9.1 Intersecting Lines and Planes

These problems taken from the Nelson text: Pg. 496 – 498

1. Tiffany is given the parametric equations for a line L and the Cartesian equation for a plane π and is trying to determine their point of intersection. She makes a substitution and gets $(1 + 5s) - 2(2 + s) - 3(-3 + s) - 6 = 0$.
   a. Give a possible equation for both the line and the plane.
   b. Finish the calculation, and describe the nature of the intersection between the line and the plane.

4. For each of the following, show that the line lies on the plane with the given equation. Explain how the equation that results implies this conclusion.
   a. $L: x = -2 + t, y = 1 - t, z = 2 + 3t, t \in \mathbf{R}$; $\pi: x + 4y + z - 4 = 0$
   b. $L: \vec{r} = (1, 5, 6) + t(1, -2, -2), t \in \mathbf{R}$; $\pi: 2x - 3y + 4z - 11 = 0$

5. For each of the following, show that the given line and plane do not intersect. Explain how the equation that results implies there is no intersection.
   a. $L: \vec{r} = (-1, 1, 0) + s(-1, 2, 2), s \in \mathbf{R}$; $\pi: 2x - 2y + 3z - 1 = 0$
   b. $L: x = 1 + 2t, y = -2 + 5t, z = 1 + 4t, t \in \mathbf{R}$;
   $\pi: 2x - 4y + 4z - 13 = 0$

6. Verify your results for question 5 by showing that the direction vector of the line and the normal for the plane meet at right angles, and the given point on the line does not lie on the plane.

8. Determine points of intersection between the following pairs of lines, if any exist:
   a. $L_1: \vec{r} = (3, 1, 5) + s(4, -1, 2), s \in \mathbf{R}$;
   $L_2: x = 4 + 13t, y = 1 - 5t, z = 5t, t \in \mathbf{R}$
   b. $L_3: \vec{r} = (3, 7, 2) + m(1, -6, 0), m \in \mathbf{R}$;
   $L_4: \vec{r} = (-3, 2, 8) + s(7, -1, -6), s \in \mathbf{R}$

9. Determine which of the following pairs of lines are skew lines:
   a. $\vec{r} = (-2, 3, 4) + p(6, -2, 3), p \in \mathbf{R}$;
   $\vec{r} = (-2, 3, -4) + q(6, -2, 11), q \in \mathbf{R}$
   b. $\vec{r} = (4, 1, 6) + t(1, 0, 4), t \in \mathbf{R}$; $\vec{r} = (2, 1, -8) + s(1, 0, 5), s \in \mathbf{R}$
   c. $\vec{r} = (2, 2, 1) + m(1, 1, 1), m \in \mathbf{R}$;
   $\vec{r} = (-2, 2, 1) + p(3, -1, -1), p \in \mathbf{R}$
   d. $\vec{r} = (9, 1, 2) + m(5, 0, 4), m \in \mathbf{R}$; $\vec{r} = (8, 2, 3) + s(4, 1, -2), s \in \mathbf{R}$

11. a. Show that the lines $L_1 : \vec{r} = (-2, 3, 4) + s(7, -2, 2), s \in \mathbf{R}$, and $L_2 : \vec{r} = (-30, 11, -4) + t(7, -2, 2), t \in \mathbf{R}$, are coincident by writing each line in parametric form and comparing components

b. Show that the point $(-2, 3, 4)$ lies on L_2. How does this show that the lines are coincident?

12. The lines $\vec{r} = (-3, 8, 1) + s(1, -1, 1), s \in \mathbf{R}$, and $\vec{r} = (1, 4, 2) + t(-3, k, 8), t \in \mathbf{R}$, intersect at a point.

a. Determine the value of k.

b. What are the coordinates of the point of intersection?

15. The lines $\vec{r} = (-1, 3, 2) + s(5, -2, 10), s \in \mathbf{R}$, and $\vec{r} = (4, -1, 1) + t(0, 2, 11), t \in \mathbf{R}$, intersect at point A.

a. Determine the coordinates of point A.

b. Determine the vector equation for the line that is perpendicular to the two given lines and passes through point A.

Answers

1. **a.** $\pi: x - 2y - 3z = 6$, $\vec{r} = (1, 2, -3) + s(5, 1, 1)\ s \in \mathbf{R}$

b. This line lies on the plane.

4. **a.** For $x + 4y + z - 4 = 0$, if we substitute our parametric equations, we have $(-2 + t) + 4(1 - t) + (2 + 3t) + 4 = 0$
All values of t give a solution to the equation, so all points on the line are also on the plane.

b. For the plane $2x - 3y + 2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$:
$2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11 = 0$
All values of t give a solution to this equation, so all points on the line are also on the plane.

5. **a.** $2(-1 - s) - 2(1 + 2s) + 3(2s) - 1 = -5$
Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.

b. $2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13 = 1$
Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. **a.** The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, $\vec{m} \cdot \vec{n} = 0$. So the line is parallel to the plane, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So, the point on the line is not on the plane.

b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, $\vec{m} \cdot \vec{n}$, $= 0$, so the line is parallel to the plane.
and
$2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$
So, the point on the line is not on the plane.

8. **a.** There is no intersection and the lines are skew.

b. $(4, 1, 2)$

9. **a.** not skew

b. not skew

c. not skew

d. skew

11. **a.** Comparing components results in the equation $s - t = -4$ for each component.

b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.

12. **a.** 3

b. $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$

15. **a.** $(4, 1, 12)$

b. $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$