9.3 Systems of Equations and Matrices

Definition 9.3.1

A **matrix** is an array (or arrangement) of numbers, arranged in rows and columns

e. g. $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 5 & 0 & 3 \end{pmatrix}$ is a matrix with 3 rows and 3 columns.

$a_{23} = 1$

$a_{22} = 2$

The numbers in a matrix are called **elements** of the matrix.

Notation: $A_{3\times 3} = [a_{ij}]$ "Row – Column"

where: a_{ij} is the element in row i and column j

eg $a_{23} = 1$, $a_{31} = 5$ $a_{12} = 1$

Using matrices to solve Systems of Linear Equations

Consider the system:

$$2x - y + z = 1$$
$$2x + y - 2z = 3$$
$$0x - y + 3z = 4$$

We can use the **coefficients** of the S.O.L.E. in an **Augmented Coefficient Matrix**.

coefficient matrix

augmented by the "answer column"

$$\begin{array}{c} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 1 & -2 & 3 \\ 0 & -1 & 3 & 4 \end{array}\right) \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

We solve systems of linear equations using matrices by employing Elementary Row Operations! There are two techniques:

1) **Gaussian Elimination**

Here the goal is to get the **ACM** into the form:

$$\left(\begin{array}{ccc|c} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{array}\right)$$

Row Echelon Form

After "simplifying" a matrix using Gaussian Elimination, the matrix is said to be in **Row Echelon Form**

If we get a row which looks like:

" 0 0 0 0 "

↳ free variable ⇒ line

2 rows

0 0 0 0
0 0 0 0

⇒ soln is a plane

2) **Guass – Jordan Elimination**

Here the goal is to get the ACM into the form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right)$$

Reduced Row Echelon Form

row that looks like

0 0 0 #

⇒ No solns

By the way:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_{3\times 3} \qquad A_{3\times 3} \times I_{3\times 3} = A_{3\times 3}$$

Example 9.3.1

Use Gaussian Elimination to solve the system:

$$2x + y - z = 1$$
$$x + y + 2z = -1$$
$$0x + y - z = 3$$

$$\begin{pmatrix} 2 & 1 & -1 & | & 1 \\ 1 & 1 & 2 & | & -1 \\ 0 & 1 & -1 & | & 3 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}$$

ERO

refers back

one step

$$\sim \; 2r_2 - r_1 \begin{pmatrix} 2 & 1 & -1 & | & 1 \\ 0 & 1 & 5 & | & -3 \\ 0 & 1 & -1 & | & 3 \end{pmatrix}$$

$$\begin{array}{c} \\ \\ r_3 - r_2 \end{array} \begin{pmatrix} 2 & 1 & -1 & | & 1 \\ 0 & 1 & 5 & | & -3 \\ 0 & 0 & -6 & | & 6 \end{pmatrix}$$

$r_3 \Rightarrow \quad -6z = 6 \Rightarrow z = -1$

$r_2 \Rightarrow \quad y + 5(-1) = -3 \Rightarrow y = 2$

$r_1 \Rightarrow \quad 2x + (2) - (-1) = 1$

$\Rightarrow \quad x = -1$

$\therefore (x, y, z) = (-1, 2, -1)$

Example 9.3.2

Use Guass – Jordan Elimination to solve:

$$x - y + z = 0$$
$$x + 2y - z = 8$$
$$2x - 2y + z = -11$$

$$\begin{pmatrix} 1 & 0 & 0 & | & \# \\ 0 & 1 & 0 & | & \# \\ 0 & 0 & 1 & | & \# \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 0 \\ 1 & 2 & -1 & | & 8 \\ 2 & -2 & 1 & | & -11 \end{pmatrix}$$

$$\sim \; \begin{array}{c} \\ r_2 - r_1 \\ r_3 - 2r_1 \end{array} \begin{pmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 3 & -2 & | & 8 \\ 0 & 0 & -1 & | & -11 \end{pmatrix}$$

$$\sim \begin{array}{r} \\ r_2 - 2r_3 \\ r_3 \times (-1) \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 30 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\sim \begin{array}{r} r_1 - r_3 \\ r_2 \div 3 \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -11 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\sim \begin{array}{r} r_1 + r_2 \\ \\ \\ \end{array} \left(\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\therefore (x, y, z) = (-1, 10, 11)$$

Class/Homework for Section 9.3

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