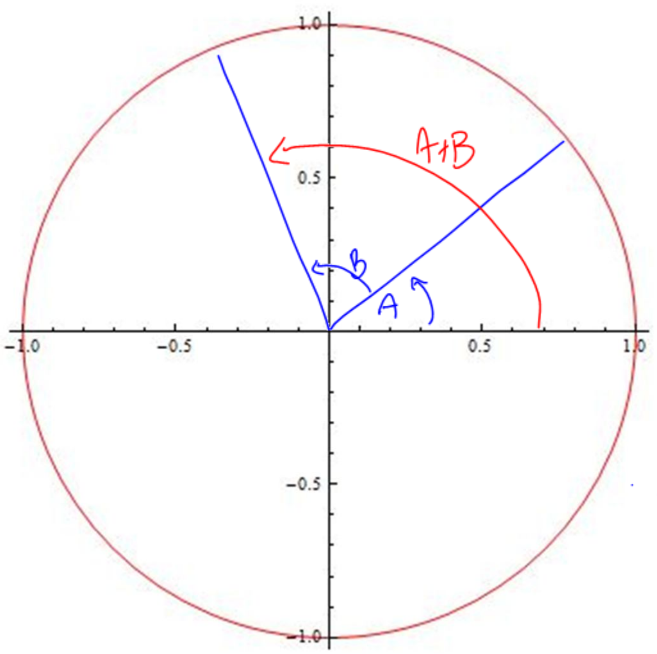
6.2 Compound Angle Formulae

Here we learn to find **exact trig ratios** for **non-special angles**!

$\underbrace{}$ ↳ No calculators.

Consider the picture:



Can we find $\sin(A+B)$ if we know $\sin(A)$ and $\sin(B)$?

or $\cos(A+B)$?

or $\tan(A+B)$?

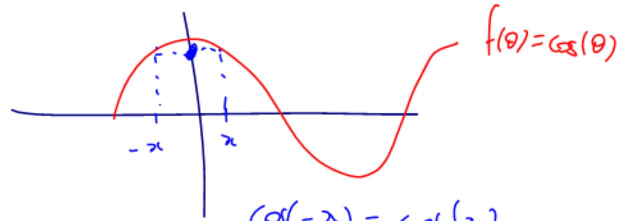
Your text has a nice proof of one of the six compound angle formulas (there are six of them!...see Pg. 394)

Namely your text proves that $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

Using some trig equivalencies (from 6.1) we will find the other 5 compound angle formulae.

Example 6.2.1

Find a formula for $\cos(A+B)$



$\cos(-x)=\cos(x)$

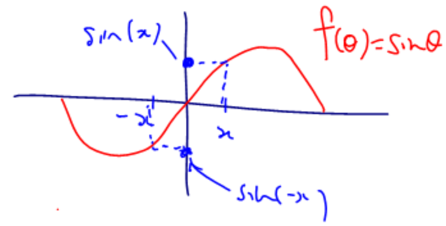
Cosine is even

$$\cos(A+B) = \cos(\underbrace{A}_{\text{1st angle}} - \underbrace{(-B)}_{\text{second angle}})$$

$$= \cos(A)\cos(-B) + \sin(A)\sin(-B)$$

$$= \cos(A)(\cos(B)) + \sin(A)(-\sin(B))$$

$$= \cos(A)\cos(B) - \sin(A)\sin(B)$$



$\sin(-x) = -\sin(x)$

(Sine is odd)

$\therefore$ We have

$$\boxed{\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)}$$

Example 6.2.2

Determine a compound angle formula for $\sin(A+B)$ using **a cofunction identity** and a **cosine compound angle formula**.

$$\sin(A+B) = \cos\left(\frac{\pi}{2} - (A+B)\right) \qquad \cos\left(\frac{\pi}{2} - A - B\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - A\right) - B\right)$$

Recall

$fn(\theta) = cofn\left(\frac{\pi}{2} - \theta\right)$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos(B) + \sin\left(\frac{\pi}{2} - A\right)\sin(B)$$

$$= (\sin(A))\cos(B) + (\cos(A))\sin B$$

(cofns!)

$$\therefore \boxed{\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)}$$

Similarly, we can show (as in ex 6.2.1)

$$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

$$\boxed{\therefore \sin(A\pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)}$$

Example 6.2.3

Using the fact that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ determine a compound angle formula for $\tan(A + B)$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Oh how we wish this was in terms of "tan"!

÷ every term (all 4) by $\cos(A)\cos(B)$

$$= \frac{\sin(A)\cos(B) + \sin(B)\cos(A)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$\frac{\underbrace{\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)}}_{\tan(A)} + \underbrace{\frac{\sin(B)\cos(A)}{\cos(A)\cos(B)}}_{\tan(B)}}{\underbrace{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)}}_{1} - \underbrace{\frac{\sin(A)}{\cos(A)}}_{\tan(A)}\underbrace{\frac{\sin(B)}{\cos(B)}}_{\tan(B)}}$$

$$\therefore \tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\cdot\tan(B)}$$

Similarly, we can show

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\cdot\tan(B)}$$

$$\therefore \boxed{\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}}$$

Example 6.2.4

From your text: Pg. 400 #3acd

Express each given angle as a compound angle using a pair of special triangle angles

a) $75^\circ = 45^\circ + 30^\circ$

c) $-\dfrac{\pi}{6} = \dfrac{\pi}{6} - \dfrac{2\pi}{6} = \dfrac{\pi}{6} - \dfrac{\pi}{3}$ or $\dfrac{2\pi}{6} - \dfrac{3\pi}{6} = \dfrac{\pi}{3} - \dfrac{\pi}{2}$

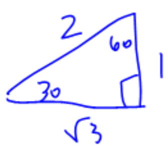
d) $\dfrac{\pi}{12} = \dfrac{4\pi}{12} - \dfrac{3\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$

Example 6.2.5

From your text: Pg. 400 #4ac, 8bd

Determine the **EXACT** value of the trig ratio

$\dfrac{5\pi}{12} = \dfrac{3\pi}{12} + \dfrac{2\pi}{12} = \dfrac{\pi}{4} + \dfrac{\pi}{6}$



a) $\sin(75^\circ)$

$= \sin(45 + 30)$

$= \sin(45)\cos(30) + \sin(30)\cos(45)$

$= \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{2}\right)\left(\dfrac{1}{\sqrt{2}}\right)$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}}$

rationalize the denominator

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{(\sqrt{3}+1)(\sqrt{2})}{2(2)}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$

b) $\tan\left(\dfrac{5\pi}{12}\right)$

$= \tan\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right)$

$= \dfrac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\cdot\tan\left(\frac{\pi}{6}\right)}$

$= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - (1)\left(\frac{1}{\sqrt{3}}\right)}$ Yikes!

$= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$ ← common denom $\sqrt{3}$

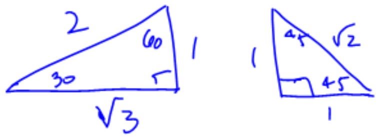
$= \dfrac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \cdot \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$ ↑ conjugate

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= \dfrac{4+2\sqrt{3}}{2}$

$= 2+\sqrt{3}$



#8b) $\tan\left(-15^{\circ}\right)$ d) $\sin\left(\frac{13\pi}{12}\right)$

$= \tan(30 - 45)$

$= \dfrac{\tan(30) - \tan(45)}{1 + \tan(30)\cdot\tan(45)}$

$= \dfrac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}} = \dfrac{\frac{1-\sqrt{3}}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3}+1}{\cancel{\sqrt{3}}}}$

$\Rightarrow = \dfrac{1-\sqrt{3}}{1+\sqrt{3}} \cdot \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$ FOIL.

$a+b$ Conjugate $\Rightarrow$ $a-b$

$= \dfrac{1 - 2\sqrt{3} + 3}{1-3}$

$= \dfrac{4 - 2\sqrt{3}}{-2}$ $(\div -2)$

$= -2 + \sqrt{3}$

Example 6.2.6

From your text: Pg. 401 #9a

If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, where $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le y \le 2\pi$ evaluate $\cos(x+y)$.

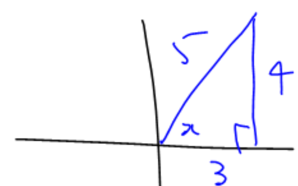
$\sin(x) = \frac{4}{5}$

$\cos(x+y) = \cos(x)\cdot\cos(y) - \sin(x)\cdot\sin(y)$

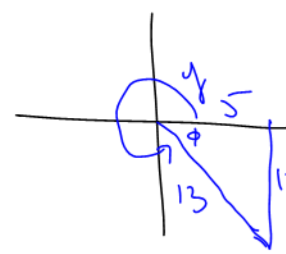
$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$

$= \frac{15}{65} + \frac{48}{65}$

$= \frac{63}{65}$



$\Rightarrow \cos(x) = \frac{3}{5}$



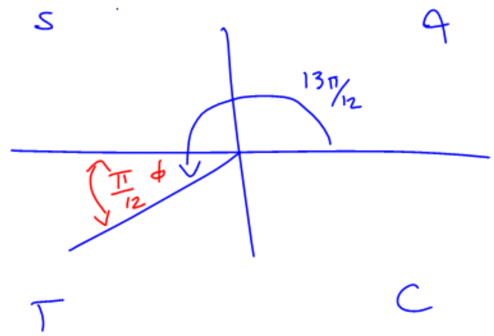
$\cos(y) = \frac{5}{13}$

Class/Homework for Section 6.2

Pg. 400 – 401 #3 – 6, 8 – 10, 13

Ex 6.2.5

e) $\sin\left(\frac{13\pi}{12}\right)$



$= -\sin\left(\frac{\pi}{12}\right)$

$= -\sin\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right)$

$= -\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= -\left(\sin\left(\frac{\pi}{4}\right)\cdot\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)\right)$

$= -\left(\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\right)$

$= -\left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)$

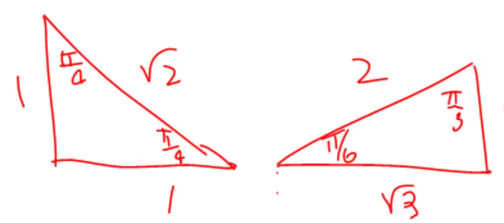
$= -\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\cdot\frac{\sqrt{2}}{\sqrt{2}}$

$= -\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

Notice that $\frac{13\pi}{2}$ cannot be "attained" by adding 2 quad I triangles.

we want these to make life "nice"



$(2\sqrt{2})(\sqrt{2})$
$= 2(\sqrt{4})$
$= 2(2) = 4$