Vectors: Awesomeness defined

These problems taken from the Nelson Text

Chapter 6 – Pg. 344 – 347

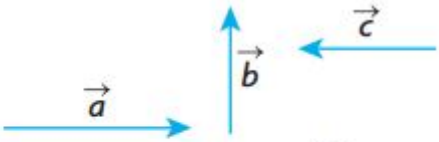
4. $X(-1, 2, 6)$ and $Y(5, 5, 12)$ are two points in R^3.
   a. Determine the components of a position vector equivalent to $\overrightarrow{YX}$.
   b. Determine the components of a *unit* vector that is in the same direction as $\overrightarrow{YX}$.

5. Find the components of the unit vector with the opposite direction to that of the vector from $M(2, 3, 5)$ to $N(8, 1, 2)$.

6. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
   a. Determine the components of the vectors representing the diagonals.
   b. Determine the angles between the sides of the parallelogram.

7. The points $A(-1, 1, 1)$, $B(2, 0, 3)$, and $C(3, 3, -4)$ are vertices of a triangle.
   a. Show that this triangle is a right triangle.
   b. Calculate the area of triangle ABC.
   c. Calculate the perimeter of triangle ABC.
   d. Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices.

8. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



   a. Construct the vector $\vec{a} - \vec{b} + \vec{c}$.
   b. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and if $|\vec{a}| = 4$ and $|\vec{b}| = 3$, determine $|\vec{a} + \vec{b}|$.

11. Calculate the values of a, b, and c in each of the following:

a. $2(a, b, 4) + \frac{1}{2}(6, 8, c) - 3(7, c, -4) = (-24, 3, 25)$

b. $2\left(a, a, \frac{1}{2}a\right) + (3b, 0, -5c) + 2\left(c, \frac{3}{2}c, 0\right) = (3, -22, 54)$

16. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, and the angle between them is 30°. Determine each of the following:

a. $|\vec{d} + \vec{e}|$ b. $|\vec{d} - \vec{e}|$ c. $|\vec{e} - \vec{d}|$

17. An airplane is headed south at speed 400 km/h. The airplane encounters a wind from the east blowing at 100 km/h.

a. How far will the airplane travel in 3 h?

b. What is the direction of the airplane?

18. a. Explain why the set of vectors: $\{(2, 3), (3, 5)\}$ spans R^2.

b. Find m and n in the following: $m(2, 3) + n(3, 5) = (323, 795)$.

21. If $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$, and $\vec{c} = 2\vec{i} + 13\vec{k}$, determine $\left|2\left(\vec{a} + \vec{b} - \vec{c}\right) - \left(\vec{a} + 2\vec{b}\right) + 3\left(\vec{a} - \vec{b} + \vec{c}\right)\right|$.

Chapter 7 – Pg. 418 – 421

2. Given that $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors, $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} - 2\vec{k}$, determine each of the following:

a. $|\vec{a}|$ c. $|\vec{a} - \vec{b}|$ e. $\vec{a} \cdot \vec{b}$

b. $|\vec{b}|$ d. $|\vec{a} + \vec{b}|$ f. $\vec{a} \cdot (\vec{a} - 2\vec{b})$

4. Determine the angle between the vectors $\vec{x} = (4, 5, 20)$ and $\vec{y} = (-3, 6, 22)$.

7. An airplane has a speed of 300 km/h and is headed due west. A wind is blowing from the south at 50 km/h. Determine the resultant velocity of the airplane.

9. Determine the components of a unit vector perpendicular to $(0, 3, -5)$ and to $(2, 3, 1)$.

10. A triangle has vertices $A(2, 3, 7)$, $B(0, -3, 4)$, and $C(5, 2, -4)$.
    a. Determine the largest angle in the triangle.
    b. Determine the area of $\triangle ABC$.

12. A particle is acted upon by the following four forces: 25 N pulling east, 30 N pulling west, 54 N pulling north, and 42 N pulling south.
    a. Draw a diagram showing these four forces.
    b. Calculate the resultant and equilibrant of these forces.

14. If $\vec{a}$ and $\vec{b}$ are unit vectors, and $|\vec{a} + \vec{b}| = \sqrt{3}$, determine $(2\vec{a} - 5\vec{b}) \cdot (\vec{b} + 3\vec{a})$. **(Tricky – Ask for help)**

18. For the vectors $\vec{m} = (\sqrt{3}, -2, -3)$ and $\vec{n} = (2, \sqrt{3}, -1)$, determine the following:
    a. the angle between these two vectors, to the nearest degree
    b. the scalar projection of $\vec{n}$ on $\vec{m}$
    c. the vector projection of $\vec{n}$ on $\vec{m}$
    d. the angle that $\vec{m}$ makes with the z-axis

20. If $\vec{p} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{q} = 2\vec{i} - \vec{j} + \vec{k}$, and $\vec{r} = \vec{j} - 2\vec{k}$, determine each of the following:
    a. $\vec{p} \times \vec{q}$
    b. $(\vec{p} - \vec{q}) \times (\vec{p} + \vec{q})$
    c. $(\vec{p} \times \vec{r}) \cdot \vec{r}$
    d. $(\vec{p} \times \vec{q}) \times \vec{r}$

22. Determine the components of a vector that is perpendicular to the vectors $\vec{a} = (3, 2, -1)$ and $\vec{b} = (5, 0, 1)$.

30. A 25 N force is applied at the end of a 60 cm wrench. If the force makes a 30° angle with the wrench, calculate the magnitude of the torque.

Chapter 8: Pg. 480 – 483

3. a. Determine the vector, parametric, and symmetric equations of the line passing through points $A(-3, 2, 8)$ and $B(4, 3, 9)$.
   b. Determine the vector and parametric equations of the plane containing the points $A(-3, 2, 8)$, $B(4, 3, 9)$, and $C(-2, -1, 3)$.
   c. Explain why a symmetric equation cannot exist for a plane.

4. Determine the vector, parametric, and symmetric equations of the line passing through the point $A(7, 1, -2)$ and perpendicular to the plane with equation $2x - 3y + z - 1 = 0$.

5. Determine the Cartesian equation of each of the following planes:
   a. through the point $P(0, 1, -2)$, with normal $\vec{n} = (-1, 3, 3)$
   b. through the points $(3, 0, 1)$ and $(0, 1, -1)$, and perpendicular to the plane with equation $x - y - z + 1 = 0$
   c. through the points $(1, 2, 1)$ and $(2, 1, 4)$, and parallel to the x-axis

6. Determine the Cartesian equation of the plane that passes through the origin and contains the line $\vec{r} = (3, 7, 1) + t(2, 2, 3)$, $t \in \mathbf{R}$.

8. Determine the Cartesian equation of the plane that contains the line $\vec{r} = (2, 3, 2) + t(1, 1, 4)$, $t \in \mathbf{R}$, and the point $(4, -3, 2)$.

9. Determine the Cartesian equation of the plane that contains the following lines:
   $L_1: \vec{r} = (4, 4, 5) + t(5, -4, 6)$, $t \in \mathbf{R}$, and
   $L_2: \vec{r} = (4, 4, 5) + s(2, -3, -4)$, $s \in \mathbf{R}$

11. A plane has $3x + 2y - z + 6 = 0$ as its Cartesian equation. Determine the vector and parametric equations of this plane.

20. Calculate the acute angle that is formed by the intersection of each pair of lines.
   a. $L_1: \dfrac{x-1}{1} = \dfrac{y-3}{5}$ and $L_2: \dfrac{x-2}{2} = \dfrac{1-y}{3}$
   b. $y = 4x + 2$ and $y = -x + 3$
   c. $L_1: x = -1 + 3t, y = 1 + 4t, z = -2t$ and
      $L_2: x = -1 + 2s, y = 3s, z = -7 + s$
   d. $L_1: (x, y, z) = (4, 7, -1) + t(4, 8, -4)$ and
      $L_2: (x, y, z) = (1, 5, 4) + s(-1, 2, 3)$

24. Determine the parametric equations of the plane that contains the following two parallel lines:
   $L_1: (x, y, z) = (2, 4, 1) + t(3, -1, 1)$ and
   $L_2: (x, y, z) = (1, 4, 4) + t(-6, 2, -2)$

29. Determine the Cartesian equation of the plane that has normal vector $(6, -5, 12)$ and passes through the point $(5, 8, -3)$.

34. Determine the Cartesian equation of the plane that

a. contains the point $P(-2, 6, 1)$ and has normal vector $(2, -4, 5)$

b. contains the point $P(-2, 0, 6)$ and the line $\frac{x-4}{3} = \frac{y+2}{-5} = \frac{z-1}{2}$

c. contains the point $P(3, 3, 3)$ and is parallel to the xy-plane

d. contains the point $P(-4, 2, 4)$ and is parallel to the plane $3x + y - 4z + 8 = 0$

e. is perpendicular to the yz-plane and has y-intercept 4 and z-intercept -2

f. is perpendicular to the plane $x - 2y + z = 6$ and contains the line $(x, y, z) = (2, -1, -1) + t(3, 1, 2)$

Chapter 9: Pg. 552 – 555

6. Determine the intersection of the plane $3x - 4y - 5z = 0$ with $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$.

8. Solve each system of equations.

a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$

12. a. Given the line $\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$, and the plane $x - 2y + z + 4 = 0$, verify that the line lies on the plane.

14. You are given the lines $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$, and $\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$.
    a. Determine the coordinates of their point of intersection.
    b. Determine a vector equation for the line that is perpendicular to both of the given lines and passes through their point of intersection.

19. Determine the point of intersection of the line $\frac{x+1}{-4} = \frac{y-2}{3} = \frac{z-1}{-2}$ and the plane with equation $x + 2y - 3z + 10 = 0$.

3. Solve each system of equations.

a. ① $x - y + 2z = 3$
② $2x - 2y + 3z = 1$
③ $2x - 2y + z = 11$

b. ① $x + y + z = 300$
② $x + y - z = 98$
③ $x - y + z = 100$

Cumulative Review: Pg. 557 – 560

1. For the vectors $\vec{a} = (2, -1, -2)$ and $\vec{b} = (3, -4, 12)$, determine the following:
    a. the angle between the two vectors
    b. the scalar and vector projections of $\vec{a}$ on $\vec{b}$
    c. the scalar and vector projections of $\vec{b}$ on $\vec{a}$

3. If $\vec{x}$ and $\vec{y}$ are unit vectors, and the angle between them is 60°, determine the value of each of the following:

a. $|\vec{x} \cdot \vec{y}|$ b. $|2\vec{x} \cdot 3\vec{y}|$ c. $|(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})|$

4. Expand and simplify each of the following, where $\vec{i}, \vec{j}$, and $\vec{k}$ represent the standard basis vectors in R^3:
    a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$
    b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$

5. Determine the angle that the vector $\vec{a} = (4, -2, -3)$ makes with the positive x-axis, y-axis, and z-axis.

6. If $\vec{a} = (1, -2, 3)$, $\vec{b} = (-1, 1, 2)$, and $\vec{c} = (3, -4, -1)$, determine each of the following:

   a. $\vec{a} \times \vec{b}$

   b. $2\vec{a} \times 3\vec{b}$

   c. the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$

   d. $\vec{c} \cdot (\vec{b} \times \vec{a})$

7. Determine the coordinates of the unit vector that is perpendicular to $\vec{a} = (1, -1, 1)$ and $\vec{b} = (2, -2, 3)$.

11. Determine the value of c such that the plane with equation $2x + 3y + cz - 8 = 0$ is parallel to the line with equation $\frac{x-1}{2} = \frac{y-2}{3} = z + 1$.

(Not the best worded question in that planes and lines aren't really "parallel". Planes contain vectors which are parallel to lines, though.)

18. An airplane heads due north with a velocity of 400 km/h and encounters a wind of 100 km/h from the northeast. Determine the resultant velocity of the airplane.

20. a. A line with equation $\vec{r} = (1, 0, -2) + s(2, -1, 2)$, $s \in \mathbf{R}$, intersects the plane $x + 2y + z = 2$ at an angle of θ degrees. Determine this angle to the nearest degree.

    b. Show that the planes π_1: $2x - 3y + z - 1 = 0$ and π_2: $4x - 3y - 17z = 0$ are perpendicular.

    c. Show that the planes π_3: $2x - 3y + 2z - 1 = 0$ and π_4: $2x - 3y + 2z - 3 = 0$ are parallel but not coincident.

21. Two forces, 25 N and 40 N, have an angle of 60° between them. Determine the resultant and equilibrant of these two vectors.

25. Solve the following systems of equations:

    a. ① $x - y + z = 2$
    ② $-x + y + 2z = 1$
    ③ $x - y + 4z = 5$

    b. ① $-2x - 3y + z = -11$
    ② $x + 2y + z = 2$
    ③ $-x - y + 3z = -12$

    c. ① $2x - y + z = -1$
    ② $4x - 2y + 2z = -2$
    ③ $2x + y - z = 5$

    d. ① $x - y - 3z = 1$
    ② $2x - 2y - 6z = 2$
    ③ $-4x + 4y + 12z = -4$

28. a. If $\vec{a}$ and $\vec{b}$ are unit vectors, and the angle between them is 60°, calculate $(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b})$.

    b. Calculate the dot product of $4\vec{x} - \vec{y}$ and $2\vec{x} + 3\vec{y}$ if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 60°.

31. A river is 2 km wide and flows at 4 km/h. A motorboat that has a speed of 10 km/h in still water heads out from one bank, which is perpendicular to the current. A marina lies directly across the river, on the opposite bank.

    a. How far downstream from the marina will the motorboat touch the other bank?

    b. How long will it take for the motorboat to reach the other bank?

34. A crate has a mass of 400 kg and is sitting on an inclined plane that makes an angle of 30° with the level ground. Determine the components of the *weight* of the mass, perpendicular and parallel to the plane. (Assume that a 1 kg mass exerts a force of 9.8 N.)

Answers

See Photocopied Sheets