

$\Rightarrow \cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

$\vec{a} \cdot \vec{b} = -13$ (from a)

$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

$= \sqrt{(3,2) \cdot (3,2)}$

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

$|\vec{b}| = \sqrt{(-5)^2 + (1)^2} = \sqrt{26}$

$\Rightarrow \cos(\theta) = \dfrac{-13}{(\sqrt{13})(\sqrt{26})}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{-13}{(\sqrt{13})(\sqrt{26})}\right) = 135°$ or $\left(\dfrac{3\pi}{4}\right)$

Example 7.4.2

Prove the commutative property for vectors in $\mathbb{R}^2$.

(*Note: we assume the commutative property true for numbers*)

Given vectors $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$

Prove $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Consider $\vec{a} \cdot \vec{b}$

$= (a_1, a_2) \cdot (b_1, b_2)$

$= a_1 b_1 + a_2 b_2$

$= b_1 a_1 + b_2 a_2$ (commutative property of $\mathbb{R}$)

$= (b_1, b_2) \cdot (a_1, a_2)$

$= \vec{b} \cdot \vec{a}$ $\square$

Read example 1 on page 380 for a proof of the distributive property.

Example 7.4.3

Are $\vec{a} = (3, 2, -1)$ and $\vec{b} = (-1, 2, 3)$ perpendicular?

Consider $\vec{a} \cdot \vec{b}$

$= (3, 2, -1) \cdot (-1, 2, 3)$

$= (3)(-1) + (2)(2) + (-1)(3)$

$= -2 \neq 0$

$\therefore \vec{a} \not\perp \vec{b}$

"not perpendicular"

Note: in $\mathbb{R}^2$

given $\vec{a} = (a_1, a_2)$

$\vec{a}_\perp = (-a_2, a_1)$

(flip components and make one negative)

ex $(3, -5) \perp (+5, 3)$

Example 7.4.4

Determine for what values of m $\vec{a}=(2m,m,5)$ and $\vec{b}=(m,-3,-1)$ are perpendicular.

$$\vec{a}\cdot\vec{b}=0$$

$$\Rightarrow (2m,m,5)\cdot(m,-3,-1)=0$$

$$\Rightarrow 2m^2-3m-5=0$$

$$\Rightarrow (2m-5)(m+1)=0$$

$$\therefore m=\frac{5}{2} \quad \text{or} \quad m=-1$$

Example 7.4.5 **(This One Is Important)**

Determine a vector $\vec{v}=(x,y,z)$ which is perpendicular to **BOTH** of the vectors $\vec{a}=(1,2,-1)$ and $\vec{b}=(3,1,1)$, $\vec{v}\neq\vec{0}$

$$\vec{v}\cdot\vec{a}=0 \quad \& \quad \vec{v}\cdot\vec{b}=0$$

$$(x,y,z)\cdot(1,2,-1)=0 \quad \& \quad (x,y,z)\cdot(3,1,1)=0$$

$$\Rightarrow x+2y-z=0 \quad (1) \qquad 3x+y+z=0 \quad (2)$$

Note: we have 2 eqns w/ 3 unknown

$\Rightarrow$ we require FREEDOM for one of the variables.

(we can choose which variable to make "free" OR we can let it happen as we do some math)

next page to find the freedom

(1) + (2) $\quad 4x + 3y = 0$

$$x = -\frac{3}{4}y$$

Let 'y' be the free variable

(1) ×3 − (2) $\quad 5y - 4z = 0 \quad$ (solve for z)

$$\Rightarrow \quad z = \frac{5}{4}y$$

we also need a relationship between z & y

$\Rightarrow$ eliminate x

$$\therefore \quad \vec{v} = (x, y, z)$$

$$= \left(-\frac{3}{4}y, y, \frac{5}{4}y\right)$$

we set $y = t$

(where t is the "free parameter)

$$\Rightarrow \vec{v} = t\left(-\frac{3}{4}, 1, \frac{5}{4}\right)$$

Note: t is a scalar giving ∞'ly many vectors $\perp \vec{a}$ & $\vec{b}$

- but we wanted a single vector

$\Rightarrow$ MAKE A CHOICE for t

Choose $t = 4$ (to give integer components)

$$\Rightarrow \quad \vec{v} = 4\left(-\frac{3}{4}, 1, \frac{5}{4}\right)$$

$$\boxed{\vec{v} = (-3, 4, 5)}$$

Class/Homework for Section 7.4

Pg. 385 – 387 #1 – 5, 6bc, 7, 9b, 10 – 13, 14, 15, 17

see example 5