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MATH@TD

MCV4U Equations of Lines and Planes: Chapter 8 Practice Test

Solutions

1. Determine which line is parallel to the line with Cartesian equation $2x + 3y + 5 = 0$.

   a. $\vec{r} = (-1,-1) + s(2,3),\ s \in \mathbf{R}$
   b. $x = 3t - 1,\ y = -2t - 1,\ t \in \mathbf{R}$
   c. $x = 2t - 1,\ y = 3t - 1,\ t \in \mathbf{R}$
   d. none of the above

normal: $\vec{n} = (2,3) \Rightarrow$ direction vector $\vec{m} = (3,-2)$ or $\vec{m} = (-3,2)$

(b) is the only eqn with direction vector $\vec{m} = (3,-2)$.

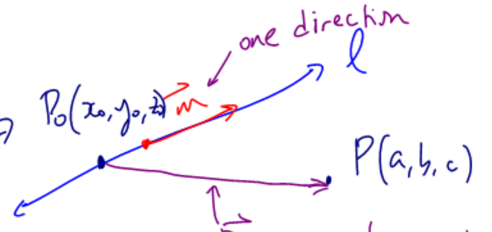
2. Which of the following determines a plane?

   a. a line and a point not on the line
   b. two intersecting lines
   c. two parallel, non-coincident lines
   d. all of the above

This is a badly worded question. My apologies

A plane requires two directions. Clearly (c) can't be correct → whoops – I confused vector w/ line see below for a picture

(a) looks like this:

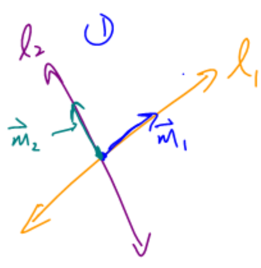


some point on the line which we "know"

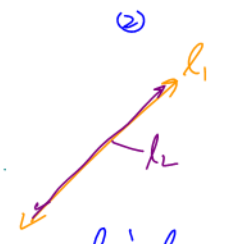
$\vec{P_0P}$ can be used as a second direction

$\Rightarrow$ (a) is a valid answer

(b) has two possibilities



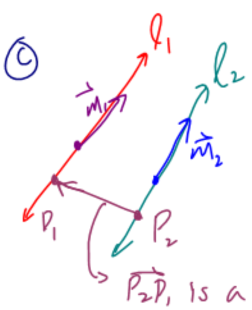
two directions → can define a plane



ℓ_1 & ℓ_2 are coincident ⇒ only one direction

(b) is sometimes valid

(c)



$\vec{P_2P_1}$ is a second direction!

Clearly we only have 'one' ($\vec{m}_1 = k\vec{m}_2$) direction – But we can get another direction between the lines, using one point on each line

(c) is valid too!

$\Rightarrow$ (d) all of the above

3. Which of the following is not a plane?

a. $\vec{r} = (1,3,4) + s(2,-1,2) + t(1,1,1),\ s,t \in \mathbf{R}$ — two non-collinear directions ⇒ plane

b. $\vec{r} = (2,4,2) + s(1,-2,3) + t(3,2,2),\ s,t \in \mathbf{R}$ — " " " " " "

c. $\vec{r} = (3,2,3) + s(4,-4,2) + t(-2,2,-1),\ s,t \in \mathbf{R}$ — Note: $(4,-4,2) = -2(-2,2,-1)$ ⇒ 'one' direction ⇒ not a plane

d. $\vec{r} = (-2,1,4) + s(2,2,-1) + t(2,2,1),\ s,t \in \mathbf{R}$

→ not collinear ⇒ 2 directions ⇒ plane

4. Which of the following lines is not parallel to the other three?

⇓ the direction vectors are not collinear

a. $\dfrac{x-3}{2} = \dfrac{y+3}{-6} = \dfrac{z+7}{4}$

b. $\dfrac{x+1}{-1} = \dfrac{y-7}{3} = \dfrac{z+11}{-2}$

c. $x = t-1,\ y = -3t+4,\ z = 2t-11,\ t \in \mathbf{R}$

d. $\vec{r} = (3,-5,-3) + s(2,-3,1),\ s \in \mathbf{R}$

Direction vectors a) $\vec{m} = (2,-6,4)$ b) $\vec{m} = (-1,3,-2)$ c) $\vec{m} = (1,-3,2)$

d) $\vec{m} = (2,-3,1)$

Clearly the directions for a), b) and c) are collinear (scalar multiples of each other). The direction of d) is not collinear with any of the others ⇒ d)

5. Which plane goes through the origin and is perpendicular to the line $\vec{r} = (2,-2,1) + s(2,3,-4),\ s \in \mathbf{R}$?

$\vec{m} = (2,3,-4)$

a. $2x - 2y + z = 0$ — $\vec{n} = (2,-2,1)$

b. $2x + 3y - 4z = 0$ — $\vec{n} = (2,3,-4) = \vec{m}$!

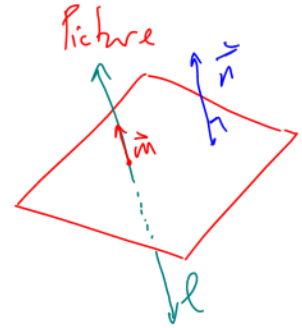
c. $2x + 3y + z - 4 = 0$ — $n = (2,3,1)$

d. none of the above

All of the planes are given in Cartesian form ⇒ we have normals

The line is $\perp$ to the plane ⇒ the direction vector of line is collinear with the normal of the plane

b)



$\vec{m} = k\vec{n}$, k is a scalar

6. Determine the vector and parametric equations for the line passing through the points $P(-1,-3)$ and $Q(3,5)$.

a. $\vec{r} = \underbrace{(3,5)}_{\text{good point}} + s\underbrace{(1,2)}_{\text{good direction}}, s \in \mathbf{R}$
   $x = 2t-1,\ y = 4t-3,\ t \in \mathbf{R}$ $\vec{m} = (2,4)$ good, point $(-1,-3)$ good

b. $\vec{r} = (1,-3) + s(4,8),\ s \in \mathbf{R}$
   $x = 4t-1,\ y = 8t-3,\ t \in \mathbf{R}$

not a point given

c. $\vec{r} = (1,0) + s(2,4),\ s \in \mathbf{R}$
   $x = t-1,\ y = 2t-3,\ t \in \mathbf{R}$

not a point given

d. all of the above

$\vec{m} = \vec{PQ}$
$= (4,8)$

we can take any scalar multiple of $\vec{m}$ as a direction

eg $\vec{m}$ could be
$\vec{m} = (1,2)$ or $\vec{m} = (2,4)$
or $\vec{m} = (8,16)$ or ....

a is the answer.

7. Which of the following equations determines a line with normal vector $\vec{n} = (4,3)$ going through the point $P(1,-1)$?

a. $4x+3y-1=0$
b. $4x+3y+1=0$
c. $3x+4y-1=0$
d. $3x+4y+1=0$

Cartesian form of a line:
$Ax + By + C = 0$ where the normal $\vec{n} = (A,B)$

$\vec{n} = (4,3)$

$\Rightarrow$ eq: $4x + 3y + C = 0$ use $P(1,-1)$ to find C

$4(1) + 3(-1) + C = 0 \Rightarrow C = -1 \Rightarrow$ eq: $4x + 3y - 1 = 0$ a

8. What value of k will make the planes π_1: $2x - ky + 3z - 1 = 0$ and π_2: $2kx + 3y - 2z = 4$ perpendicular?

a. 2
b. 4
c. 6
d. 8

The planes are given in Cartesian form $\Rightarrow$ we have the normals

$\pi_1: \vec{n}_1 = (2,-k,3) \qquad \pi_2: \vec{n}_2 = (2k,3,-2)$

Since the planes are $\perp \Rightarrow \vec{n}_1 \perp \vec{n}_2$

$\Rightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$

$\Rightarrow (2,-k,3)\cdot(2k,3,-2) = 0$

$\Rightarrow 4k - 3k - 6 = 0 \Rightarrow k = 6$

c

9. Which of the following is a unit normal vector to the plane with Cartesian equation $2x - y + 2z - 5 = 0$?

a. $\vec{n} = (2,-1,2)$

b. $\vec{n} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

c. $\vec{n} = \left(\frac{2}{5}, -\frac{1}{5}, \frac{2}{5}\right)$

d. $\vec{n} = (-2,1,-2)$

The normal to the plane is $\vec{n} = (2,-1,2)$

To get a unit normal we take $\vec{n}_u = \frac{\vec{n}}{|\vec{n}|}$ (so that $\vec{n}_u$ has a magnitude of 1)

$|\vec{n}| = \sqrt{2^2 + (-1)^2 + (2)^2} = \sqrt{9}$
$= 3$

b

$\therefore \vec{n}_u = \frac{\vec{n}}{3} = \frac{(2,-1,2)}{3} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

so we have the normal

10. Which of the following is a direction vector for the plane with Cartesian equation $3x - 2y - 5z + 3 = 0$?

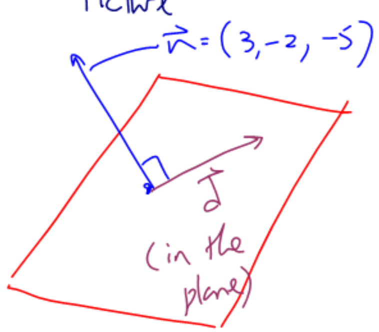
a. $\vec{d} = (3,-2,-5)$

b. $\vec{d} = (1,-1,1)$

c. $\vec{d} = (-1,1,1)$

d. $\vec{d} = (3,2,5)$

Picture



$\vec{n} = (3,-2,-5)$ is $\perp$ to the plane

Since $\vec{d}$ is in the plane

$\Rightarrow \vec{n} \perp \vec{d}$

$\Rightarrow \vec{n} \cdot \vec{d} = 0$

a) $\vec{n} \cdot \vec{d} = (3,-2,-5) \cdot (3,-2,-5) = 38 \neq 0$ No

b) $\vec{n} \cdot \vec{d} = (3,-2,-5) \cdot (1,-1,1) = 3 + 2 - 5 = 0 \therefore$ Yes!

b

11. Determine the parametric equations for the line perpendicular to $L: \vec{r} = (5,6) + s(1,-5),\ s \in \mathbf{R}$ containing the point $P(2,4)$.

L has direction $\vec{m} = (1,-5)$

only in $\mathbb{R}^2$ can we do the "negative reciprocal" trick

$\Rightarrow$ a line $\perp$ to L will have direction $\vec{m}_{\perp} = (5,1)$

$L_{\perp}: \vec{r} = \vec{r}_0 + t\vec{m}_{\perp},\ t \in \mathbb{R}$

$\Rightarrow (x,y) = (2,4) + t(5,1)$

$\therefore$ parametric eq'ns are: $\left.\begin{array}{l} x = 2 + 5t \\ y = 4 + t \end{array}\right\}\ t \in \mathbb{R}$.

12. Determine Vector, Parametric and if possible Symmetric equations of the line going through the points $P(-2,0,3)$ and $Q(1,3,7)$. BEFORE ANSWERING, A MINI-LESSON IS PRESENTED

KNOW THE FORMS OF EQUATIONS AND THE 'INFORMATION' EACH FORM CONTAINS

		LINES	PLANES
$\mathbb{R}^2$	VECTOR	$\vec{r} = \vec{r}_0 + t\vec{m},\ t \in \mathbb{R}$ $(x,y) = (x_0,y_0) + t(a,b)$ — KNOW: POINT $P_0(x_0,y_0)$ direction $\vec{m} = (a,b)$	No planes in $\mathbb{R}^2$
	PARAMETRIC	$x = x_0 + at$ $y = y_0 + bt$ — KNOW: POINT $P_0(x_0,y_0)$ DIRECTION $\vec{m} = (a,b)$	
	SYMMETRIC	$\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b},\ a,b \neq 0$ — KNOW SAME AS ABOVE	
	CARTESIAN	$Ax + By + C = 0$ — KNOW Normal $\vec{n} = (A,B)$	
$\mathbb{R}^3$	VECTOR	$\vec{r} = \vec{r}_0 + t\vec{m},\ t \in \mathbb{R}$ $(x,y,z) = (x_0,y_0,z_0) + t(a,b,c)$ — KNOWN POINT; DIRECTION VECTOR	$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b},\ t,s \in \mathbb{R}$ $(x,y,z) = (x_0,y_0,z_0) + t(a_1,a_2,a_3) + s(b_1,b_2,b_3)$ — KNOWN POINT; 2 DIRECTION VECTORS
	PARAMETRIC	$x = x_0 + at$ $y = y_0 + bt$ $z = z_0 + ct$ — KNOWN: $P_0(x_0,y_0,z_0)$ direction $\vec{m} = (a,b,c)$	$x = x_0 + a_1t + b_1s$ $y = y_0 + a_2t + b_2s$ $z = z_0 + a_3t + b_3s$ — KNOWN: $P_0(x_0,y_0,z_0)$ 2 DIRECTIONS $\vec{a}$ & $\vec{b}$
	SYMMETRIC	$\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$ — KNOWN $P_0(x_0,y_0,z_0)$ DIRECTION $\vec{m} = (a,b,c)$	NONE
	CARTESIAN	NONE	$Ax + By + Cz + D = 0$ KNOWN: normal $\vec{n} = (A,B,C)$

I hope the above chart helps. Back to #12

12. Determine Vector, Parametric and if possible Symmetric equations of the line going through the points $P(-2,0,3)$ and $Q(1,3,7)$.

$\vec{m} = \vec{PQ} = (3,3,4)$

vector eqn: $\vec{r} = \vec{r}_0 + t\vec{m}$ (using $P(-2,0,3)$ as the known point)

$\Rightarrow (x,y,z) = (-2,0,3) + t(3,3,4)$, $t \in \mathbb{R}$

parametric:
$$\left.\begin{array}{l} x = -2 + 3t \\ y = 3t \\ z = 3 + 4t \end{array}\right\} t \in \mathbb{R}$$

Symmetric:
$$\frac{x+2}{3} = \frac{y}{3} = \frac{z-3}{4}$$

13. Determine a Vector equation and the Cartesian equation of a plane that contains the points $P(3,-1,-2)$, $Q(2,2,0)$, and $R(-5,2,1)$.

VECTOR: need two direction vectors

take $\vec{a} = \vec{PQ} = (-1,3,2)$, $\vec{b} = \vec{PR} = (-8,3,3)$, known point $P(3,-1,-2)$

vector eqn: $\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}$, $t, s \in \mathbb{R}$

$\Rightarrow \vec{r} = (3,-1,-2) + t(-1,3,2) + s(-8,3,3)$

CARTESIAN: need a normal (which is $\perp$ to the plane), and a known point

take $\vec{n} = \vec{a} \times \vec{b}$

$= (-1,3,2) \times (-8,3,3)$

$= \big((3)(3) - (2)(3),\ (2)(-8) - (-1)(3),\ (-1)(3) - 3(-8)\big)$

$= (3, -13, 21)$

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eqn (two methods!!! pick whichever you prefer)

① Using the general eqn

$Ax + By + Cz + D = 0$ $\quad \vec{n} = (A,B,C) = (3,-13,21) \quad P_0(3,-1,-2)$

$\Rightarrow 3x - 13y + 21z + D = 0$ use P_0 to find D

$3(3) - 13(-1) + 21(-2) + D = 0$

$\Rightarrow D = 20$

$\therefore \boxed{3x - 13y + 21z + 20 = 0}$

② using $P(x,y,z)$, $P_0(3,-1,-2)$ and the fact that $\vec{n} \cdot \overrightarrow{P_0P} = 0$

$\Rightarrow (3,-13,21) \cdot (x-3, y+1, z+2) = 0$

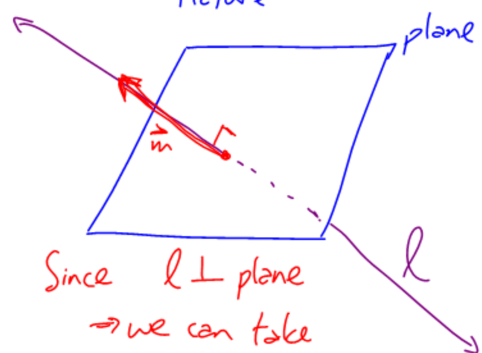
$\Rightarrow 3x - 13y + 21z + 20 = 0$

14. Determine ~~the~~ an equation of a plane containing the point $P_0(1,-1,0)$ and which is perpendicular to the line given by $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{3}$.

Line has direction vector $\vec{m} = (2,-1,3)$

Take $\vec{n} = \vec{m} = (2,-1,3)$

Picture



Since $\ell \perp$ plane $\rightarrow$ we can take $\vec{m}$ as the normal!

$\therefore$ Cartesian eqn of the plane is

$2x - y + 3z + D = 0$: use $P_0(1,-1,0)$ to find D

$2(1) - (-1) + 3(0) + D = 0$

$\Rightarrow D = -3$

$$\boxed{2x - y + 3z - 3 = 0}$$

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15. Determine Parametric and Catersian equations of the plane containing the lines
$l_1: (x,y,z) = \underset{P_1}{(3,-4,1)} + s\underset{\vec{m}_1}{(1,-3,-5)}, \quad s \in \mathbb{R}, \quad l_2: \underset{P_2}{(7,-1,0)} + t\underset{\vec{m}_2}{(-2,6,10)}, \quad t \in \mathbb{R}.$

$\vec{m}_1 = (1,-3,-5)$

$\vec{m}_2 = (-2,6,10) = -2\vec{m}_1$

$\Rightarrow \vec{m}_1$ & $\vec{m}_2$ are collinear $\Rightarrow$ only 'one' direction

Parametric eq'ns of a plane requires TWO DIRECTIONS

Take $\vec{a} = (1,-3,-5)$, $\vec{b} = \vec{P_1P_2}$
$= (4,3,-1)$ (not collinear w/ $\vec{a}$!)

Take $P_1(3,-4,1)$ as our known point

parametric eq'ns

$$\begin{aligned} x &= x_0 + a_1 t + b_1 s \\ y &= y_0 + a_2 t + b_2 s \\ z &= z_0 + a_3 t + b_3 s \end{aligned} \Rightarrow \boxed{\left.\begin{aligned} x &= 3 + t + 4s \\ y &= -4 - 3t + 3s \\ z &= 1 - 5t - s \end{aligned}\right\} t, s \in \mathbb{R}}$$

Cartesian

need a normal: Take $\vec{n} = \vec{a} \times \vec{b}$
$= (1,-3,-5) \times (4,3,-1)$
$= (18,-19,15)$

~~reducing (÷−3) take prefer "A" to be positive~~ ~~$\vec{n} = (4,-7,-5)$~~

not needed

$\therefore$ eq'n is $18x - 19y + 15z + D = 0$ use $P(3,-4,1)$ to find D

$18(3) - 19(-4) + 15(1) + D = 0$

$\Rightarrow D = -145$

$$\therefore \boxed{18x - 19y + 15z - 145 = 0}$$