7.7 Applications of the Dot and Cross Products

Given two non collinear vectors $\vec{a},\vec{b} \in \mathbb{R}^3$, with $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$ then:

(with θ being the angle between $\vec{a}$ & $\vec{b}$)

Dot Product

Geometric Form	Algebraic Form
$\vec{a}\cdot\vec{b} = \lvert\vec{a}\rvert\,\lvert\vec{b}\rvert\cos(\theta)$	$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Cross Product

Geometric Form	Algebraic Form
?	$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

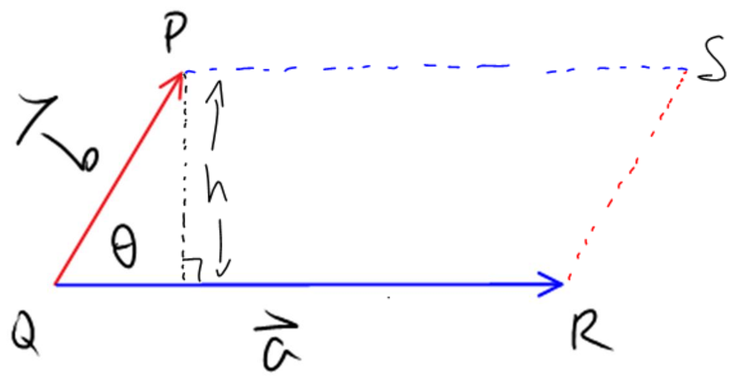
baseball stinks

Consider the fact that $\lvert\vec{a}\times\vec{b}\rvert = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin(\theta)$ (see Pg. 411 for a proof). Further consider a **unit vector** $\hat{n}$ perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Then the geometric form of $\vec{a}\times\vec{b}$ is

$$\vec{a}\times\vec{b} = \left(\lvert\vec{a}\rvert\,\lvert\vec{b}\rvert\sin(\theta)\right)\hat{n}$$

geometric form of cross product

Consider the following picture (note: $\vec{a}, \vec{b} \in \mathbb{R}^3$) :



$$\sin(\theta) = \frac{h}{|\vec{b}|}$$

$$\Rightarrow h = |\vec{b}| \sin\theta$$

The parallelogram arising from the vectors $\vec{a}$ and $\vec{b}$ has an area given by:

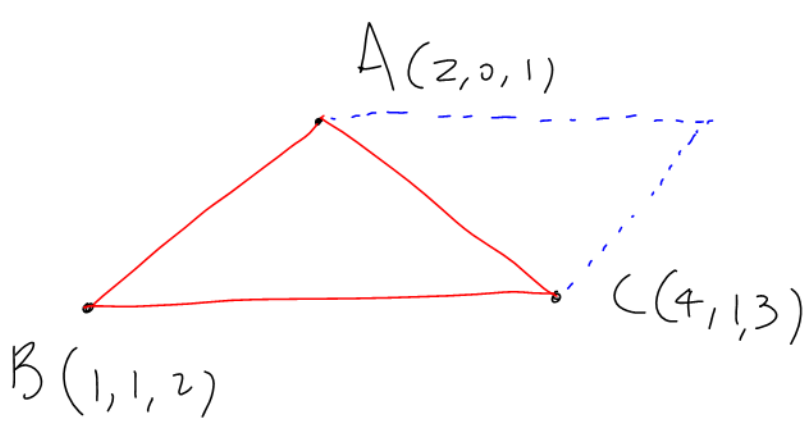
$$A = b \times h \qquad (\text{base} \times \text{height})$$

$$= |\vec{a}||\vec{b}| \sin\theta$$

$$= |\vec{a} \times \vec{b}|$$

Example 7.7.1

Calculate the area of a triangle with vertices $A(2,0,1)$, $B(1,1,2)$ and $C(4,1,3)$.



$$A_{\triangle} = \frac{1}{2}|\vec{BA} \times \vec{BC}|$$

$$= \frac{1}{2}|(1,-1,-1) \times (3,0,1)|$$

$$= \frac{1}{2}|(-1,-4,3)|$$

$$\vec{BA} = (1,-1,-1) \qquad \vec{BC} = (3,0,1)$$

$$= \frac{1}{2}\sqrt{(-1)^2 + (-4)^2 + 3^2}$$

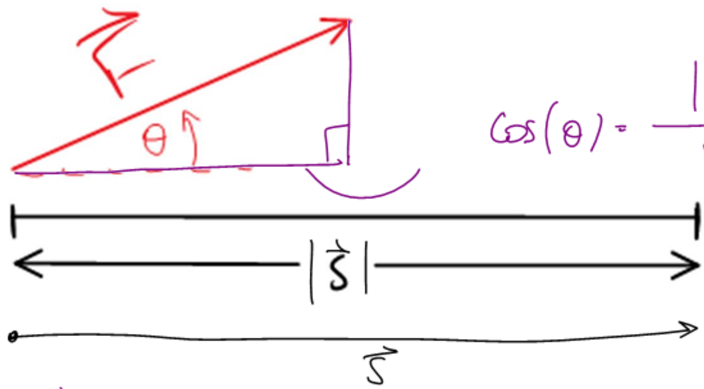
$$= \frac{1}{2}\sqrt{26} \text{ units}^2$$

Two Applications

Dot Product and Work

in the direction of motion

Physics tells us that $Work = (Force\ applied)(\underbrace{displacement}_{\text{distance.}})$. Consider the picture:



$$\cos(\theta) = \frac{|\vec{F}_a|}{|\vec{F}|}$$

$|F_{app}| = |\vec{F}|\cos(\theta)$

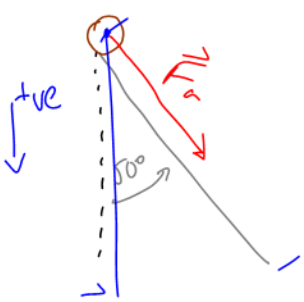
Thus

$$W = |\vec{F}|\cos(\theta)\,|\vec{s}|$$
$$= |\vec{F}||\vec{s}|\cos(\theta) = \vec{F}\cdot\vec{s}$$

Example 7.7.2

From your text: Pg. 415 #3b

Calculate the amount of work done when a $40kg$ rock falls $40m$ down a slope at 50° to the vertical.



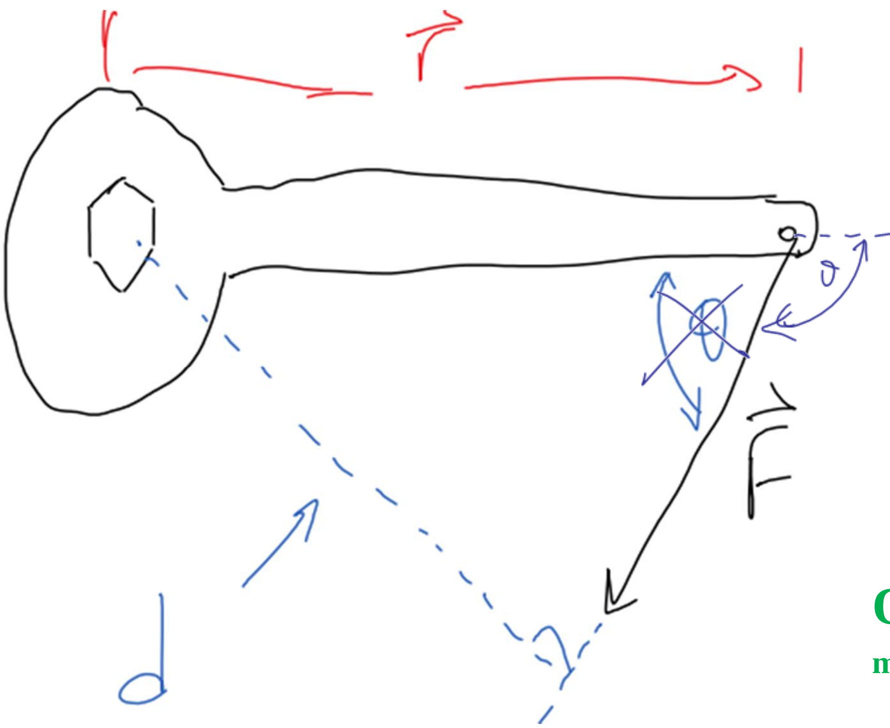
$\vec{F}_g = mg$
$= (40)(9.8)$
$= 392\ N$

$$W = \vec{F}\cdot\vec{s}$$
$$= |\vec{F}||\vec{s}|\cos(50)$$
$$= (392)(40)\cos(50)$$
$$= 10079 \text{ Joules}$$

(Joules are N·m)

Cross Product and Torque

Torque is the “twisting” force around a turning point. (e.g. the force on a bolt by a wrench.) Consider the (poorly drawn) picture:



Torque is the force applied at the **bolt**.

Q. How can we **maximize** torque?

make $\vec{r}$ as long as possible

apply $\vec{F}$ $\perp$ to $\vec{r}$

$$|\vec{T}| = |\vec{r} \times \vec{F}|$$

Class/Homework for Section 7.7

Pg. 415 #2, 3, 5, 6, 8, 10