8.5 The Cartesian Equation of a Plane

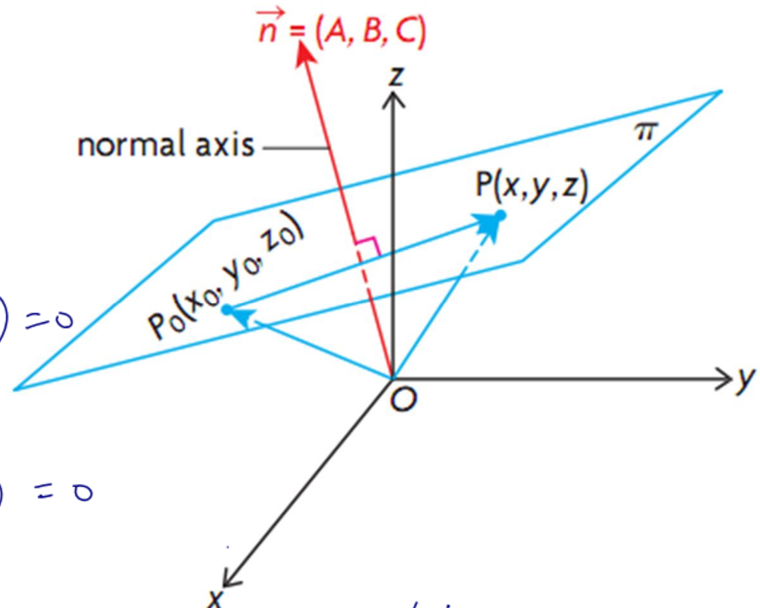
information $\vec{n} = (A, B)$

In $\mathbb{R}^2$ we saw that the Cartesian Equation of a **Line** is given by: $Ax + By + C = 0$, where the vector $\vec{n} = (A, B)$ was a **normal** to the line (recall that **normal** means

$\Rightarrow$ direction vector $\vec{m} = (-B, A)$ or $\vec{m} = (B, -A)$

In $\mathbb{R}^3$ a line has no normal (or infinitely many normals, depending on your perspective) which gives another reason that there is no "scalar" equation of a line in $\mathbb{R}^3$. However, we can obtain a normal to a plane in $\mathbb{R}^3$!

Consider the diagram:



Since $\vec{n} \perp \overrightarrow{P_0P}$

$\Rightarrow \vec{n} \cdot \overrightarrow{P_0P} = 0$

$\Rightarrow (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$ Let $D = -(Ax_0 + By_0 + Cz_0)$

$$\Rightarrow \boxed{Ax + By + Cz + D = 0}$$

with normal $\vec{n} = (A, B, C)$

Example 8.5.1

Determine the Cartesian equation of the plane containing the point $P_0(3,-1,2)$ and with normal vector $\vec{n}=(2,-1,5)$.

Two Methods

① Use the general eqn $Ax+By+Cz+D=0$

$\vec{n}=(A,B,C)$

$2x-y+5z+D=0$ use $P_0(3,-1,2)$ to find D

$\Rightarrow 2(3)-(-1)+5(2)+D=0$

$\Rightarrow D=-17$

$\therefore \boxed{2x-y+5z-17=0}$

Consider $P(x,y,z)$ in the plane

then $\vec{n}\cdot\overrightarrow{P_0P}=0$

$(2,-1,5)\cdot(x-3,y+1,z-2)=0$

$\Rightarrow \boxed{2x-y+5z-17=0}$

Example 8.5.2

Determine Vector, Parametric and Cartesian equations of the plane through the three (coplanar) points $P_1(3,-1,2)$, $P_2(0,2,-1)$ and $P_3(-1,2,3)$

Vector: need $\rightarrow$ point: take $P_1(3,-1,2)$ & 2 direction vectors

let $\vec{a}=\overrightarrow{P_1P_2}$ $\quad$ $\vec{b}=\overrightarrow{P_1P_3}$

$=(-3,3,-3)$ $\quad$ $=(-4,3,1)$

take $\vec{a}=(-1,1,-1)$

$\vec{r}=\vec{r_0}+t\vec{a}+s\vec{b}$, $t,s\in\mathbb{R}$

$\Rightarrow (x,y,z)=(3,-1,2)+t(-1,1,-1)+s(-4,3,1)$

Parametric:

$$\left.\begin{aligned} x &= 3-t-4s \\ y &= -1+t+3s \\ z &= 2-t+s \end{aligned}\right\} \quad t,s\in\mathbb{R}$$

Cartesian: We need a normal and a known point

Take $P_1(3,-1,2)$ as $P_0(x_0, y_0, z_0)$

Take $\vec{n} = \vec{a} \times \vec{b}$

$= (-1,1,-1) \times (-4,3,1)$

$= (4, 5, 1)$

Using "method ②"

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$P(x,y,z)$

$\Rightarrow (4,5,1) \cdot (x-3, y+1, z-2) = 0$

$\Rightarrow 4x + 5y + z - 9 = 0$

Example 8.5.3

Given the Cartesian equation of the plane $\pi: 2x + y - 3z + 2 = 0$, find a vector equation for π.

If we had 3 points in the plane we could construct 2 directions

$\Rightarrow$ a point & 2 directions

P_1 : let $x=0, z=0 \Rightarrow y=-2$ $P_1(0,-2,0)$

P_2 : let $x=0, y=1 \Rightarrow z=1$ $P_2(0,1,1)$

P_3 : let $y=0, z=0 \Rightarrow x=-1$ $P_3(-1,0,0)$

Direction vectors

$\vec{a} = \overrightarrow{P_1P_2} = (0,3,1)$

$\vec{b} = \overrightarrow{P_1P_3} = (-1,2,0)$

Take $P_1(0,-2,0)$ as our point

$\Rightarrow \vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}, \quad t,s \in \mathbb{R}$

$\Rightarrow (x,y,z) = (0,-2,0) + t(0,3,1) + s(-1,2,0)$

$t,s \in \mathbb{R}$

Class/Homework for Section 8.5

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