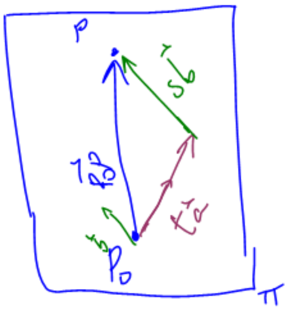
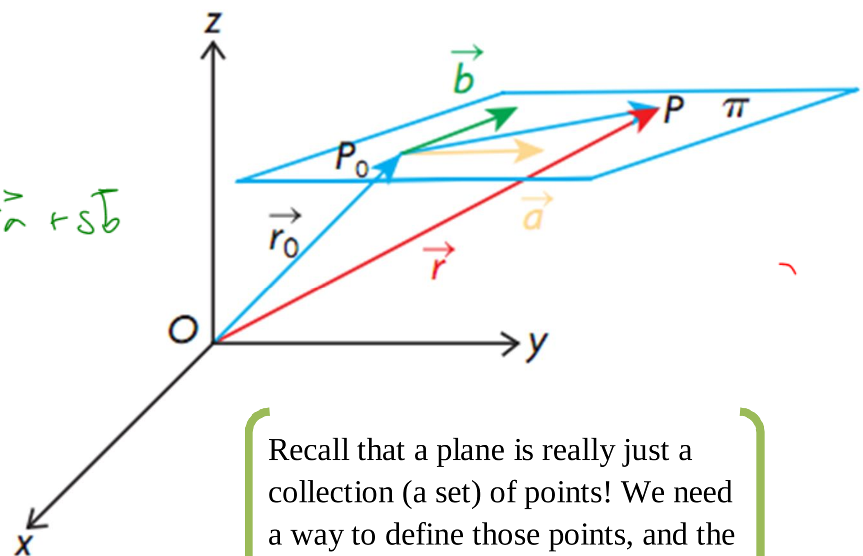
8.4 Vector and Parametric Equations of Planes

Here we will be working in $\mathbb{R}^3$. Recall that any **two non-collinear vectors will span a plane**. Thus, we can say that a **plane**, which we might call π, can be "defined" by **two direction vectors**. Consider the sketch (from your text):

$\vec{r} = \vec{r_0} + \vec{P_0P}$



$\vec{P_0P} = t\vec{a} + s\vec{b}$



$\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}$

Recall that a plane is really just a collection (a set) of points! We need a way to define those points, and the way we normally use is algebraic equations!

Vector and Parametric Equations of a Plane

Given a **known point** $P_0(x_0, y_0, z_0)$,

$\vec{a}$ & $\vec{b}$ are non-collinear

and **two direction vectors** $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$,

and **two scalars**, s and t (both real numbers), then

Vector Equation

$$\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}, \quad t, s \in \mathbb{R}$$

$$(x, y, z) = (x_0, y_0, z_0) + t(a_1, a_2, a_3) + s(b_1, b_2, b_3), \quad s, t \in \mathbb{R}$$

Parametric Equations

$$\left.\begin{aligned} x &= x_0 + ta_1 + sb_1 \\ y &= y_0 + ta_2 + sb_2 \\ z &= z_0 + ta_3 + sb_2 \end{aligned}\right\} \quad s, t \in \mathbb{R}$$

Q. What about symmetric equations of a plane?

These do not exist because we cannot simultaneously solve for two scalars, t & s.

Example 8.4.1

From your text: Pg. 459 #1

State which of the following equations define lines and which define planes. Explain how you made your decision.

a. $\vec{r} = (1, 2, 3) + s(1, 1, 0) + t(3, 4, -6),\ s, t \in \mathbf{R}$

b. $\vec{r} = (-2, 3, 0) + m(3, 4, 7),\ m \in \mathbf{R}$

c. $x = -3 - t, y = 5, z = 4 + t,\ t \in \mathbf{R}$

d. $\vec{r} = m(4, -1, 2) + t(4, -1, 5),\ m, t \in \mathbf{R}$

a) Plane (2 directions) b) line c) line $\vec{m} = (-1, 0, 1)$ d) Plane (two directions!)

Example 8.4.2

From your text: Pg. 459 #5

Explain why the equation $\vec{r} = (-1,0,-1) + s(2,3,-4) + t(4,6,-8)$ does not describe a plane.

$\vec{m}_1 = (2, 3, -4)$

$\vec{m}_2 = (4, 6, -8) = 2\vec{m}_1$

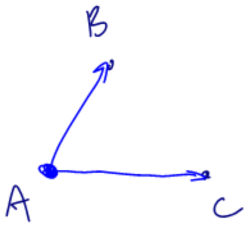
$\Rightarrow$ $\vec{m}_1$ & $\vec{m}_2$ are collinear $\Rightarrow$ they cannot span a plane

$\Rightarrow$ $\vec{r}$ defines a line in $\mathbb{R}^3$!

Note: 3 points are always coplanar.
We have to make sure that the points are not collinear

Example 8.4.3

Determine vector and parametric equations of a plane containing the points $A(1,2,0)$, $B(3,-2,1)$ and $C(0,2,1)$.



We need 2 directions

$\vec{m}_1 = \vec{AB} = (2,-4,1)$

$\vec{m}_2 = \vec{AC} = (-1,0,1)$

vector use $A(1,2,0)$ as the known pt

$\vec{r} = (1,2,0) + t(2,-4,1) + s(-1,0,1)$, $t,s \in \mathbb{R}$

parametric

$$\left.\begin{aligned} x &= 1 + 2t - s \\ y &= 2 - 4t \\ z &= t + s \end{aligned}\right\} t,s \in \mathbb{R}$$

Clearly $\vec{m}_1 \neq k\vec{m}_2$ (not collinear)

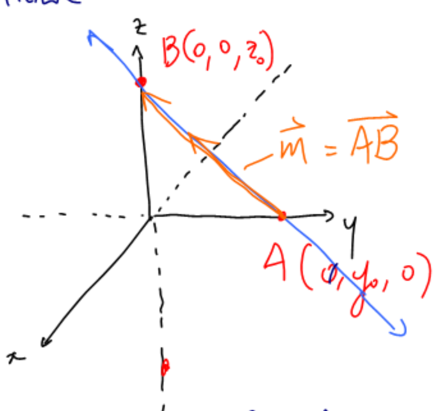
$\therefore$ they span a plane

Example 8.4.4

From your text: Pg. 460 #15

The plane with equation $\vec{r} = (1, 2, 3) + m(1, 2, 5) + n(1, -1, 3)$ intersects the y- and z-axes at the points A and B, respectively. Determine the equation of the line that contains these two points.

Picture



A & B are both on the plane

Parametric eqns of plane:

$x = 1 + m + n$

$y = 2 + 2m - n$

$z = 3 + 5m + 3n$

Point A

$x = 0 \Rightarrow m + n = -1$ ①

$z = 0 \quad 5m + 3n = -3$ ②

② − 3① $\quad 2m = 0$

$m = 0 \Rightarrow n = -1$

$y_0 = 2 + 2(0) - (-1) = 3$

$\therefore A(0,3,0)$

Point B

$x = 0 \Rightarrow m + n = -1$ ①

$y = 0 \Rightarrow 2m - n = -2$ ②

①+② $\quad 3m = -3 \Rightarrow m = -1 \Rightarrow n = 0$

$z_0 = 3 + 5(-1) + 3(0)$

$= -2$

$\therefore B(0,0,-2)$

Class/Homework for Section 8.4

Read Example 4 on Pg. 458

Pg. 459 – 460 #3, 4, 6, 7 (beautiful), 9 – 11, 13

$\therefore \vec{m} = \vec{AB} = (0,-3,-2)$

$\therefore$ vector eqn of a line (using $A(0,3,0)$)

$\vec{r} = (0,3,0) + t(0,-3,-2), t \in \mathbb{R}$