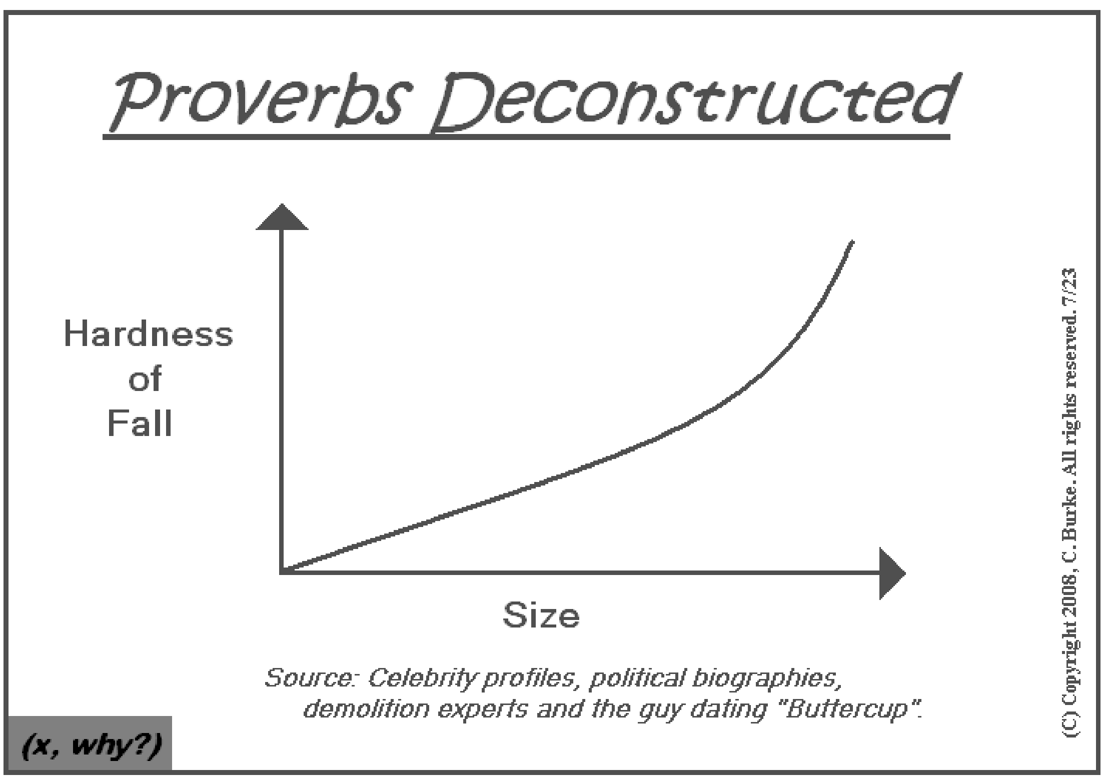
# Functions & Applications 11

## MCF3M

Course Notes

## Chapter 7: Exponential Functions



# Homework

*Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check. Alternatively, you may be handing your homework in more regularly for me to correct.*

Section 7.2 – Page 399
#1-9, 11, 17 (do only some of each. Three is a good number)

Section 7.3 – Page 407
#1-9, 11 (do only some of each)

Section 7.4 – Page 415
#1-3, 6, 7, 9, 10 (do only some of each)

Section 7.6 – Page 429
#1-11 (14 and 15 are interesting too)

Section 7.7 – Page 437
#1, 3, 4, 7-10, 12, 14

# 7.2 The Laws of Exponents

**Learning Goal:** We are learning to simplify expressions using the laws of exponents.

Exponent → 5
$3^5$
Base → 3

When working with exponents, there are three main laws that we can apply to simplify the expressions. These laws only work when the bases are the same.

**Product Law:** when you multiply like bases, add the exponents

$\cancel{3^2 \times 4^3}$ Can't combine

$= 3^2 \times 3^3$

$= (3\times3)\times(3\times3\times3)$

$= 3^5$

$$\boxed{a^m \times a^n = a^{m+n}}$$

**Quotient Law:** when you divide like bases, subtract the exponents

$= \dfrac{3^5}{3^3} = \dfrac{3\times3\times\cancel{3}\times\cancel{3}\times\cancel{3}}{\cancel{3}\times\cancel{3}\times\cancel{3}} = 3^2$

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

**Power Rule:** when raising a power to another power, multiply the exponents.

$= (3^2)^3$

$= (3^2)\times(3^2)\times(3^2)$

$= (3\times3)\times(3\times3)\times(3\times3) = 3^6$

$$\boxed{(a^m)^n = a^{mn}}$$

$(2x^2)^4$

$= 2^4 \cdot (x^2)^4$

$= 16x^8$

**Okay, one more… Zero Rule:**

$$\boxed{a^0 = 1}$$

$3^0 = 1$ $(\text{Anything})^0 = 1$

$1{,}000{,}000^0 = 1$

$(\text{NOAH})^0 = 1$

$5^0 = 1$

$0^0 = 1$

Mathematical Conundrum

DON'T FORGET BEDMAS

$$3^4(3^8)\div 3^7$$

$= 3^{12} \div 3^7$

$\boxed{= 3^5}$

$$\frac{6(6^7)}{(6^3)^2}$$

$= \dfrac{6^8}{6^6}$

$\boxed{= 6^2}$

$$\frac{(5^4)^2(5^5)^2}{5^2(5^{13})}$$

$= \dfrac{(5^8)(5^{10})}{(5^2)(5^{13})}$ *

$= \dfrac{5^{18}}{5^{15}}$

$= 5^3$

$$(4x^6y^3)^3$$

$= 4^3 \cdot (x^6)^3 \cdot (y^3)^3$

$= 64x^{18}y^9$ ✓

**Success Criteria**

- I can use the product, quotient, power, and zero rules to simplify exponential expressions

# 7.3 Working with Integer (Negative) Exponents

**Learning Goal:** We are learning to simplify exponential expressions that contain negative exponents.

What does $4^{-1}$ or $3^{-2}$ mean? Exponents are just repeated multiplication, but how do negatives work?

$7^3 = 7 \times 7 \times 7$

$4^4 = 256$
$4^3 = 64$
$4^2 = 16$
$4^1 = 4$
$4^0 = 1$
$4^{-1} = \frac{1}{4}$
$4^{-2} = \frac{1}{16}$
$4^{-3} = \frac{1}{64}$

×4 ×4 ×4

÷4 ÷4 ÷4 ÷4 ÷4 ÷4

$\frac{1}{4} \div \frac{4}{1} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

$\frac{1}{16} \div \frac{4}{1} = \frac{1}{16} \times \frac{1}{4} = \frac{1}{64}$

A negatie exponent means the reciprocal.

$a^{-1} = \frac{1}{a^1}$

$$\boxed{\begin{array}{c} a^{-m} = \frac{1}{a^m} \\ \frac{1}{a^{-m}} = a^m \end{array}}$$

$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$\left(\frac{a}{b}\right)^{-m} = \frac{a^{-m}}{b^{-m}} = \frac{b^m}{a^m}$

Further notes on Zero exponent

$\frac{7^2}{7^2} = \frac{49}{49} = 1$ $\frac{7^2}{7^2} = 7^{2-2} = 7^0 = 1$

must be the same.

Examples:

neg means reciprocal, so flip it!

1. Rewrite the expression with a positive exponent: $\left(\frac{3}{5}\right)^{-3} = \left(\frac{5}{3}\right)^{3}$

2. Simplify. Write expression with a single positive power:

$$\underbrace{2^{-3}(2^{5})}\div 2^{4}$$

$$= 2^{2} \div 2^{4} \quad = 2^{2-4} \quad = 2^{-2} \quad = \left(\frac{1}{2}\right)^{2} \quad \boxed{= \frac{1}{2^{2}}}$$

negative means flip it!

3. Simplify. Write expression with a single positive power:

$$\left(\frac{(3^{4})(3^{-3})^{2}}{3(3^{-7})}\right)^{-3} = \left[\frac{(3^{4})\cancel{(3^{-6})}}{\cancel{(3^{-6})}}\right]^{-3} = \left[\frac{(3^{-2})}{(3^{-6})}\right]^{-3} = \left[3^{(-2)-(-6)}\right]^{-3}$$

$$= (3^{4})^{-3} \quad = (3^{4})^{-3}$$

$$= 3^{-12} = \frac{1}{3^{12}}$$

4. Evaluate. Leave answers as fractions or integers:

work it out!

$$4^{-1} + 3^{0} - \left(\frac{2}{3}\right)^{-2}$$

$$= \left(\frac{1}{4}\right)^{1} + 1 - \left(\frac{3}{2}\right)^{2} \quad = \frac{1}{4} + 1\tfrac{\times 4}{\times 4} - \frac{9}{4}$$

$$= \frac{1}{4} + \frac{4}{4} - \frac{9}{4} \quad = \frac{-4}{4} \quad \boxed{= -1}$$

**Success Criteria:**

- I can change a negative exponent into a positive one by writing it as a reciprocal
- I can use the negative exponent rule, along with the other exponent rules to simplify expressions containing exponents

# 7.4 Working with Rational Exponents

**Learning Goal:** We are learning to simplify expressions that contain rational exponents.

First a refresher on fractions:

1. When you multiply fractions, multiply the numerators and multiply the denominators.

$$\frac{3}{5}\times\frac{2}{3} = \frac{6^{\div 3}}{15_{\div 3}} = \frac{2}{5}$$

$$\frac{\cancel{3}^{1}}{5}\times\frac{2}{\cancel{3}} = \frac{2}{5}$$

2. When adding or subtracting fractions, you first need a common denominator, then you add or subtract the numerators.

$$\frac{3^{\times 3}}{5_{\times 3}}+\frac{2^{\times 5}}{3_{\times 5}} = \frac{9}{15}+\frac{10}{15} = \frac{19}{15}$$

only can cancel when fractions are multiplied

3. When dividing fractions, flip the second fraction and multiply. "Kiss n' Flip"

$$\frac{3}{5}\div\frac{2}{3} = \frac{3}{5}\times\frac{3}{2} = \frac{9}{10}$$

Fractions are, unfortunately for some, a part of exponents. What do fractions in an exponent mean?

$4^{\frac{1}{2}} = ?$

normal exponent

a "root"

So, $4^{\frac{1}{2}} = \sqrt{4^1} = \sqrt{4}$

$4^{\frac{1}{3}} = \sqrt[3]{4^1}$

$\sqrt[3]{27} = 3$, $\therefore 27^{\frac{1}{3}} = 3$

$4^{\frac{1}{4}} = \sqrt[4]{4}$

More Complicated Examples

$27^{\frac{4}{3}} = \left(\sqrt[3]{27}\right)^4 = (3)^4 = 81$

OR

$= \sqrt[3]{27^4} = \sqrt[3]{531,441} = 81$

$$\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ OR } \left(\sqrt[n]{a}\right)^m}$$

When working with rational exponents, all of the other exponent rules remain true. Use your techniques for combining fractions to simplify them.

what about $32^{0.2} = 32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

Examples which relate to your textbook questions:

1. $36^{\frac{1}{2}} = \sqrt{36} = 6$

2. $\sqrt[4]{2401} = 7$

OR

$2401^{\frac{1}{4}} = 7$

3. $\sqrt[6]{236} = 2.4869\ldots$

OR

$236^{\frac{1}{6}} = 2.4869$

Write as a single term

6. $5^{\frac{1}{2}} \times 5^{3} \div 5^{\frac{3}{2}} = 5^{\frac{1}{2} + \frac{6}{2} - \frac{3}{2}} = 5^{\frac{4}{2}} = \boxed{5^2} = 25$

7. $\left(3^{\frac{2}{3}}\right)^{\frac{6}{5}} = 3^{\frac{2}{3} \cdot \frac{6}{5}} = 3^{\frac{12}{15}} = \boxed{3^{\frac{4}{5}}} \doteq 2.40822\ldots$

9. $\frac{\sqrt{147}}{\sqrt{3}} = \frac{12.12435}{1.732} = 7.00020 \quad \boxed{=7}$

10. $\frac{\left(8^{\frac{1}{6}}\right)^7}{8^{\frac{1}{2}}8^{\frac{1}{3}}} = \frac{8^{\frac{1}{6} \cdot \frac{7}{1}}}{8^{\frac{1}{2} + \frac{1}{3}}} = \frac{8^{\frac{7}{6}}}{8^{\frac{5}{6}}} = 8^{\frac{7}{6} - \frac{5}{6}} = 8^{\frac{2}{6}} = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

$\frac{3}{6} + \frac{2}{6}$

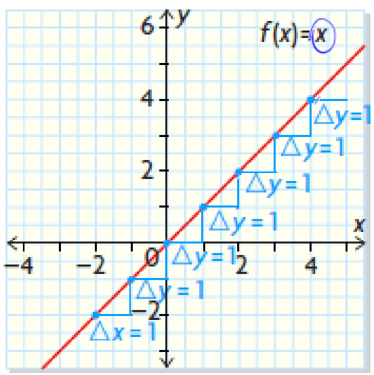
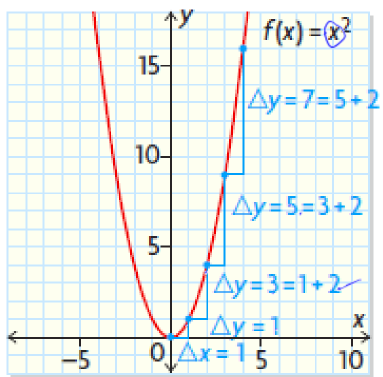
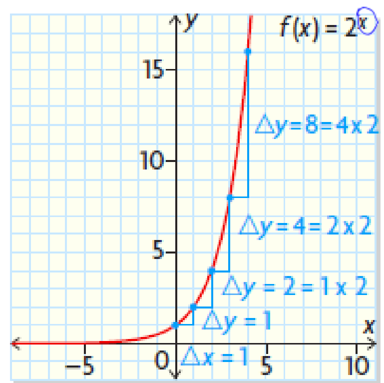
**Success Criteria:**

- I can understand that the numerator of a fractional exponent is the power, while the denominator is the root
- I can apply the exponent rules, if they contain fractional exponents, to simplify expressions

# 7.6 Solving Problems with Exponential Growth

**Learning Goal:** We are learning to solve problems that are modelled by exponential growth.

Exponential growth is when something grows at, well, an exponential rate.

| Linear | Quadratic | Exponential |
|---|---|---|
| Linear functions have constant first differences. | Quadratic functions have first differences that are related by an addition pattern. As a result, their second differences are constant. | Exponential functions have first differences that are related by a multiplication pattern. As a result, their second differences are not constant. |
|  |  |  |

Must be the same units. "per year" "per month" "per day"

An exponential growth function is defined as:

$$P(n)=P_0(1+r)^n$$

$n$ → growth periods

$r$ → rate, written as a decimal

turn %'s into decimals: $\frac{\%}{100}$

8% becomes 0.08 → $\frac{8}{100}$

$= (1+1)^{24}$

$167,772.16 = 0.01(2)^{24}$

$1$ → One

$P_0$ → Initial number/amount

$P(n)$ → Final number/Amount

Example 1: A population of frogs is estimated to be 320. This type of frog population grows at a rate of 15% per year. How many frogs will there be in 5 years?

$P_{(n)} = ?$

$P_0 = 320$

$r = 15\% = \frac{15}{100} = 0.15$

$n = 5$

$P_n = P_0(1+r)^n$

$P_{(5)} = 320(1+0.15)^5$

$P_{(5)} = 320(1.15)^5$

$P_{(5)} = 643.63$

$\therefore$ In 5 years, there will be 644 frogs.

Example 2: In 2005, a town had a population of 50,000 people. If the town has a population growth rate of 5%:

a) What will the population be in 2015?

$P_0 = 50{,}000$

$r = 0.05$

$n = (2015 - 2005)$

$\boxed{n = 10}$

$P_{(n)} = P_0(1+r)^n$

$P_{(10)} = 50{,}000(1.05)^{10}$

$P_{(10)} = 81{,}444.7$

$\therefore$ In 10 years, there will be 81,445 people.

b) When is the population estimated to reach 150,000?

$P_0 = 50{,}000$

$P_{(n)} = 150{,}000$

$\boxed{n = ?}$

$r = 0.05$

Get rid of!

$\frac{150{,}000}{50000} = \frac{50{,}000(1.05)^n}{50{,}000}$

$3 = (1.05)^n$

Guess & Check

$n = 21, \quad (1.05)^{21} = 2.78$

$n = 25, \quad (1.05)^{25} = 3.38$

$\boxed{n = 23, \quad (1.05)^{23} = 3.07}$

$n = 22, \quad (1.05)^{22} = 2.93$

$\therefore$ In 23 years, or in 2028, the population reaches 150,000.

**Success Criteria:**

- I can recognize the difference between linear, quadratic, and exponential functions
- I can use the exponential growth function to model and solve problems that involve exponential growth

# 7.7 Problems Involving Exponential Decay

$3\% \rightarrow \frac{3}{100} = 0.03$

**Learning Goal:** We are learning to solve problems that are modelled by exponential decay.

Decay works very similar to growth, but instead of the "population" increasing, it is decreasing.

The decay function is: $P(n) = P_0(1-r)^n$

Ending Amount; rate. write as a decimal.; Amount of time that passes; Initial Amount

r + n have to be the same units.

Everything is the same as the growth function, but this time you **subtract the rate.**

Example 1: A new car depreciates at a rate of 20% per year. If Melanie bought a new car for $25,500, how much is her car worth in 4 years?

$r = 20\% = \frac{20}{100} = \boxed{0.2}$

$P_0 = 25{,}500$

$n = 4$

$P(n) = P_0(1-r)^n$

$P(4) = 25{,}500(1-0.2)^4$

$P(4) = 25{,}500(0.8)^4$

$P(4) = \$10{,}444.80$

Example 2: The half-life of caffeine is approximately 5 hours. A cup of coffee can have 150 mg of caffeine. If you drink a cup of coffee at 9 am, how much caffeine is in your body at:

n = 1 (5 hrs)
n = 2 (10 hrs)
n = 3 (15 hrs)

a) noon = 12 pm (3 hrs)

$n = \frac{\text{time}}{\text{Period}}$ (1 complete cycle)

$= \frac{3}{5}$

$\boxed{n = 0.6}$

$P_0 = 150$ mg

r = lose 50%

r = 0.5

$P(0.6) = 150(1-0.5)^{0.6}$

$P(0.6) = 150(0.5)^{0.6}$

= 99 mg

b) 8 pm → 11 hrs passed

$\therefore n = \frac{11}{5} = \boxed{2.2}$

$P(2.2) = 150(0.5)^{2.2}$

= 33 mg

c) midnight

→ 15 hrs

$n = \frac{15}{5} = 3$

$P(3) = 150(0.5)^3$

= 18.75 mg

= 19 mg

**Success Criteria:**

- I can use the exponential decay function to model and solve problems that involve exponential decay