7.4 The Laws of Logarithms

In terms of human understanding, new knowledge can be built from previous knowledge. If this weren't the case, then humans would either have no knowledge at all, or we would prepossess all knowledge without any learning. Now, I know that I personally don't have all knowledge. I'm fairly certain that you do not have all knowledge either. At the same time, I know that I know more today than I did in the past. That is to say, I know that my knowledge has increased. I can only hope that yours has too, at least in terms of mathematics. If it hasn't, then there is only one thing that can be done. Work. Amen and amen.

At this point in our mathematical discussions, we have the knowledge that:

- There are exponent laws
- Logarithms are the inverse of exponentials

We expect, then, that there will be some **Laws of Logarithms** which will **allow us to manipulate logarithms algebraically**. To find (build?) these elusive Log Laws, we will begin by assuming the following to be true:

1) The Exponent Rules
2) Logarithms are the inverses of exponentials
3) If $M = N$, then $\log_a(M) = \log_a(N)$

The Logarithm Laws

1. Evaluate $\log_a(a)$

Let $x = \log_a(a)$

$a^x = a \Rightarrow x = 1$

$\log_a(a) = 1$

2. Evaluate $\log_a(a^b)$

Let $x = \log_a(a^b)$

$\Rightarrow a^x = a^b$

$\Rightarrow x = b$

$\log_a(\underbrace{a^b}) = b$

same base

3. The Product Law of Logarithms

Prove $\log_a(m \cdot n) = \log_a(m) + \log_a(n) \quad a, m, n > 0$

Let $m = a^x \qquad n = a^y$

$\Rightarrow x = \log_a(m) \;|\; y = \log_a(n)$

Consider

$$\log_a(m \cdot n) = \log_a(a^x \cdot a^y)$$

$$= \log_a\left(a^{(x+y)}\right)$$

$$= x + y = \log_a(m) + \log_a(n) \;\square$$

> **Note: Logarithm Laws go backwards too!!!**

4. The Quotient Law of Logarithms

Prove $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n) \quad a, m, n > 0$

Let $m = a^x, \; n = a^y$

$\Rightarrow x = \log_a(m), \; y = \log_a(n)$

Consider $\log_a\left(\frac{m}{n}\right) = \log_a\left(\frac{a^x}{a^y}\right)$

$$= \log_a\left(a^{x-y}\right)$$

$$= x - y$$

$$= \log_a(m) - \log_a(n) \;\square$$

5. The Power Law of Logarithms use the Product Law (over & over)

Prove $\log_a(m^n) = n \cdot \log_a(m) \quad a, m > 0$

Note $\log_a(m^n) \neq (\log_a(m))^{(n)}$ does not move down

Consider $\log_a(m^n)$

$= \log_a(\underbrace{m \cdot m \cdot m \cdot m \cdot \ldots \cdot m}_{\text{"n" m's}})$

$\log_a(m \cdot m) = \log_a(m) + \log_a(m)$

$= \log_a(m) + \log_a(m) + \log_a(m) + \cdots + \log_a(m)$

$= n \cdot \log_a(m)$

Change of Base

Changing base allows one to use a calculator to evaluate numbers like $\log_3 7$ without graphing, or using "guess and check" (unless you happen to possess a cheater calculator like Patrick Clark). Our goal will be to change the "base 3 log" to a "base 10 log".

Procedure

1) Invert the non-base 10 log to an exponential equation.
2) Take the log $(\log_{10})$ of "both sides"
3) Solve using your calculator.

e.g. Evaluate $\log_3 7$

Let $x = \log_3(7)$

$3^x = 7$

$\Rightarrow \log(3^x) = \log(7)$

$\Rightarrow x \cdot \log(3) = \log(7)$

$\therefore x = \dfrac{\log(7)}{\log(3)}$

$= 1.77$

This idea leads to another Log Law: The Change of Base Law

$$\boxed{\text{If } x = \log_a(b) \text{, then } x = \frac{\log(b)}{\log(a)}}$$

Example 7.4.1

From your text: Pg. 475 #6a)

Evaluate $\log_{25}(5^3)$

$$= \log_{25}\left(\left(25^{\frac{1}{2}}\right)^3\right)$$

$$= \log_{25}\left(25^{3/2}\right)$$

$$= \frac{3}{2}\cdot\log_{25}(25) = \frac{3}{2}$$

$= 1$

$$\frac{\log(5^3)}{\log(25)} = \frac{3}{2}$$

Example 7.4.2

Evaluate $\log_3(54) - \frac{1}{2}\log_3(36)$

"$\frac{1}{2}$" in the way!

$$\log_a(M) - \log_a(N) = \log_a\left(\frac{M}{N}\right)$$

$$= \log_3(54) - \log_3\left(36^{\frac{1}{2}}\right)$$

$$= \log_3(54) - \log_3(6)$$

$$= \log_3\left(\frac{54}{6}\right) = \log_3(9) = \log_3(3^2)$$

$$= 2$$

Example 7.4.3

Evaluate $\log_8(2)+3\log_8(2)+\frac{1}{2}\log_8(16)$

$= \log_8(2) + \log_8(2^3) + \log_8(16^{\frac{1}{2}})$

$= \log_8(2 \cdot 2^3 \cdot \overset{16^{\frac{1}{2}}}{4})$

$= \log_8(64) = \log_8(8^2) = 2$

Example 7.4.4

Express as a single logarithm:

a) $\log(3)+\log(9)$

$= \log(27)$

b) $\log_5(6)+\log_5(4)-\log_7(7)$

$= \log_5(\overset{6\cdot 4}{24}) - 1$

$= \log_5(24) - \log_5(5)$

$= \log_5\left(\frac{24}{5}\right)$

Example 7.4.5

Simplify and evaluate:

a) $\log_4(432) - \log_4(27)$

$= \log_4\left(\frac{432}{27}\right)$

$= \log_4(16)$ $\quad 4^2$

$= 2$

b) $\log\left(\sqrt[7]{1000}\right)$

$= \log_{10}\left(\left(10^3\right)^{\frac{1}{7}}\right)$

$= \log_{10}\left(10^{3/7}\right)$

$= 3/7$

Example 7.4.6

Express **IN** **TERMS** **OF** $\log_b(x)$, $\log_b(y)$, and $\log_b(z)$

$$\log_b\left(\frac{z^3y^2}{x^5}\right) = \log_b(z^3) + \log_b(y^2) - \log_b(x^5)$$

$$= 3\log_b(z) + 2\log_b(y) - 5\log_b(x)$$

Example 7.4.7

Solve the equation:

$$\log_3(x)+\log_3(10)=5\log_3(2)+\log_3(5)$$

$\log_a(M) = \log_a(N)$

$\Rightarrow \log_3(x) = \log_3(2^5) + \log_3(5) - \log_3(10) \rightarrow M = N$

$\Rightarrow \log_3(x) = \log_3\left(\frac{2^5 \cdot 5}{10}\right)$

$\rightarrow \log_3(x) = \log_3(16)$

$\therefore x = 16.$

Class/Homework for Section 7.4

Pg. 475 – 476 #2, 4 – 8, 10, 11