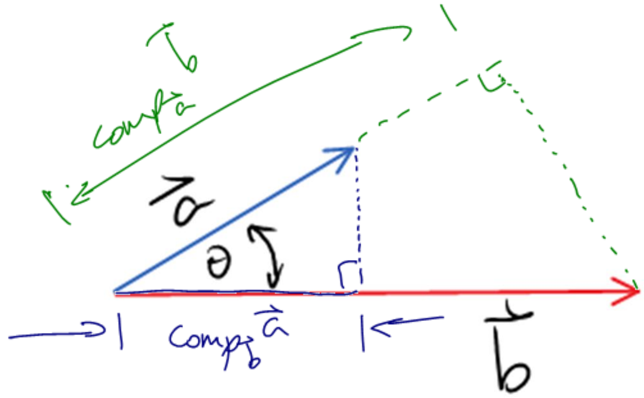
7.5 Projections: Scalar and Vector

Scalar Projections

Consider the vector diagram:



Notation: The "amount of $\vec{a}$-ness" along the direction of $\vec{b}$ is called the component of $\vec{a}$ on $\vec{b}$

$$\cos(\theta) = \frac{\text{comp}_{\vec{b}}\vec{a}}{|\vec{a}|} \Rightarrow \boxed{\text{comp}_{\vec{b}}\vec{a} = |\vec{a}|\cos(\theta)}$$

$$\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos(\theta)$$

Now, we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, and so we can write

$$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{b}|\,\text{comp}_{\vec{b}}\vec{a}$$

$$\Rightarrow \text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

similarly $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Example 7.5.1

Given $\vec{a} = (1,2,3)$ and $\vec{b} = (-2,1,4)$ determine the scalar projections of:

a) $\vec{a}$ on $\vec{b}$ b) $\vec{b}$ on $\vec{a}$

$$\vec{a}\cdot\vec{b} = (1,2,3)\cdot(-2,1,4) = -2+2+12 = 12$$

$$|\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

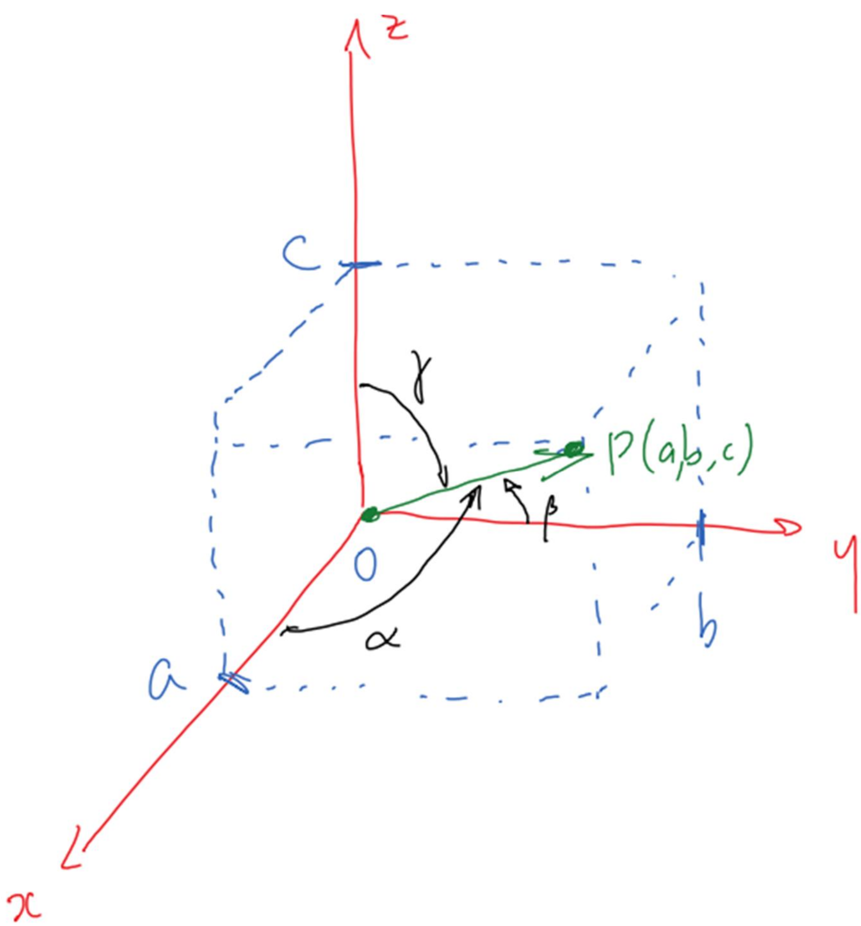
$$|\vec{b}| = \sqrt{(-2)^2+1^2+4^2} = \sqrt{21}$$

a)
$$\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{12}{\sqrt{21}} = \frac{12\sqrt{21}}{21} = \frac{4\sqrt{21}}{7}$$

b)
$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{12}{\sqrt{14}} = \frac{12\sqrt{14}}{14} = \frac{6\sqrt{14}}{7}$$

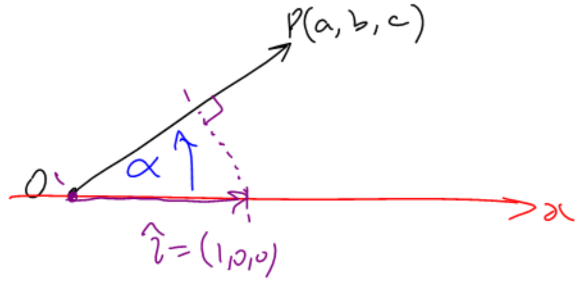
Direction Cosines

Sometimes, especially in $\mathbb{R}^3$, the idea of "direction" is difficult to pin down (or better – direction can be difficult to describe mathematically). Consider the sketch of a position vector in $\mathbb{R}^3$.



Consider view relative to the x-axis

$\vec{OP} = (a, b, c)$



$$\text{Comp}_{\vec{OP}}\, \hat{i} = \frac{\vec{OP} \cdot \hat{i}}{|\vec{OP}|}$$

$$\text{Comp}_{\vec{OP}}\, \hat{i} = |\hat{i}| \cos(\alpha) = \cos(\alpha)$$

$$\rightarrow \quad \cos\alpha = \frac{\vec{OP} \cdot \hat{i}}{|\vec{OP}|} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

Similarly

$$\cos(\beta) = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$\cos(\gamma) = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Example 7.5.2

Given $\vec{u} = (3, -2, 1)$ determine the angle $\vec{u}$ makes with the *x*-axis and the *y*-axis to the nearest degree.

Let α be the angle between x-axis and $\vec{u} = (3,-2,1) = (a,b,c)$

" β " " " y-axis " "

$$\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{14}}$$

$$\alpha = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

$$= 37^\circ$$

$$\cos(\beta) = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-2}{\sqrt{14}}$$

$$\Rightarrow \beta = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right)$$

$$= 122^\circ$$

Class/Homework for Section 7.5

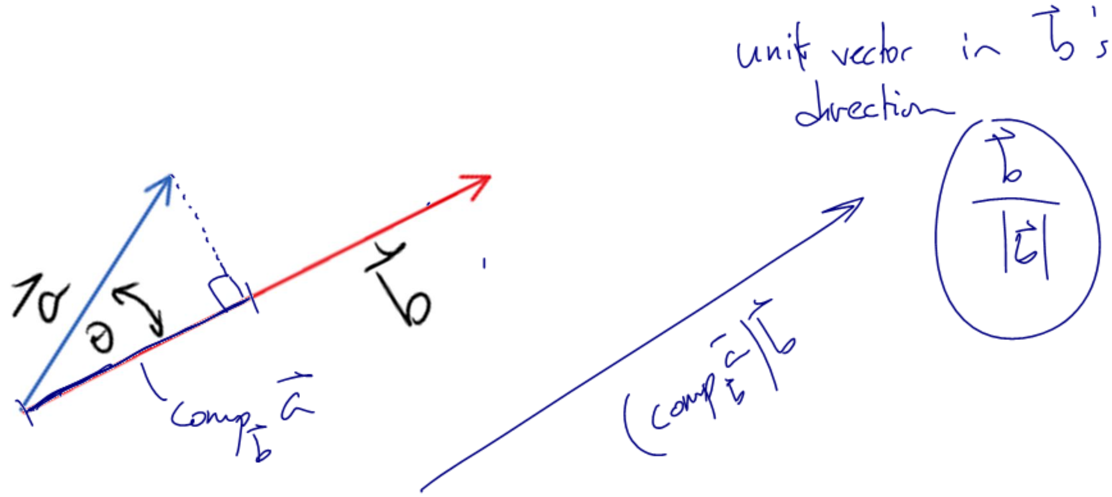
Pg. 398 – 400 #3, 4, 6 (scalar only), 7 (scalar only)

7.5b Projections: Scalar and Vector (2)

Vector Projections

Recall that given two vectors $\vec{a}$ and $\vec{b}$ we can find how much "$\vec{a}$-ness" there is along the vector $\vec{b}$ (or vice versa).

Picture



We can **construct** a **vector** along $\vec{b}$ with a length (magnitude) of $comp_{\vec{b}}\vec{a}$. But to do this, we will need the "pure direction" of $\vec{b}$. We will need a **unit vector** along $\vec{b}$ which we can "scale" with the **scalar** $comp_{\vec{b}}\vec{a}$.

That is, we want to use the vector $\frac{\vec{b}}{|\vec{b}|}$ (unit vector w/ $\vec{b}$'s direction)

Thus, a vector of length ~~$comp_{\vec{a}}\vec{b}$~~ $comp_{\vec{b}}\vec{a}$ in the direction of $\vec{b}$ is given by

$$proj_{\vec{b}}\vec{a} = \left(comp_{\vec{b}}\vec{a}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$$

Similarly

$$proj_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$$

Example 7.5.3

Given $\vec{a} = (3,5,-1)$ and points $B(2,-1,3)$ and $C(5,3,-2)$ determine the vector projection of $\vec{a}$ on $\overrightarrow{BC}$.

$$\overrightarrow{BC} = (5-2,\ 3-(-1),\ -2-3) = (3,4,-5)$$

$$\operatorname{proj}_{\overrightarrow{BC}}\vec{a} = \left(\frac{\vec{a}\cdot\overrightarrow{BC}}{\overrightarrow{BC}\cdot\overrightarrow{BC}}\right)\overrightarrow{BC}$$

$$= \left(\frac{9+20+5}{9+16+25}\right)(3,4,-5)$$

$$= \frac{34}{50}(3,4,-5) = \frac{17}{25}(3,4,-5)$$

$$= \left(\frac{51}{25}, \frac{68}{25}, -\frac{17}{5}\right)$$

Example 7.5.4

Given $P(3,-4,-6)$ find the vector projection of $\overrightarrow{OP}$ onto the z-axis.

$$\overrightarrow{OP} = (3,-4,-6)$$

$$\operatorname{proj}_{\hat{k}}\overrightarrow{OP} = \left(\frac{\overrightarrow{OP}\cdot\hat{k}}{\hat{k}\cdot\hat{k}}\right)\hat{k}$$

$$= \left(\frac{-6}{1}\right)(0,0,1) = (0,0,-6)$$

$$= -6\hat{k}$$

Class/Homework for Section 7.5b

Pg. 398 – 400 #8, 11, 12, 14, 15