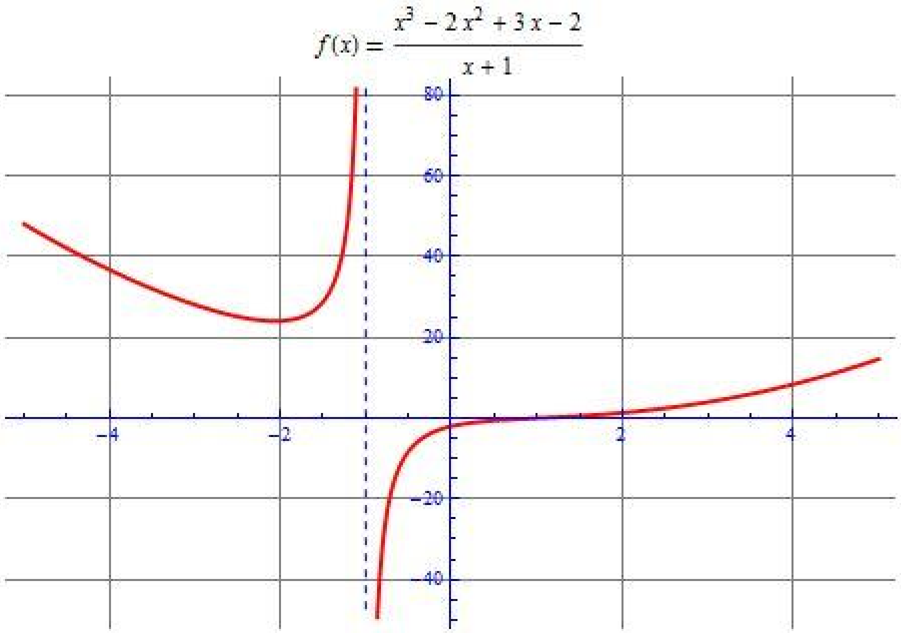
ADVANCED FUNCTIONS

Chapter 4 – Rational Functions, Equations and Inequalities

(*Material adapted from Chapter 5 of your text*)



Chapter 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

4.1 Rational Functions, Domain and Asymptotes

Definition 4.1.1

A **Rational Function** is of the form $R(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where $p(x)$ & $q(x)$ are polynomial fns.

e.g. $f(x) = \frac{3x^2 - 5x + 1}{2x - 1}$ is a rational fn

$g(x) = \frac{\sqrt{2x+5}}{3x-2}$ — not a polynomial $\Rightarrow$ $g(x)$ is not a rational fn.

Domain

Definition 4.1.2

Given a rational function $f(x) = \frac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f : \{ x \in \mathbb{R} \mid q(x) \neq 0 \}$

Example 4.1.1

Determine the natural domain of $f(x) = \frac{x^2 - 4}{x - 3}$.

$D_f = \{ x \in \mathbb{R} \mid x \neq 3 \}$ (set)

OR

$x \in (-\infty, 3) \cup (3, \infty)$ (interval)

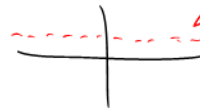
Asymptotes

There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes

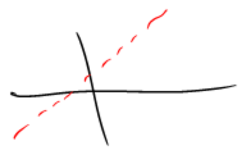
vertical line → eg: "$x = \#$"

2) Horizontal Asymptotes



horizontal line → "$y = \#$"

3) Oblique Asymptotes



line of the form $y = mx + b$

Vertical Asymptotes

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x) = 0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.

Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x) = \dfrac{5x}{x^2 - x - 6}$

V.A.'s occur when denominator = 0

$$f(x) = \frac{5x}{(x-3)(x+2)} \qquad D_f = \{ x \in \mathbb{R} \mid x \neq 3,\ x \neq -2 \}$$

∴ V.A. $x = 3$, $x = -2$.

b) $h(x)=\dfrac{x+3}{x^2-9}$

$$h(x)=\frac{\cancel{x+3}\ 1}{(x-3)\cancel{(x+3)}}\ ,\quad D_h=\{x\in\mathbb{R}\mid x\neq 3,\ x\neq -3\}$$

V.A.

hole discontinuity

$$\Rightarrow\ h(x)=\frac{1}{x-3}$$

(we cancelled the "offending factor away)

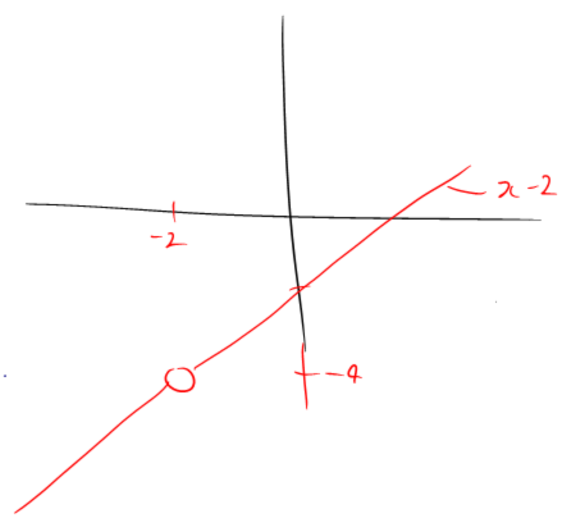
V.A. $x=3$

c) $g(x)=\dfrac{x^2-4}{x+2}$

$$D_g=\{x\in\mathbb{R}\mid x\neq -2\}$$

$$g(x)=\frac{(x-2)\cancel{(x+2)}}{\cancel{x+2}}$$

$$g(x)=x-2$$



$x=-2$ is NOT a V.A.

it's, instead, a hole discontinuity

Horizontal Asymptotes

Here we are concerned with END BEHAVIOUR of the rational fn.

i.e. We are asking, given a rational function $f(x)=\frac{p(x)}{q(x)}$, how is $f(x)$ behaving as $x \to \pm\infty$.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $p(x)$ > Order of $q(x)$

e.g. $f(x)=\frac{x^3-2}{x^2+1}$ $\frac{\text{order } 3}{\text{order } 2}$

Consider $\lim_{x\to\infty}\left(\frac{x^3-2}{x^2+1}\right) = \infty$

÷ every term by the highest power x^3

$$\lim_{x\to\infty}\left(\frac{1-\cancel{\frac{2}{x^3}}^{\to 0}}{\cancel{\frac{1}{x}}^{\to 0}+\cancel{\frac{1}{x^3}}^{\to 0}}\right) = \frac{1}{0} = \infty$$

$\therefore$ No H.A.

2) Order of numerator = Order of denominator

e.g. $f(x)=\frac{2x^2-3x+1}{3x^2+4x-5}$ $\frac{\text{order } 2}{\text{order } 2}$

Consider $\lim_{x\to\infty}\left(\frac{2x^2-3x+1}{3x^2+4x-5}\right)$

÷ each term by x^2

$$\lim_{x\to\infty}\left(\frac{2-\cancel{\frac{3}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}}{3+\cancel{\frac{4}{x}}^{\to 0}-\cancel{\frac{5}{x^2}}^{\to 0}}\right) = \frac{2}{3}$$

$\therefore$ The H.A. is $y=\frac{2}{3}$

e.g. Determine the horizontal asymptote of $g(x)=\frac{3x-4x^5}{5x^5+2x-1}$ $\frac{\text{order } 5}{\text{order } 5}$

$y=-\frac{4}{5}$ ⟸ ratio of lead coefficients

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x) = \dfrac{x^2 - 5x + 6}{x^5 + 7}$ $\dfrac{\text{order } 2}{\text{order } 5}$

Here we also get a H.A.

$y = 0$

Oblique Asymptotes

These occur when the order of the numerator is one more than the order of denominator

e.g. $f(x) = \dfrac{x^2 - 2x + 3}{x - 1}$ $\dfrac{\text{order } 2}{\text{order } 1}$

With Oblique Asymptotes we are still dealing with end behaviour

O.A. have the form $y = mx + b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: By dividing!

eg $\dfrac{x^2 - 2x + 3}{x - 1}$. Divide by Synthetic Div

$$\begin{array}{c|ccc} 1 & 1 & -2 & 3 \\ & & 1 & -1 \\ \hline & 1 & -1 & 2 \end{array}$$

order 2

order 1

$\therefore \dfrac{x^2 - 2x + 3}{x - 1} = x - 1 + \dfrac{2}{x - 1}$

$\therefore f(x) = x - 1 + \dfrac{2}{x - 1}$

$\therefore$ OA.

$y = x - 1$

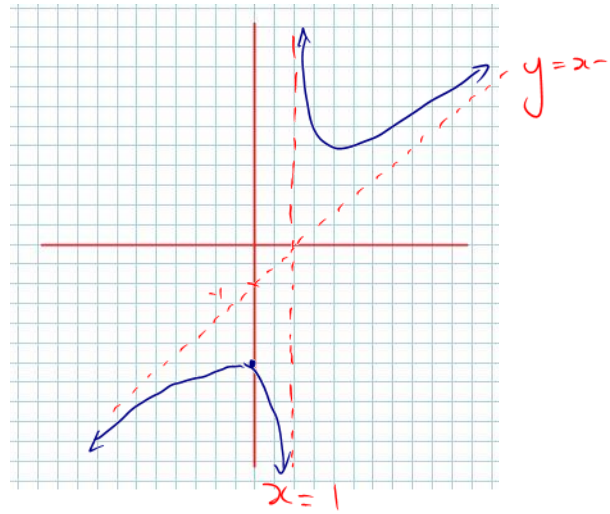
(Rough) Sketch of $f(x) = \dfrac{x^2 - 2x + 3}{x - 1}$ $\dfrac{\text{order } 2}{\text{order } 1}$

V.A. $x = 1$

H.A.   NONE

O.A. $y = x - 1$

y int $f(0) = -3$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x) = \dfrac{x+2}{x^2 + 3x + 2}$ $\dfrac{\text{order } 1}{\text{order } 2}$

$f(x) = \dfrac{x+2}{(x+2)(x+1)}$ $D_f: \{x \in \mathbb{R} \mid x \neq -2, x \neq -1\}$

V.A.	Hole dscty at	H.A.
$x = -1$	$x = -2$	$y = 0$

b) $g(x) = \dfrac{4x^2 - 25}{1x^2 - 9}$ $\dfrac{\text{order } 2}{\text{order } 2}$

$g(x) = \dfrac{(2x-5)(2x+5)}{(x-3)(x+3)}$

VA	H.A.	Holes
$x = -3$, $x = 3$	$y = 4$	none.

c) $h(x)=\dfrac{x^2}{x+3}$ $\quad \dfrac{\text{order } 2}{\text{order } 1}$ $\quad$ OA.

$$\begin{array}{c|ccc} -3 & 1 & 0 & 0 \\ & & -3 & 9 \\ \hline & 1 & -3 & 9 \end{array}$$

$y = x-3$

V.A	HA	Holes
$x=-3$	NONE	none

Example 4.1.4

Determine an equation for a function with a $\underbrace{\text{vertical asymptote at } x=-3}_{\text{need } (x+3) \text{ as a non-cancellable factor}}$, and a $\underbrace{\text{horizontal asymptote at } y=0}_{\text{order numerator} < \text{order of denom}}$.

$f(x)=\dfrac{x}{(x+3)^2}$, $g(x)=\dfrac{1}{x+3}$, $h(x)=\dfrac{x-5}{(x+3)(x-2)}$

Example 4.1.5

Determine an equation for a function with a $\underbrace{\text{hole discontinuity at } x=3}_{\text{need a cancellable factor } (x-3)}$.

$$f(x)=\frac{x-3}{(x-3)(x+2)}$$

Class/Homework for Section 4.1

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