8.2 Cartesian and Symmetric Equations of Lines

$y = mx + b$

Example 8.2.1

Determine vector and parametric equations for the line through $P_0(2,-1)$ and with direction vector $\vec{m} = (2,-3)$. direction #s $a = 2$, $b = -3$

general

vector $\vec{r} = \vec{r}_0 + t\vec{m}$, $t \in \mathbb{R}$

$(x,y) = (x_0, y_0) + t(a,b)$

parametric $\left.\begin{array}{l} x = x_0 + at \\ y = y_0 + bt \end{array}\right\} t \in \mathbb{R}$

Given $P_0(2,-1)$, $\vec{m}(2,-3)$

v: $\vec{r} = (2,-1) + t(2,-3)$, $t \in \mathbb{R}$

p: $\left.\begin{array}{l} x = 2 + 2t \\ y = -1 - 3t \end{array}\right\} t \in \mathbb{R}$

Wait what if we solve both parametric equations for *t*?

Given $x = 2 + 2t$ $\quad$ $y = -1 - 3t$

$\Rightarrow t = \dfrac{x-2}{2}$ $\quad$ $\Rightarrow t = \dfrac{y+1}{-3}$

$\Rightarrow \dfrac{x-2}{2} = \dfrac{y+1}{-3}$ symmetric eqn of line

In general the symmetric equation for a line $\left(\text{in } \mathbb{R}^2\right)$ given $P_0(x_0, y_0)$ and $\vec{m} = (a,b)$ is:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

Problem: If a or $b = 0$ we have no symmetric eqn

Example 8.2.2

Determine vector, parametric and symmetric equations for the line (in "scalar" form)

$$y = -\frac{2}{3}x + 4 \quad (y = mx + b)$$

we need a point & direction vector

since (scalar)

$m = -\frac{2}{3}$ "$\frac{y}{x}$" $\Rightarrow \vec{m} = (3, -2)$

(or $\vec{m} = (-3, 2)$)

$P_0(0,4)$

$\vec{m} = (3,-2)$

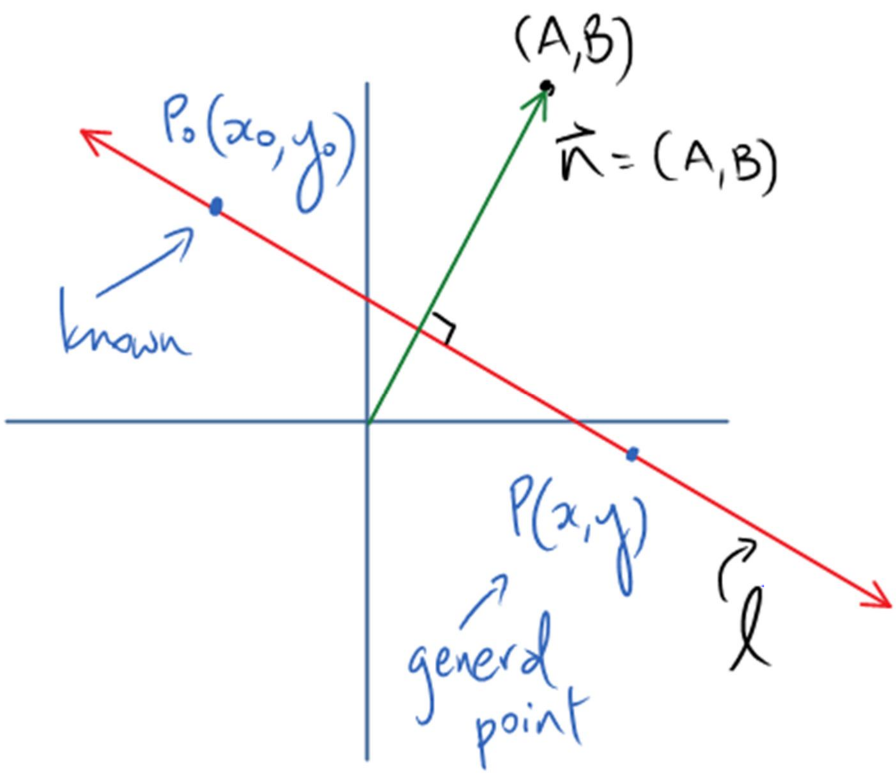
vector: $\vec{r} = (0,4) + t(3,-2),\ t \in \mathbb{R}$

para: $\left.\begin{aligned} x &= 3t \\ y &= 4 - 2t \end{aligned}\right\}\ t \in \mathbb{R}$

symm: $\dfrac{x-0}{3} = \dfrac{y-4}{-2}$

The Cartesian Equation of a Line

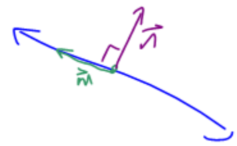
Consider the following sketch:



Note: We DO NOT know the direction vector for line *l*!

we can construct it

$\vec{m} = \overrightarrow{P_0P}$



Definition 8.2.1

A **normal vector** $\vec{n}=(A,B)$ to a line with direction vector $\vec{m}=(a,b)$ is $\perp$ to $\vec{m}$ (to the line) ie $\vec{n}\perp\vec{m} \Rightarrow \vec{n}\cdot\vec{m}=0$

Using the 'info' from above: $\vec{n}=(A,B)$

$$\vec{m}=\overrightarrow{P_0P}=(x-x_0,\ y-y_0)$$

$$\vec{n}\cdot\vec{m}=0$$

$$\Rightarrow (A,B)\cdot(x-x_0,\ y-y_0)=0$$

$$A(x-x_0)+B(y-y_0)=0$$

$$\Rightarrow Ax+By-Ax_0-By_0=0$$

Let $-Ax_0-By_0=C$

$$\boxed{Ax+By+C=0}$$

The Cartesian eqn of a line w/ $\vec{n}=(A,B)$

Example 8.2.3

Determine an equation of a line which is perpendicular to $3x-4y+5=0$. ℓ

We do not have a known point $\Rightarrow$ use $P_0(x_0,y_0)$

$\vec{n}_\ell=(3,-4)$

$\vec{m}_\ell=(4,3)$

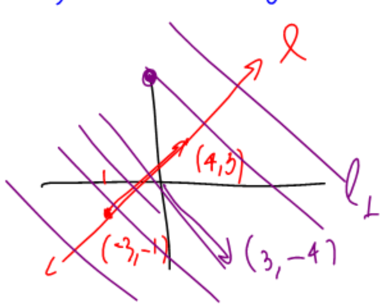
$\vec{n}_\ell\perp\ell \Rightarrow \vec{m}_{\ell_\perp}=(3,-4)$

vector: $(x,y)=(x_0,y_0)+t(3,-4)$

Cartesian

$Ax+By+C=0$

$$\boxed{4x+3y+C=0}$$



Example 8.2.4

Determine the Cartesian equation of a line in $\mathbb{R}^2$ passing through $P_0(5,-2)$ with normal $\vec{n}=(\underset{A}{2},\underset{B}{-7})$.

Cartesian $Ax+By+C=0$, $\vec{n}=(A,B)$

Cartesian eqn: $2x-7y+C=0$ Use $P_0(5,-2)$ to find C

$2(5)-7(-2)+C=0 \Rightarrow C=-24$

$\therefore \boxed{2x-7y-24=0}$

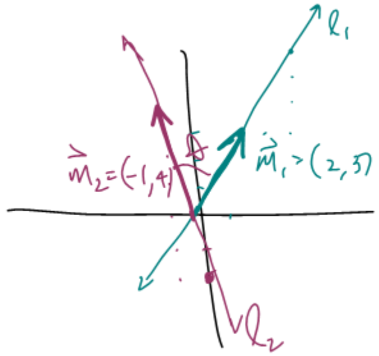
Example 8.2.5

Given lines

$l_1:(x,y)=(1,3)+t(2,3)$

$l_2:(x,y)=(0,-2)+s(-1,4)$

Determine the angle between l_1 and l_2.



Note: the angle θ, between the lines is just the angle between the direction vector

$$\cos(\theta)=\frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1|\,|\vec{m}_2|}=\frac{(2,3)\cdot(-1,4)}{(\sqrt{13})(\sqrt{17})}$$

$$\rightarrow \theta=\cos^{-1}\left(\frac{10}{\sqrt{13}\sqrt{17}}\right)=47.7^\circ$$

Class/Homework for Section 8.2

Pg. 442 Investigation (whistle a happy tune)
Pg. 443 – 444 #1 – 3, 5 – 7, 9, 10, 12

8.3 Lines in Three Space

As in $\mathbb{R}^2$, to get an **equation for a line** we will need **two bits of information**. We need either:

A) A known point and a direction vector.
B) Two known points (from which we get a direction vector).

Note: "Slope" has no real meaning in $\mathbb{R}^3$, and so there are **no scalar equations in 3-Space!**

Vector Equation

A line L passes through the point $P_0(x_0, y_0, z_0)$ and has direction vector $\vec{m} = (a, b, c)$ determine a vector equation of L.

$$\vec{r} = \vec{r_0} + t\vec{m}, \quad t \in \mathbb{R}$$

$$\Rightarrow \boxed{(x, y, z) = (x_0, y_0, z_0) + t(a, b, c), \quad t \in \mathbb{R}}$$

Parametric Equations

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} t \in \mathbb{R}$$

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: If 2 of the direction numbers (a, b, or c) are zero
$\Rightarrow$ No symmetric eqn

But if only 1 direction # is zero (eg $b = 0$) the symmetric eqn. is: $\boxed{\frac{x - x_0}{a} = \frac{z - z_0}{c}, y = y_0}$

Example 8.3.1

From your text: Pg. 449 #5f

Determine the vector, parametric and symmetric equations for the line passing through the point $Q(1,2,4)$ and which is parallel to the z-axis.

for $\vec{m}$, take $\vec{m} = (0,0,1)$

Note: there are no symmetric eq'ns

vector: $\vec{r} = \vec{r}_0 + t\vec{m}$, $t \in \mathbb{R}$

$\Rightarrow (x,y,z) = (1,2,4) + t(0,0,1)$, $t \in \mathbb{R}$

parametric

$$\left.\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 4 + t \end{aligned}\right\} t \in \mathbb{R}.$$

Class/Homework for Section 8.3

KNOW the equations for line in 3-Space
Pg. 449 – 450 #1, 2, 4 – 6, 8 – 11