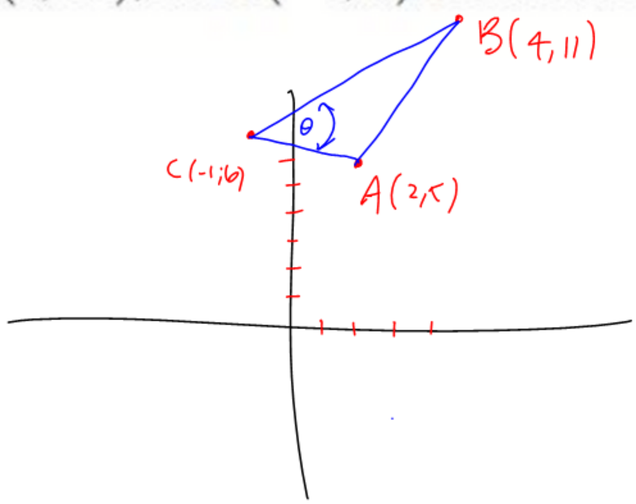
Hwk Check

§7.4

11. $\triangle ABC$ has vertices at $A(2, 5)$, $B(4, 11)$, and $C(-1, 6)$. Determine the angles in this triangle.



$$\cos(\theta) = \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}||\vec{CB}|}$$

$$= \frac{(3,-1)\cdot(5,5)}{(\sqrt{10})(\sqrt{50})}$$

$$= \frac{10}{10\sqrt{5}}$$

$$\rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$= 26.6°.$$

etc.

§7.5

15. a. If α, β, and γ represent the direction angles for vector $\vec{OP}$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

$\vec{OP} = (a, b, c)$

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}} \qquad \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}} \qquad \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

LHS

$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)^2 + \left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)^2$$

$$= 1 = \text{RHS}$$

b. Determine the coordinates of a vector $\overrightarrow{OP}$ that makes an angle of 30° with the y-axis, 60° with the z-axis, and 90° with the x-axis.

$\alpha = 90°$ $\beta = 30°$ $\gamma = 60°$

$\cos\alpha = 0$ $\cos\beta = \frac{\sqrt{3}}{2}$ $\cos\gamma = \frac{1}{2}$

$\therefore \frac{a}{\sqrt{a^2+b^2+c^2}} = 0$

$\Rightarrow \boxed{a = 0}$

$\Rightarrow \frac{b}{\sqrt{b^2+c^2}} = \frac{\sqrt{3}}{2}$

$\frac{b^2}{b^2+c^2} = \frac{3}{4}$

$4b^2 = 3b^2 + 3c^2$

$\Rightarrow b^2 = 3c^2$

$b = \pm\sqrt{3}\,c$

$\frac{c}{\sqrt{b^2+c^2}} = \frac{1}{2}$

$\frac{c^2}{b^2+c^2} = \frac{1}{4}$

$4c^2 = b^2 + c^2$

$\Rightarrow 3c^2 = b^2$.

Free variable let $c = t$

Then $\vec{OP} = (0, \sqrt{3}\,t, t)$ or $\vec{OP} = (0, -\sqrt{3}\,t, t)$

$= t(0, \sqrt{3}, 1)$ or $\vec{OP} = t(0, -\sqrt{3}, 1)$

Choose $t = 1$ $\vec{OP} = (0, \sqrt{3}, 1)$ or $\vec{OP} = (0, -\sqrt{3}, 1)$.

$(t \neq 0)$

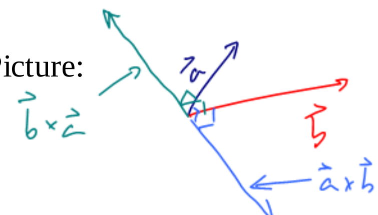
7.6 The Cross Product

The **Cross Product** is sometimes called the Vector Product because it produces a **vector**.

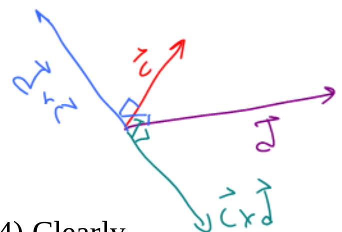
Note: 1) The cross product is only used in $\mathbb{R}^3$.

2) Given two non collinear vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ the cross product produces a vector which is perpendicular to the plane spanned by $\vec{a}$ & $\vec{b}$

Picture:



3) We determine the **direction** of $\vec{a}\times\vec{b}$ using the $\boxed{\text{right hand rule}}$



4) Clearly

NOTE: $\vec{c}\times\vec{d} \neq \vec{d}\times\vec{c}$

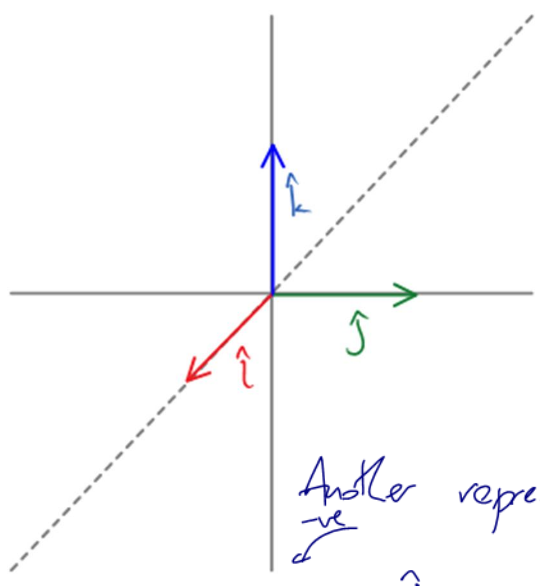
$\rightarrow$ the cross product does not commute.

Algebraic View of the Cross Product

Consider: $\hat{i}\times\hat{j} = \hat{k}$

$\hat{j}\times\hat{k} = \hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$



Another representation

$\hat{i}\times\hat{k} = -\hat{j}$

Note: Given two vectors

$\vec{a}$ & $m\vec{a}$, m a scalar

then $\vec{a}\times m\vec{a} = \vec{0}$

Consider two general vectors in $\mathbb{R}^3$: $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. If we write the vectors as linear combinations of the standard unit vectors we can use the above ideas to calculate $\vec{a} \times \vec{b}$.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then:

$$\vec{a} \times \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \times \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1b_1 \cancelto{\vec{0}}{\hat{i}\times\hat{i}} + a_1b_2 \cancelto{\hat{k}}{\hat{i}\times\hat{j}} + a_1b_3 \cancelto{-\hat{j}}{\hat{i}\times\hat{k}}$$

$$+ a_2b_1 \cancelto{-\hat{k}}{\hat{j}\times\hat{i}} + a_2b_2 \cancelto{\vec{0}}{\hat{j}\times\hat{j}} + a_2b_3 \cancelto{\hat{i}}{\hat{j}\times\hat{k}}$$

$$+ a_3b_1 \cancelto{\hat{j}}{\hat{k}\times\hat{i}} + a_3b_2 \cancelto{-\hat{i}}{\hat{k}\times\hat{j}} + a_3b_3 \cancelto{\vec{0}}{\hat{k}\times\hat{k}}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Another "pattern" for developing the algebraic cross product

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$$\begin{array}{ccc} \vec{a} & \times & \vec{b} \\ a_2 & & -\,b_2 \\ a_3 & & +\,b_3 \\ & & - \\ a_1 & & +\,b_1 \\ & & - \\ a_2 & & +\,b_2 \end{array}$$

$(a_2b_3 - a_3b_2)\hat{i}$

$(a_3b_1 - a_1b_3)\hat{j}$

$(a_1b_2 - a_2b_1)\hat{k}$

Example 7.6.1

Given $\vec{a} = (3, -2, 5)$ and $\vec{b} = (2, 1, 0)$ determine $\vec{a} \times \vec{b}$.

$(3, -2, 5) \times (2, 1, 0)$

$= ((-2)(0) - (5)(1), (5)(2) - (3)(0), (3)(1) - (2)(-2))$

$= (-5, 10, 7)$

Example 7.6.2

Given $\vec{a} = (3, 1, 0)$ and $\vec{b} = (4, -1, 3)$, find:

a) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

(triple scalar product)

(In fact for any 3 vectors, $\vec{a}, \vec{b}, \vec{c}$ if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ then $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar)

b) $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

c) $\vec{a} \times (\vec{a} \cdot \vec{b})$ = nonsense

cross product is between vectors

Q. Of the following two vector expressions, which is meaningful?

i) $(\vec{a} \cdot \vec{b})(\vec{a} \times \vec{b})$, or *ii*) $\vec{a} \times (\vec{a} \times \vec{b})$

Both are.

Class/Homework for Section 7.6

Pg. 405 Investigation (optional)
Pg. 407 – 408 #1, 3, 4def, 5 – 7, 8a, 11