7.6 The Cross Product

The problems taken from the Nelson Text: Pg. 407 – 408

1. The two vectors $\vec{a}$ and $\vec{b}$ are vectors in R^3, and $\vec{a} \times \vec{b}$ is calculated.
   a. Using a diagram, explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$.
   b. Draw the parallelogram determined by $\vec{a}$ and $\vec{b}$, and then draw the vector $\vec{a} + \vec{b}$. Give a simple explanation of why $(\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$.
   c. Why is it true that $(\vec{a} - \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$? Explain.
3. For each of the following calculations, say which are possible for vectors in R^3 and which are meaningless. Give a brief explanation for each.
   a. $\vec{a} \cdot (\vec{b} \times \vec{c})$
   b. $(\vec{a} \cdot \vec{b}) \times \vec{c}$
   c. $(\vec{a} \times \vec{b}) \cdot (\vec{c} + \vec{d})$
   d. $(\vec{a} \cdot \vec{b})(\vec{c} \times \vec{d})$
   e. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$
   f. $\vec{a} \times \vec{b} + \vec{c}$
4. Calculate the cross product for each of the following pairs of vectors, and verify your answer by using the dot product.
   a. $(2, -3, 5)$ and $(0, -1, 4)$
   b. $(2, -1, 3)$ and $(3, -1, 2)$
   c. $(5, -1, 1)$ and $(2, 4, 7)$
5. If $(-1, 3, 5) \times (0, a, 1) = (-2, 1, -1)$, determine a.
6. a. Calculate the vector product for $\vec{a} = (0, 1, 1)$ and $\vec{b} = (0, 5, 1)$.
   b. Explain geometrically why it makes sense for vectors of the form $(0, b, c)$ and $(0, d, e)$ to have a cross product of the form $(a, 0, 0)$.
7. a. For the vectors $(1, 2, 1)$ and $(2, 4, 2)$, show that their vector product is $\vec{0}$.
   b. In general, show that the vector product of two collinear vectors, (a, b, c) and (ka, kb, kc), is always $\vec{0}$.

8. In the discussion, it was stated that $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$ for vectors in R^3. Verify that this rule is true for the following vectors.

   a. $\vec{p} = (1, -2, 4)$, $\vec{q} = (1, 2, 7)$, and $\vec{r} = (-1, 1, 0)$

11. You are given the vectors $\vec{a} = (2, 0, 0)$, $\vec{b} = (0, 3, 0)$, $\vec{c} = (2, 3, 0)$, and $\vec{d} = (4, 3, 0)$.

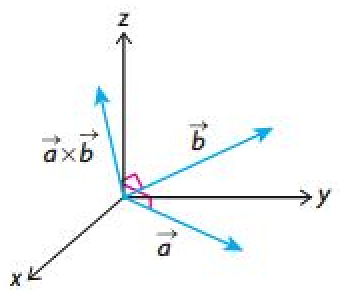
   a. Calculate $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$.

   b. Calculate $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

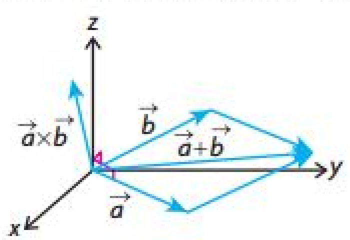
   c. Without doing any calculations (that is, by visualizing the four vectors and using properties of cross products), say why $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d}) = \vec{0}$.

Answers to Selected Problems

1. a.



$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Thus, their dot product must equal 0. The same applies to the second case.



b. $\vec{a} + \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

c. Once again, $\vec{a} - \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} - \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

4. **a.** $(-7, -8, -2)$
b. $(1, 5, 1)$
c. $(-11, -33, 22)$

5. 1

6. **a.** $(-4, 0, 0)$
b. Vectors of the form $(0, b, c)$ are in the yz-plane. Thus, the only vectors perpendicular to the yz-plane are those of the form $(a, 0, 0)$ because they are parallel to the x-axis.

7. **a.** $(1, 2, 1) \times (2, 4, 2)$
$= (2(2) - 1(4), 1(2) - 1(2), 1(4) - 2(2))$
$= (0, 0, 0)$
b. $(a, b, c) \times (ka, kb, kc)$
$= (b(kc) - c(kb), c(ka) - a(kc), a(kb) - b(ka))$
Using the associative law of multiplication, we can rearrange this:
$= (bck - bck, ack - ack, abk - abk)$
$= (0, 0, 0)$

11. **a.** $(0, 0, 6), (0, 0, -6)$
b. $(0, 0, 0)$
c. All the vectors are in the xy-plane. Thus, their cross product in part b. is between vectors parallel to the z-axis and so parallel to each other. The cross product of parallel vectors is $\vec{0}$.