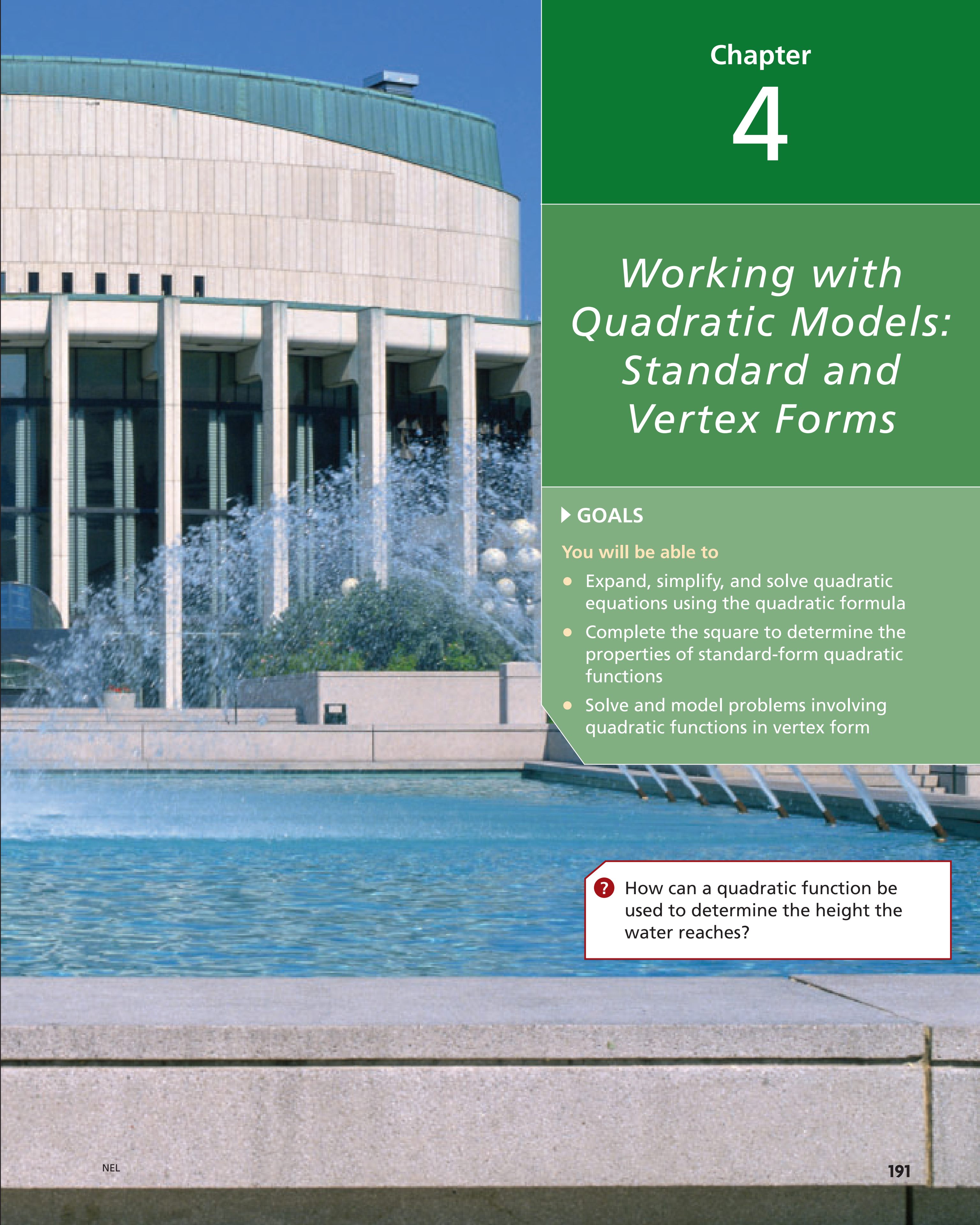


Chapter

4

Working with Quadratic Models: Standard and Vertex Forms

GOALS

You will be able to

- Expand, simplify, and solve quadratic equations using the quadratic formula
- Complete the square to determine the properties of standard-form quadratic functions
- Solve and model problems involving quadratic functions in vertex form

? How can a quadratic function be used to determine the height the water reaches?

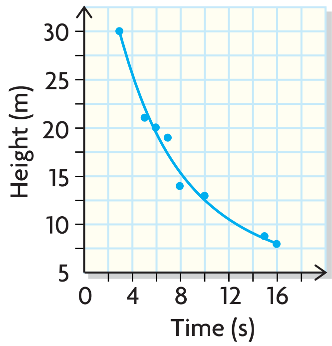
4 Getting Started

WORDS You Need to Know

1. Match the term with the picture or example that best illustrates its definition.

a) quadratic equation
b) solutions to a quadratic equation
c) quadratic function
d) perfect square trinomial
e) curve of good fit
f) vertex

i)



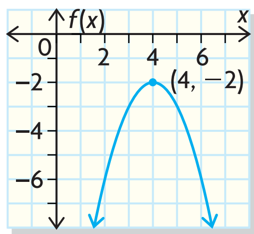
ii) $x^2 + 10x + 25$

iii) $x^2 - 5x - 24 = 0$
$(x - 8)(x + 3) = 0$
$x = 8$ and $x = -3$

iv) $-12x^2 - 31x + 27 = 0$

v) $f(x) = x^2 - 3x + 9$

vi)



SKILLS AND CONCEPTS You Need

Study ***Aid***
For help, see Essential Skills Appendix, A-8.

Solving Quadratic Equations by Graphing and Factoring

If the value of y is known for the function $y = ax^2 + bx + c$, then the corresponding values of x can be determined either graphically or algebraically.

EXAMPLE

Solve $x^2 - 2x - 15 = 0$ by a) factoring and b) graphing.

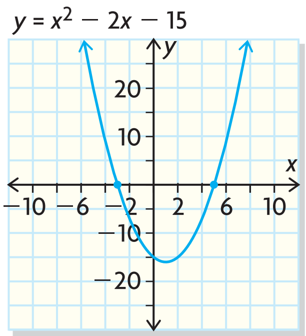
Solution

a) Factoring

$x^2 - 2x - 15 = 0$
$(x - 5)(x + 3) = 0$
$(x - 5) = 0$ and $(x + 3) = 0$
$x = 5$ and $x = -3$

Factor the trinomial.
Set each factor equal to zero.
Solve.

b) Graphing



The x-intercepts, or zeros, of the graph are the solutions to the equation. In this case, the zeros are $x = -3$ and $x = 5$.

2. Solve by factoring. Confirm your results by graphing.

a) $x^2 + 7x - 30 = 0$

b) $x^2 + 8x + 15 = 0$

c) $x^2 - x - 6 = 0$

d) $x^2 - 5x + 6 = 0$

Identifying and Factoring Perfect Square Trinomials

A trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect square. It has two identical factors of the form $(a + b)^2$ or $(a - b)^2$, respectively.

Study *Aid*

For help, see Essential Skills Appendix, A-10.

EXAMPLE

Factor the following perfect square trinomials.

a) $x^2 - 10x + 25$

b) $16x^2 + 24x + 9$

Solution

a) $a = 1x, b = 5$

$= (x - 5)(x - 5)$

or $(x - 5)^2$

b) $a = 4x, b = 3$

$= (4x + 3)(4x + 3)$

or $(4x + 3)^2$

3. Factor the following perfect square trinomials.

a) $x^2 + 6x + 9$

b) $x^2 - 8x + 16$

c) $9x^2 + 6x + 1$

d) $4x^2 - 12x + 9$

PRACTICE

Study Aid

For help, see Essential Skills Appendix.

Question	Appendix
4	A-9
5, 7	A-10
9	A-8

4. Simplify.

a) $(2x + 5 - 6x^2) + (5x - 9x^2 + 3)$
b) $(4x^2 - 3x + 7) - (2x^2 - 3x + 1)$
c) $(3x + 5)(4x - 7)$
d) $(2x - 1)(2x + 1)$

5. Factor.

a) $x^2 + 3x - 40$
b) $6x^2 + 5x - 6$
c) $81x^2 - 49$
d) $9x^2 + 6x + 1$

6. What term will make each expression a perfect trinomial square?

a) $x^2 + 6x + \square$
b) $x^2 + \square + 25$
c) $4x^2 + \square + 49$
d) $9x^2 - 24x + \square$

7. Write in factored form.

a) $f(x) = x^2 - 7x - 18$
b) $g(x) = -2x^2 + 17x - 8$
c) $h(x) = 4x^2 - 25$
d) $y = 6x^2 + 13x - 5$

8. Determine the vertex of each quadratic function, and state the domain and range of each.

a) $y = x^2 + 6x + 5$
b) $f(x) = 2x^2 - 5x - 12$
c) $g(x) = -6x^2 - 7x + 3$
d) $h(x) = -3x^2 + 9x + 30$

9. Sketch the graph of each function by hand. Start with $y = x^2$ and use the appropriate transformations.

a) $f(x) = x^2 - 5$
b) $y = (x - 2)^2 + 1$
c) $f(x) = 2(x + 3)^2 - 4$
d) $y = -\frac{1}{2}(x - 4)^2 + 2$

10. The height of a diver above the water is given by the quadratic function $h(t) = -5t^2 + 5t + 10$, with t in seconds and $h(t)$ in metres. When will the diver reach the maximum height?

11. Use what you know about quadratic equations to complete the chart.

Essential characteristics:	Non-essential characteristics:
Quadratic Equation	
Examples:	Non-examples:

APPLYING What You Know

Cutting the Cake

Kommy wants to share his birthday cake with as many people as possible.

YOU WILL NEED

- graph paper or graphing calculator
- ruler

? If Kommy makes 12 cuts in the cake, what is the maximum number of people he can serve?

A. Copy and complete the table by drawing a diagram for each case.

Number of Cuts	0	1	2	3	4	5	6
Diagram of Cake with Cuts							
Maximum Number of Pieces of Cake							

B. Compare your table with that of other classmates. Do you have the same answers? Discuss and make any necessary changes.

C. How should the cake be cut to get the maximum number of pieces?

D. Graph the data from your table, with number of cuts along the horizontal axis and pieces of cake along the vertical axis. What relationship is there between the number of cuts made and the maximum number of pieces of cake? Explain how you know.

E. With a partner, repeat part A with a circular cake and then a triangular cake. (You do one type of cake, your partner another.) Does the shape of the cake affect the relationship between the number of cuts and the maximum number of pieces? Explain.

F. If Kommy makes 12 cuts in the cake, what is the maximum number of people he can serve? Explain your reasoning.

4.1 The Vertex Form of a Quadratic Function

YOU WILL NEED

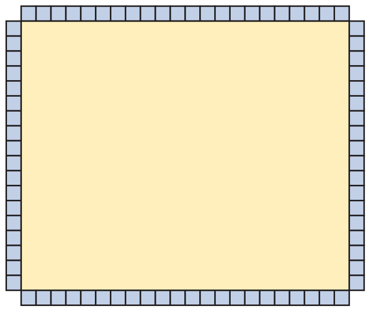
- graph paper
- graphing calculator

GOAL

Compare the standard and vertex forms of a quadratic function.

LEARN ABOUT the Math

The school environment club will use 80 square concrete slabs 1 m in length to surround a rectangular garden. The largest possible area must be enclosed.



The math team has suggested two quadratic functions to model the enclosed area:

$$f(w) = -w^2 + 40w \quad \text{and}$$

$$g(w) = -(w - 20)^2 + 400$$

Here, w is the width in metres, and $f(w)$ and $g(w)$ are the areas in square metres.

? Which function should the environment club use?

EXAMPLE 1 Connecting functions in standard and vertex forms

Compare the two functions suggested by the math team.

Kirsten's Solution: Using Algebra

$$g(w) = -(w - 20)^2 + 400$$

If the functions have the same form, it's easier to compare them. I expanded $g(w)$ to get it into standard form, like $f(w)$.

$$g(w) = -(w - 20)(w - 20) + 400$$

$$g(w) = -(w^2 - 20w - 20w + 400) + 400$$

$$g(w) = -(w^2 - 40w + 400) + 400$$

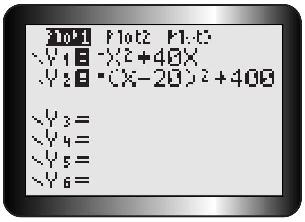
$$g(w) = -w^2 + 40w - 400 + 400$$

$$g(w) = -w^2 + 40w$$

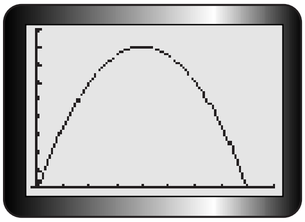
This is the same as $f(w)$.

The environment club can use either equation.

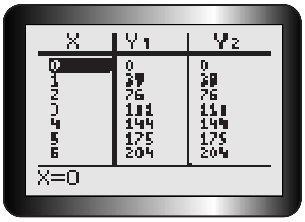
Marc's Solution: Using a Table of Values and Graphing



I entered the functions $f(w) = -w^2 + 40w$ and $g(w) = -(w - 20)^2 + 400$ in Y1 and Y2, changing the w to an x.



When I graphed the functions, it looked like there was only one graph. So either the graphs are the same, or the second graph does not appear in the window I'm using.



I used the table feature to look at the table of values for each function and compared them.

Each number for the independent variable x gives the same value for the dependent variable y for both functions. So the functions are the same.

The environment club can use $f(w)$ or $g(w)$.

The vertex form of a quadratic function provides useful information about some of the characteristics of its graph.

EXAMPLE 2 Connecting information about the parabola to the vertex form of the function

What is the maximum area of a garden defined by $f(w) = -w^2 + 40w$? How does this relate to the function $g(w) = -(w - 20)^2 + 400$?

Kirsten's Solution: Using Algebra

$$f(w) = -w^2 + 40w$$

$f(w)$ is a quadratic function, and the leading coefficient is less than zero. The graph is a parabola that opens down. I factored the function. The two zeros occur when $w = 0$ or $w = 40$, and the maximum will be halfway between them.

$$f(w) = -w(w - 40)$$

zeros are $w = 0$ and $w = 40$

x-coordinate of vertex

$$x = \frac{0 + 40}{2}$$

$$= 20$$

I used the zeros to find the axis of symmetry by adding them and dividing by 2. This corresponds to the x-coordinate of the vertex.

$$f(20) = -(20)^2 + 40(20)$$

$$= -400 + 800$$

$$= 400$$

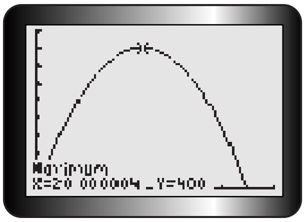
I substituted 20 into $f(w)$ to get the y-coordinate, which is the maximum value.

The maximum area is 400 m^2 when the width is 20 m. Both numbers appear in the function $g(w)$, which is written in vertex form.

vertex form

a quadratic function in the form $f(x) = a(x - h)^2 + k$, where the vertex is (h, k)

Marc's Solution: Using a Graph



Since the graph of the function $f(w)$ is a parabola that opens down, the maximum area corresponds to the y-coordinate of the vertex.

I used the maximum operation on my graphing calculator to get the vertex. The maximum point is (20, 400).

The maximum occurs at width 20 m and the area is 400 m^2. Both numbers appear in $g(w)$.

Reflecting

A. How can a quadratic function in vertex form be written in standard form?

B. What is the significance of a when the quadratic function is written in vertex form? Does a change when you change from vertex to standard form?

C. Can you always easily identify the zeros from vertex form? Explain.

D. What is the significance of h and k in vertex form?

APPLY the Math

EXAMPLE 3 Identifying features of the parabola from a quadratic function in vertex form

a) Determine the direction of opening, the axis of symmetry, the minimum value, and the vertex of the quadratic function $f(x) = 3(x - 5)^2 + 7$.
b) Without using graphing technology, use the information you determined to sketch a graph of $f(x)$.
c) State the domain and range of $f(x)$.

Steve's Solution

a) $f(x) = a(x - h)^2 + k$

$f(x) = 3(x - 5)^2 + 7$

The function is written in vertex form. Since $a = 3$, a positive number, the parabola opens up.

opens up

$x = 5$

The equation of the axis of symmetry equals the value of *h* because a parabola is symmetric about the vertical line that passes through its vertex.

minimum of 7

The vertex is (5, 7).

The parabola opens up, so the minimum value of 7 is at the vertex, when $x = 5$.

b) When $x = 0$,

$y = 3(0 - 5)^2 + 7$

$y = 3(-5)^2 + 7$

$y = 3(25) + 7$

$y = 75 + 7$

$y = 82$

I need some other points to graph the parabola. I substituted $x = 0$ into the quadratic function and solved to find the *y*-intercept.

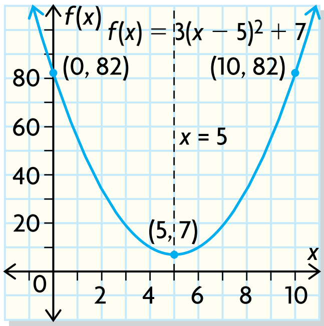
A point on the curve is (0, 82).

Another *x*-coordinate that has a *y*-value of 82 is $x = 5 + 5$.

$x = 10$

Another point on the curve is (10, 82).

I know that there's another point at $(x, 82)$ on the opposite branch of the parabola. Since 0 is 5 units to the left of the axis of symmetry, $x = 5$, another *x*-value that will have 82 as its *y*-coordinate must be 5 units to the right of the axis of symmetry.



I plotted the vertex and my two points and connected them with a parabola.

c) domain: $\{x \in \mathbf{R}\}$

The function represents a parabola and is defined for any real number.

range: $\{y \in \mathbf{R} \mid y \geq 7\}$

The range depends on where the vertex is and which way the parabola opens. Since the vertex is at (5, 7) and the parabola opens up, the range is all values of *y* greater than or equal to 7.

EXAMPLE 4 Selecting a strategy to determine the zeros from a quadratic function in vertex form

Determine the zeros of the function $f(x) = (x - 5)^2 - 36$.

Marc's Solution: Expanding and Factoring

$$(x - 5)^2 - 36 = 0$$

To find the zeros, I let *y* or *f*(*x*) equal zero.

$$(x - 5)(x - 5) - 36 = 0$$

I wrote the squared term as $(x - 5)(x - 5)$ to help me expand.

$$x^2 - 5x - 5x + 25 - 36 = 0$$

I multiplied the binomials and collected like terms.

$$x^2 - 10x - 11 = 0$$

$$(x - 11)(x + 1) = 0$$

I factored by finding two numbers that multiply to −11 and add to −10. They are −11 and 1.

$$x - 11 = 0 \quad \text{and} \quad x + 1 = 0$$

$$x = 11 \quad \text{and} \quad x = -1$$

I set each factor equal to zero and solved.

are the zeros of this function.

Beth's Solution: Using Inverse Operations

$$(x - 5)^2 - 36 = 0$$
$$(x - 5)^2 - 36 + 36 = 0 + 36$$
$$(x - 5)^2 = 36$$

To get the zeros, I set the function equal to zero.

I used inverse operations to isolate the term containing *x*.

$$(x - 5) = \pm\sqrt{36}$$

I took the square root of both sides because that's the inverse of squaring. I remembered that the square root can be positive or negative.

$$x - 5 = \pm 6$$

I took the square root of 36.

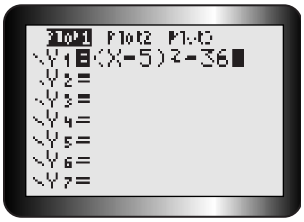
$$x - 5 + 5 = \pm 6 + 5$$
$$x = 5 \pm 6$$
$$x = 5 + 6 \quad \text{and} \quad x = 5 - 6$$
$$x = 11 \quad \text{and} \quad x = -1$$

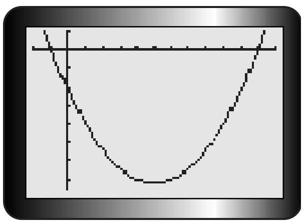
are the zeros of $f(x)$.

I isolated *x* by adding 5 to both sides.

I split up the expression and solved for *x* in each.

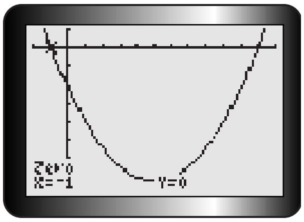
Rachel's Solution: Using a Graphing Calculator

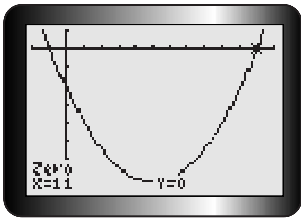




I entered the function in Y1.

I adjusted the window to see both zeros.





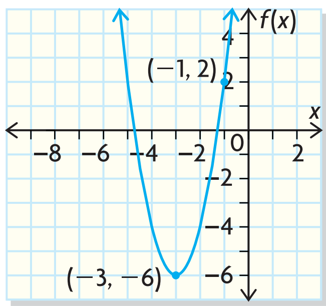
I used the zero operation twice to find the zeros of the function.

The zeros are at $x = -1$ and $x = 11$.

If you are given the graph of a quadratic function and you know the coordinates of the vertex and another point on the parabola, you can use the vertex form to determine the function's equation.

EXAMPLE 5 Using the vertex form to write the equation of a quadratic function from its graph

Determine the equation in vertex form of the quadratic function shown.



Veena's Solution

$$f(x) = a(x - h)^2 + k$$
$$f(x) = a(x - (-3))^2 + (-6)$$
$$f(x) = a(x + 3)^2 - 6$$

The vertex is $(-3, -6)$, so I replaced h with -3 and k with -6.

$$2 = a(-1 + 3)^2 - 6$$
$$2 = a(2)^2 - 6$$
$$2 = 4a - 6$$
$$2 + 6 = 4a$$
$$8 = 4a$$
$$2 = a$$

To get the value of a, I needed another point on the parabola. I used $(-1, 2)$. I replaced $f(x)$ with 2 and x with –1 and solved for a.

$$f(x) = 2(x + 3)^2 - 6$$

I rewrote the quadratic function in vertex form with my values for a, h, and k.

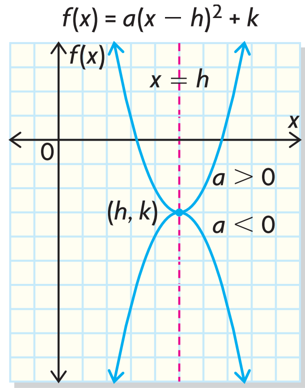
In Summary

Key Idea

- A quadratic function in vertex form, $f(x) = a(x - h)^2 + k$, can be expressed in standard form, $f(x) = ax^2 + bx + c$, by expanding and simplifying. The two forms are equivalent.

Need to Know

- If a quadratic function is expressed in vertex form, $f(x) = a(x - h)^2 + k$, then
  - the vertex is located at (h, k)
  - the equation of the axis of symmetry is $x = h$
  - the function has a maximum value of k when $a < 0$
  - the function has a minimum value of k when $a > 0$



CHECK Your Understanding

1. For each function, state the vertex and whether the function has a maximum or minimum value. Explain how you decided.

a) $f(x) = 2(x - 3)^2 - 5$

b) $f(x) = -3(x - 5)^2 - 1$

c) $f(x) = -(x + 1)^2 + 6$

d) $f(x) = (x + 5)^2 - 3$

2. For each function in question 1, identify the equation of the axis of symmetry, and determine the domain and range.

3. Graph and then write in standard form.

a) $f(x) = 2(x - 3)^2 + 1$

b) $f(x) = -(x + 1)^2 - 3$

PRACTISING

4. Complete the table.
K

	Function	Vertex	Axis of Symmetry	Opens Up/Down	Range	Sketch
a)	$f(x) = (x - 3)^2 + 1$					
b)	$f(x) = -(x + 1)^2 - 5$					
c)	$y = 4(x + 2)^2 - 3$					
d)	$y = -3(x + 5)^2 + 2$					
e)	$f(x) = -2(x - 4)^2 + 1$					
f)	$y = \frac{1}{2}(x - 4)^2 + 3$					

Tech *Support*

For help determining the zeros using a graphing calculator, see Technical Appendix, B-8.

5. Use a graphing calculator to determine the zeros for each function.

a) $f(x) = (x - 3)^2 - 121$
b) $g(x) = 2(x + 5)^2 - 98$
c) $f(x) = 3(x - 1)^2$
d) $g(x) = (x + 3)^2 - 15$
e) $f(x) = -2(x - 4)^2 + 29$
f) $g(x) = -3(x - 7)^2 - 121$

6. The same quadratic function $f(x)$ can be expressed in three different forms:

$$f(x) = (x - 7)^2 - 25$$
$$f(x) = x^2 - 14x + 24$$
$$f(x) = (x - 12)(x - 2)$$

What information about the parabola does each form provide?

7. The height above the ground of a bungee jumper is modelled by the quadratic function $h(t) = -5(t - 0.3)^2 + 110$, where height, $h(t)$, is in metres and time, t, is in seconds.
A

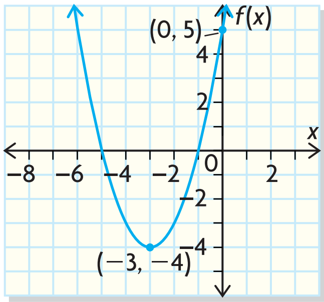
a) When does the bungee jumper reach maximum height? Why is it a maximum?
b) What is the maximum height reached by the jumper?
c) Determine the height of the platform from which the bungee jumper jumps.

8. Write the equation of the quadratic function, first in vertex form and then in standard form.

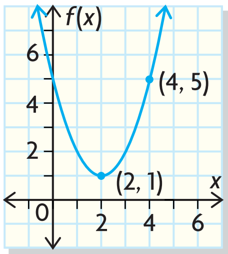
a) vertex $(-4, 8)$ and passing through $(2, -4)$
b) vertex $(3, 5)$ and passing through $(1, 1)$
c) vertex $(1, -7)$ and passing through $(-2, 29)$
d) vertex $(-6, -5)$ and passing through $(-3, 4)$

9. For each graph, write the quadratic equation in vertex form.

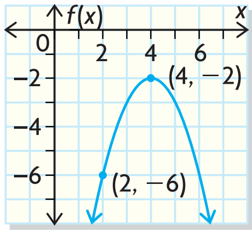
a)



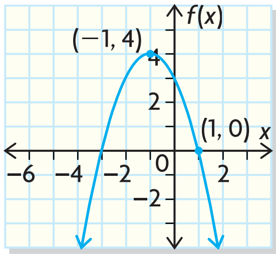
b)



c)



d)



10. The path of a ball is modelled by the quadratic function $h(t) = -5(t - 2)^2 + 23$, where height, $h(t)$, is in metres and time, t, is in seconds.

a) What is the maximum height the ball reaches?

b) When does it reach the maximum height?

c) When will the ball reach a height of 18 m?

11. For each function, determine the vertex and two points that satisfy the equation. Use the information to sketch the graph of each.

a) $y = (x - 4)^2$ **c)** $y = 2(x - 1)^2 - 3$

b) $y = x^2 + 4$ **d)** $y = -2(x + 3)^2 + 5$

12. T A quadratic function has zeros at 1 and -3 and passes through the point (2, 10). Write the equation in vertex form.

13. An equation is given in vertex form. Explain how to write it in standard form. Illustrate with an example.

14. C What information about the graph do you know immediately when a quadratic function is written in

a) standard form?

b) vertex form?

Extending

15. The table at the right shows the number of cigarettes sold from 1994 to 2005. Determine the equation of a curve of good fit, in vertex form, that can be used to model the data.

Year	Cigarettes (millions of units)
1994	18.2
1995	18.9
1996	19.5
1997	18.6
1998	19.0
1999	18.8
2000	18.2
2001	17.5
2002	16.3
2003	15.7
2004	14.4
2005	13.2

16. Write the function $f(x) = (x - 7)(x + 5)$ in vertex form.

4.2 Relating the Standard and Vertex Forms: Completing the Square

YOU WILL NEED

- graphing calculator

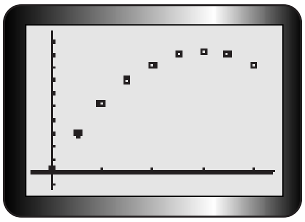
GOAL

Write a quadratic function, given in standard form, in vertex form.

LEARN ABOUT the Math

A model rocket is shot from a platform 1 m above the ground. The height of the rocket above the ground is recorded in the table.

Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (m)	1.00	14.75	26.00	34.75	41.00	44.75	46.00	44.75	41.00



Sally and Jake used a graphing calculator to create a scatter plot from the data.

They decided that a quadratic function models the data. They used different methods to determine the function and got different results. The method each used, along with their reasoning, is shown.

Sally's Solution: Using a Table of Values and a Graph

$$y = a(x - h)^2 + k$$
$$y = a(x - 3)^2 + 46$$

From the table of values and my graph, it looks like the vertex is (3, 46), so I replaced *h* with 3 and *k* with 46.

$$1 = a(0 - 3)^2 + 46$$
$$1 = a(-3)^2 + 46$$
$$1 = 9a + 46$$
$$1 - 46 = 9a$$
$$-5 = a$$

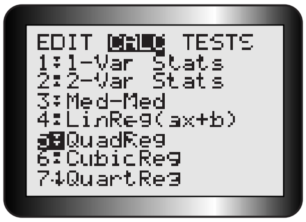
To find the value of *a*, I picked the point (0, 1) and substituted 0 for *x* and 1 for *y*. Then I solved for *a*.

$$y = -5(x - 3)^2 + 46$$

I replaced *a* with −5 and wrote the equation.

This equation gives the height of the rocket during its flight.

Jake's Solution: Using Quadratic Regression



Since I already had the data in my calculator, I used the quadratic regression operation.

The calculator gave me the coefficient of x^2, -5, and the coefficient of x, 30, as well as the constant 1.

$y = -5x^2 + 30x + 1$

This is the equation that gives the height of the rocket during its flight.

Tech Support

For help using the quadratic regression operation on a graphing calculator, see Technical Appendix, B-10.

? Do both equations represent the same function?

EXAMPLE 1 Comparing the two equations

Tim's Solution

$h(x) = -5(x - 3)^2 + 46$

$h(x) = -5(x - 3)(x - 3) + 46$

If I expand Sally's equation and get Jake's equation, then I know they're equal.

I rewrote the squared term and then multiplied the two binomials together.

$h(x) = -5(x^2 - 3x - 3x + 9) + 46$

$h(x) = -5(x^2 - 6x + 9) + 46$

$h(x) = -5x^2 + 30x - 45 + 46$

$h(x) = -5x^2 + 30x + 1$

I collected like terms and multiplied by -5.

Then I simplified.

Sally's equation is the same as Jake's equation.

$$h(x) = -5x^2 + 30x + 1$$
$$h(x) = -5x^2 + 30x - 45 + 46$$
$$h(x) = -5(x^2 - 6x + 9) + 46$$
$$h(x) = -5(x - 3)(x - 3) + 46$$
$$h(x) = -5(x - 3)^2 + 46$$

I wrote the solution in reverse to help me see how to go from standard form to vertex form.

Reflecting

A. When changing from standard form to vertex form, why do the brackets appear?

B. What type of trinomial resulted when the common factor was divided out?

C. In the trinomial $x^2 - 6x + 9$, how is the coefficient 6 related to 9?

D. Why must we use a perfect-square trinomial to write the equation of a quadratic function in vertex form?

APPLY the Math

EXAMPLE 2 Selecting a strategy to write a quadratic function in vertex form

completing the square

the process of adding a constant to a given quadratic expression to form a perfect trinomial square

for example, $x^2 + 6x + 2$ is not a perfect square, but if 7 is added to it, it becomes $x^2 + 6x + 9$, which is $(x + 3)^2$

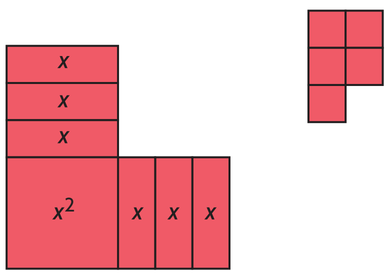
Write each quadratic function in vertex form by **completing the square.** Then use your graphing calculator to verify that the standard form and your vertex form are equivalent.

a) $y = x^2 + 6x + 5$

b) $f(x) = -2x^2 + 16x + 1$

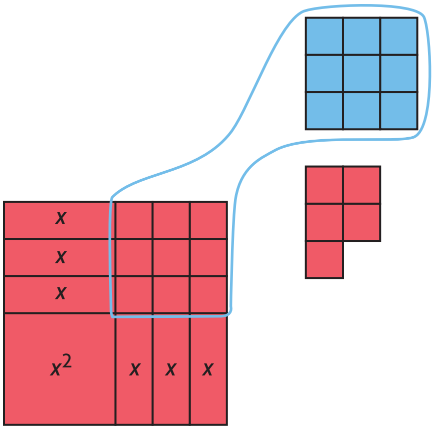
Tommy's Solution

a) $y = x^2 + 6x + 5$



I used algebra tiles to model the expression. The vertex form has two identical factors, so I had to arrange my tiles to form a square. To do this, I divided the x tiles in half. $6 \div 2 = 3$.

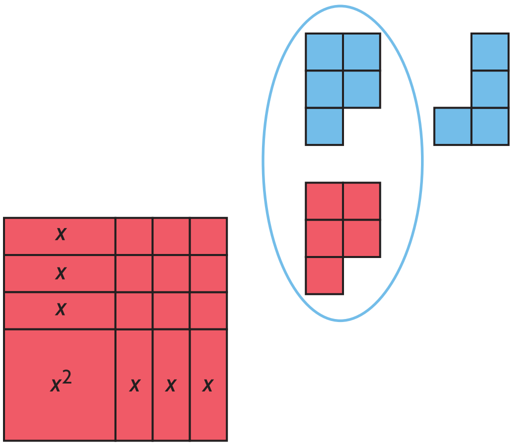
This arrangement is not a perfect square because the 5 unit tiles can't be used to create a square. But if I add 9 tiles to the model, I can create a perfect square. Dividing the coefficient of x and squaring it gave me that number. $(6 \div 2)^2 = 3^2 = 9$



Now $x^2 + 6x + 9$ is a perfect square: $(x + 3)^2 = x^2 + 6x + 9$.

If I add 9 tiles, then I'll change the original expression, so I also added 9 negative tiles since $9 + (-9) = 0$.

$$y = x^2 + 6x + 9 - 9 + 5$$

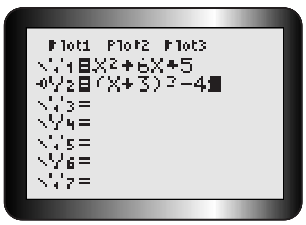


To simplify, I grouped the tiles that form the blue square and the remaining red unit tiles together $(-9) + 5 = -4$, since $(-5) + 5 = 0$.

$$y = (x^2 + 6x + 9) - 9 + 5$$

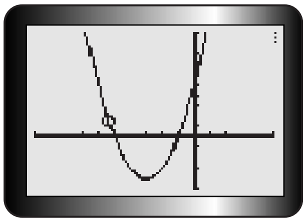
$$y = (x + 3)^2 - 4$$

I factored the trinomial to write the function in vertex form.



To check my solution on the graphing calculator, I entered both equations.

I used line mode for Y1 and path mode for Y2.



Since the cursor traces the parabola they're equal.

b) $f(x) = -2x^2 + 16x + 1$

$f(x) = -2x^2 + 16x + 1$

$f(x) = -2(x^2 - 8x) + 1$

Since $a \neq 1$, I divided the first two terms by the common factor -2. I used only the first two terms because they're all I need to complete the square.

$f(x) = -2(x^2 - 8x + 16 - 16) + 1$

I completed the square for the terms in the brackets. Half of -8 is -4, and -4 squared is 16, so I added 16 and subtracted 16 inside the brackets.

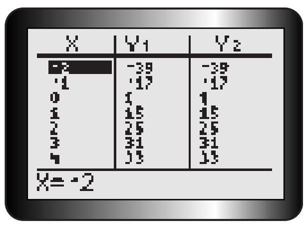
$f(x) = -2(x^2 - 8x + 16) + 32 + 1$

The first three terms inside the brackets form a perfect square, so I multiplied the last term in the brackets by -2 to group the three terms together.

$f(x) = -2(x - 4)(x - 4) + 32 + 1$

I factored the perfect square and added the constants.

$f(x) = -2(x - 4)^2 + 33$



To see if the two equations were equal, I entered the standard form in Y1 and the vertex form in Y2. Then I looked at the table the calculator creates and compared the *y*-values for each *x*-value.

Since the *y*-values are equal, the equations are equivalent.

EXAMPLE 3 Completing the square to write a quadratic function in vertex form

Write the quadratic function $y = 2x^2 - 3x - 7$ in vertex form.

Marc's Solution: Using Fractions

$$y = 2x^2 - 3x - 7$$

$$y = 2\left(x^2 - \frac{3}{2}x\right) - 7$$

I factored 2 out of the first two terms of the expression because it's easier to complete the square if the coefficient of x^2 is 1. Then I completed the square.

$$y = 2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) - 7$$

Half of $-\frac{3}{2}$ is $-\frac{3}{4}$. If I square $-\frac{3}{4}$ I get $\frac{9}{16}$, so I added and subtracted $\frac{9}{16}$ inside the brackets.

$$y = 2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{8} - 7$$

The perfect square consists of only the first three terms, so I multiplied $-\frac{9}{16}$ by 2 to get $-\frac{9}{8}$ and moved it outside of the brackets.

$$y = 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} - 7$$

I factored the expression in the brackets.

$$y = 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} - \frac{56}{8}$$

$$y = 2\left(x - \frac{3}{4}\right)^2 - \frac{65}{8}$$

To add the remaining constants, I used the common denominator 8 and left the sum as an improper fraction.

Rachel's Solution: Using Decimals

$$y = 2x^2 - 3x - 7$$

$$y = 2(x^2 - 1.5x) - 7$$

I factored 2 from the first two terms so that the coefficient of x^2 would be 1. I decided to write -3 divided by 2 as the decimal -1.5.

$$y = 2(x^2 - 1.5x + 0.5625 - 0.5625) - 7$$

Half the coefficient of x, -1.5, is -0.75, so I squared -0.75 and got 0.5625. I added and subtracted this to keep my equation the same.

$$y = 2(x^2 - 1.5x + 0.5625) - 1.125 - 7$$

Only the first three terms in the brackets form a perfect square, so I multiplied -0.5625 by 2 and moved it out of the brackets.

$$y = 2(x - 0.75)^2 - 1.125 - 7$$
$$y = 2(x - 0.75)^2 - 8.125$$

The perfect square factored to $(x - 0.75)^2$. Then I collected like terms.

EXAMPLE 4 Solving a problem by completing the square

Judy wants to fence three sides of the yard in front of her house. She bought 60 m of fence and wants the maximum area she can fence in. The quadratic function $f(x) = 60x - 2x^2$, where x is the width of the yard in metres, represents the area to be enclosed. Write an equation in vertex form that gives the maximum area that can be enclosed.

Asif's Solution

$$f(x) = -2x^2 + 60x$$

I rearranged the equation.

$$f(x) = -2(x^2 - 30x)$$

I factored -2 from both terms.

$$f(x) = -2(x^2 - 30x + 225 - 225)$$

Half the coefficient of x, -30, is -15, so I squared -15 and got 225. I added and subtracted this to keep my equation the same.

$$f(x) = -2(x^2 - 30x + 225) + 450$$

Only the first three terms in brackets form a perfect square, so I multiplied -225 by -2 and moved it out of the brackets.

$$f(x) = -2(x - 15)^2 + 450$$

The perfect square factored to $(x - 15)^2$. The vertex is (15, 450).

A maximum area of 450 m^2 can be fenced in.

In Summary

Key Ideas

- All quadratic functions in standard form can be written in vertex form by completing the square. The equations are equivalent.
- Both the standard form and the vertex form provide useful information for graphing the parabola.

Need to Know

- To complete the square, follow these steps:

$$f(x) = ax^2 + bx + c$$

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$
Factor the coefficient of x^2 from the first two terms.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$
Add and subtract the square of half the coefficient of x inside the brackets.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] - a\left(\frac{b}{2a}\right)^2 + c$$
Group the three terms that form the perfect square. Multiply the fourth term by a, and move it outside the brackets.

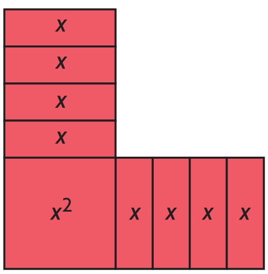
$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$
Factor the perfect square and simplify.

- Any quadratic function in standard form with $a \neq 1$ can be expressed in vertex form when fractions are used to complete the square. Decimals can be used only if the coefficient of x results in a terminating decimal when a is factored from the x^2- and x-terms of the quadratic.

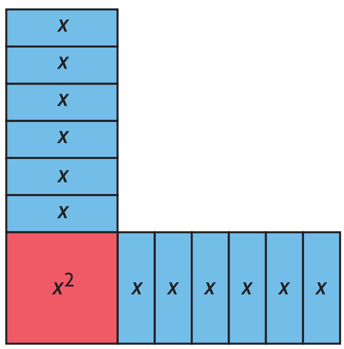
CHECK Your Understanding

1. What number must you add to the following to create a perfect square?

a)



b)



c) $x^2 + 10x$

d) $x^2 - 5x$

2. Determine the values of m and n required to create a perfect-square trinomial.
   a) $x^2 - 10x = x^2 - mx + n - n$
   b) $x^2 + 6x = x^2 + mx + n - n$
   c) $5x^2 + 60x = 5(x^2 + mx + n - n)$
   d) $2x^2 - 7x = 2(x^2 - mx + n - n)$

3. Factor.
   a) $x^2 + 14x + 49$
   b) $x^2 - 18x + 81$
   c) $x^2 - 20x + 100$
   d) $x^2 + 6x + 9$

4. Complete the square.
   a) $y = x^2 + 12x + 40$
   b) $y = x^2 - 6x + 2$
   c) $y = x^2 - 10x + 29$
   d) $y = x^2 - x - 3$

PRACTISING

5. Determine the values of a, h, and k that make the equation true.
   a) $3x^2 - 12x + 17 = a(x - h)^2 + k$
   b) $-2x^2 - 20x - 53 = a(x - h)^2 + k$
   c) $2x^2 - 12x + 23 = a(x - h)^2 + k$
   d) $\frac{1}{2}x^2 + 3x - \frac{1}{2} = a(x - h)^2 + k$

6. Write the function in vertex form.
   a) $f(x) = x^2 + 8x + 3$
   b) $f(x) = x^2 - 12x + 35$
   c) $f(x) = 2x^2 + 12x + 7$
   d) $f(x) = -x^2 + 6x + 7$
   e) $f(x) = -x^2 + 3x - 2$
   f) $f(x) = 2x^2 + 3x + 1$

7. Complete the square to express each function in vertex form. Then graph each, and state the domain and range.
   a) $f(x) = x^2 - 4x + 5$
   b) $f(x) = x^2 + 8x + 13$
   c) $f(x) = 2x^2 + 12x + 19$
   d) $f(x) = -x^2 + 2x - 7$
   e) $f(x) = -3x^2 - 12x - 11$
   f) $f(x) = \frac{1}{2}x^2 + 3x + 4$

8. For the quadratic function $g(x) = 4x^2 - 24x + 31$:
   K
   a) Write the equation in vertex form.
   b) Write the equation of the axis of symmetry.
   c) Write the coordinates of the vertex.
   d) Determine the maximum or minimum value of $g(x)$. State a reason for your choice.
   e) Determine the domain of $g(x)$.
   f) Determine the range of $g(x)$.
   g) Graph the function.

9. Colin completed the square to write $y = 2x^2 - 6x + 5$ in vertex form. Is his solution correct or incorrect? If incorrect, identify the error and show the correct solution.

$$y = 2(x^2 - 3x) + 5$$
$$y = 2\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 5$$
$$y = 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{2} + 5$$
$$y = 2\left(x - \frac{3}{4}\right)^2 + \frac{1}{2}$$

10. A lifeguard wants to rope off a rectangular area for swimmers to swim in. She has 700 m of rope. The area, $A(x)$, that is to be enclosed can be modelled by the function $A(x) = 700x - 2x^2$, where x is the width of the rectangle. What is the maximum area that can be enclosed?

11. **A** A theatre company's profit, $P(x)$, on a production is modelled by $P(x) = -60x^2 + 1800x + 16\,500$, where x is the cost of a ticket in dollars. According to the model, what should the company charge per ticket to make the maximum profit?

12. **a)** Write the quadratic equation $y = 3x^2 - 30x + 73$ in vertex form.
b) What information does the vertex form give that is not obvious from the standard form?

13. **T** What transformations must be applied to the graph of $y = x^2$ to produce the graph of $y = -2x^2 + 16x - 29$? Justify your reasoning.

14. Why express quadratic equations in several different equivalent forms?

15. **C** List the steps followed to change a quadratic function in standard form with $a = 1$ to vertex form. Illustrate with an example.

Extending

16. Betty shoots an arrow into the air. The height of the arrow is recorded in the table. What is the equation of a curve of good fit for the height of the arrow in terms of time?

Time (s)	0	0.5	1.0	1.5	2.0
Height (m)	1.00	4.75	6.00	4.75	1.00

17. The points $(-2, -12)$ and $(2, 4)$ lie on the parabola $y = a(x - 1)^2 + k$. What is the vertex of this parabola?

4.3 Solving Quadratic Equations Using the Quadratic Formula

YOU WILL NEED

- graphing calculator

GOAL

Understand and apply the quadratic formula.

LEARN ABOUT the Math

A quarter is thrown from a bridge 15 m above a pool of water. The height of the quarter above the water at time t is given by the quadratic function $h(t) = -5t^2 + 10t + 15$, where time, t, is in seconds and height, $h(t)$, is in metres.

? How can you determine when the quarter hits the water?

EXAMPLE 1 Selecting a strategy to determine the zeros of a quadratic function

Determine when the quarter will hit the water.

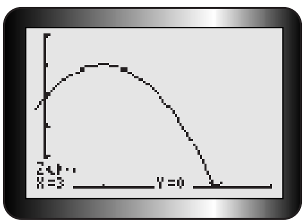
Ben's Solution: Using a Table of Values

t	$h(t)$
0	15
1	20
2	15
3	0

When the quarter hits the water, its height will be 0. I made a table of values. I substituted different values of t into the original function. Time can't be negative, so I chose numbers greater than or equal to zero. I stopped when I found an $h(t)$ value equal to zero.

The quarter will hit the water at 3 s.

Talia's Solution: Graphing



When the quarter hits the water, its height will be 0. I graphed the function to get its zeros. I looked at the part of the graph where $t \geq 0$ because time can't be negative.

The graph crosses the horizontal axis at $t = 3$.

The quarter will hit the water at 3 s.

Jim's Solution: Using Factoring

$$-5t^2 + 10t + 15 = 0$$

The quarter will hit the water when the height is zero, so I let $h(t) = 0$.

$$-5(t^2 - 2t - 3) = 0$$
$$-5(t - 3)(t + 1) = 0$$

I factored -5 from all terms in the equation. I need two numbers that multiply to give -3 and add to -2. The numbers are -3 and 1.

$$t - 3 = 0 \text{ or } t + 1 = 0$$
$$t = 3 \quad \text{or} \quad t = -1$$

I set each of the factors equal to zero and solved.

$$t = 3 \quad \text{because} \quad t \geq 0$$

The quarter hits the water at 3 s.

Since t represents time, it must be greater than or equal to zero. So $t = 3$ is the only acceptable solution.

Elva's Solution: Using the Vertex Form

$$-5t^2 + 10t + 15 = 0$$

The height of the quarter will be zero when it hits the water, so I let $h(t) = 0$.

$$-5(t^2 - 2t) + 15 = 0$$

I factored -5 from the first two terms.

$$-5(t^2 - 2t + 1 - 1) + 15 = 0$$

I completed the square by adding and subtracting half the coefficient of the term containing t.

$$-5(t^2 - 2t + 1) + 5 + 15 = 0$$
$$-5(t^2 - 2t + 1) + 20 = 0$$

I rewrote the expression so that the terms of the perfect square were in the brackets. Then I collected like terms.

$$-5(t - 1)^2 + 20 = 0$$
$$-5(t - 1)^2 = -20$$

I factored the perfect square and rewrote it with the term involving t on one side.

$$(t - 1)^2 = 4$$
$$(t - 1) = \pm\sqrt{4}$$

Then I divided both sides by -5. I took the square root, which can be positive or negative, of both sides.

$$(t - 1) = \pm 2$$

I simplified by taking the square root of 4.

$t = 1 \pm 2$ — I isolated the variable.

$t = 1 + 2$ or $t = 1 - 2$ — I separated the right side to make two equations.

$t = 3$ or $t = -1$

The quarter hits the water at 3 s. — Since time can't be negative, $t = 3$.

EXAMPLE 2 Developing a formula to determine the solutions to a quadratic equation

Given the standard form of the equation $ax^2 + bx + c = 0$, follow Elva's solution in Example 1 to help complete the square of the general quadratic equation $ax^2 + bx + c = 0$ in order to express it in vertex form. Then use it to determine the roots.

Terri's Solution

$$ax^2 + bx + c = 0$$

I factored the coefficient of x^2 from the first two terms.

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$

I completed the square by adding and subtracting the square of half the coefficient of the x-term.

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + \frac{4ac}{4a} = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a} - \frac{4ac}{4a}\right) = 0$$

I rewrote the expression so that only three terms were in the brackets.

I gathered together like terms. I used a common denominator before I added the like terms.

I factored the perfect square in the brackets.

$$a\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a} - \frac{4ac}{4a}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$

I used inverse operations so that the variable was on one side.

I divided both sides by a.

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

I took the square root of both sides. I remembered that it could be the positive or negative square root.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

I simplified and isolated the variable.

I determined the zeros.

Reflecting

A. What is the maximum number of solutions the quadratic formula gives?

B. Why is it easier to use the quadratic formula if the quadratic equation you are solving is in the form $ax^2 + bx + c = 0$?

C. What do the solutions determined using the quadratic formula represent in the original function?

D. How did solving the specific quadratic equation in Example 1 help you understand the development of the quadratic formula in Example 2?

quadratic formula

a formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0$. The formula uses the coefficients of the terms in the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

APPLY the Math

Unlike factoring, you can use the quadratic formula to solve any quadratic equation. The number of solutions depends on the coefficients of the terms in the equation.

EXAMPLE 3 Solving a quadratic equation using the quadratic formula

Use the quadratic formula to solve each equation.

a) $x^2 - 30x + 225 = 0$ **b)** $3x^2 + 2x + 15 = 0$ **c)** $2x^2 - 5x = 1$

Steve's Solution

a) $x^2 - 30x + 225 = 0$

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(225)}}{2(1)}$$

$$x = \frac{30 \pm \sqrt{900 - 900}}{2}$$

I substituted $a = 1$, $b = -30$, and $c = 225$ into the quadratic formula and solved for x.

$$x = \frac{30 \pm \sqrt{0}}{2}$$

$$x = \frac{30}{2}$$

$$x = 15$$

There is only one solution.

b) $3x^2 + 2x + 15 = 0$

I substituted $a = 3$, $b = 2$, and $c = 15$ into the quadratic formula and solved for x.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(15)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{4 - 180}}{6}$$

$$x = \frac{-2 \pm \sqrt{-176}}{6}$$

I can't take the square root of a negative number.

There's no real solution.

c) $2x^2 - 5x - 1 = 0$

I rearranged the equation to get zero on the right side.

Then I substituted $a = 2$, $b = -5$, and $c = -1$ into the quadratic formula and solved for x.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 + 8}}{4}$$

$$x = \frac{5 \pm \sqrt{33}}{4}$$

$$x = \frac{5 + \sqrt{33}}{4} \quad \text{or} \quad x = \frac{5 - \sqrt{33}}{4}$$

$$x \doteq 2.69 \quad \text{or} \quad x \doteq -0.19$$

There are two solutions.

EXAMPLE 4 Applying the quadratic formula to solve a problem

The profit on a school drama production is modelled by the quadratic equation $P(x) = -60x^2 + 790x - 1000$, where $P(x)$ is the profit in dollars and x is the price of the ticket, also in dollars.

a) Use the quadratic formula to determine the break-even price for the tickets.

b) At what price should the drama department set the tickets to maximize their profit?

Julia's Solution

a) $-60x^2 + 790x - 1000 = 0$

At the break-even price there is no profit. So I set the profit function equal to zero.

$$x = \frac{-790 \pm \sqrt{790^2 - 4(-60)(-1000)}}{2(-60)}$$

I substituted $a = -60$, $b = 790$, and $c = -1000$ into the quadratic formula and solved for x.

$$x = \frac{-790 \pm \sqrt{624\,100 - 240\,000}}{-120}$$

$$x = \frac{-790 \pm \sqrt{384\,100}}{-120}$$

$$x = \frac{-790 \pm 619.758}{-120}$$

$$x = \frac{-790 + 619.758}{-120} \quad \text{or} \quad x = \frac{-790 - 619.758}{-120}$$

$$x \doteq 1.42 \quad \text{or} \quad x \doteq 11.75$$

The price of the tickets could be either \$1.42 or \$11.75.

b) $x = \dfrac{1.42 + 11.75}{2}$

The maximum profit is halfway between the break-even prices. I added the two prices and then divided by 2 to get the price that should be charged.

$$x = \frac{13.17}{2} = 6.585$$

The price will be \$6.59.

I can't have three decimal places because I'm dealing with money, so I rounded to \$6.59.

In Summary

Key Ideas

- All quadratic equations of the form $ax^2 + bx + c = 0$ can be solved using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The quadratic formula is derived by completing the square for $ax^2 + bx + c = 0$ and solving for x. It is a direct way of calculating roots without graphing or algebraic manipulation.

Need to Know

- A quadratic equation can have 2, 1, or 0 real solutions, depending on the values of a, b, and c.
- The solutions generated by the quadratic formula for the equation $ax^2 + bx + c = 0$ correspond to the zeros, or x-intercepts, of the function $f(x) = ax^2 + bx + c$.

CHECK Your Understanding

1. Identify the values of a, b, and c you would substitute into the quadratic formula to solve each of the following.

a) $3x^2 - 5x + 2 = 0$

b) $7 - 3x + 5x^2 = 0$

c) $16x^2 + 7 = -24x - 2$

d) $(x - 3)(2x + 1) = 5(x - 2)$

2. a) Set up the quadratic formula for each equation in question 1.

b) Solve the equations in part (a).

PRACTISING

3. **K** Use the quadratic formula to solve each quadratic real equation. Round your answers to two decimal places. If there is no real solution, say so.

a) $x^2 - 5x + 11 = 0$

b) $-2x^2 - 7x + 15 = 0$

c) $4x^2 - 44x + 121 = 0$

d) $4x^2 - 5x - 7 = 0$

e) $-3x^2 - 5x - 11 = 0$

f) $-8x^2 + 5x + 2 = 0$

4. Use a graphing calculator to graph the corresponding function for each equation in question 3. Determine the zeros for each to verify that your solutions are correct.

Tech *Support*

For help using the graphing calculator to determine zeros, see Technical Appendix, B-8.

5. Identify a method that could be used to determine the roots of the given equations. Then use it to determine the roots.

a) $x^2 = 15x$

b) $x^2 = 115$

c) $2x^2 = 19x - 24$

d) $(x - 5)(2x - 3) = (x + 4)(x - 3)$

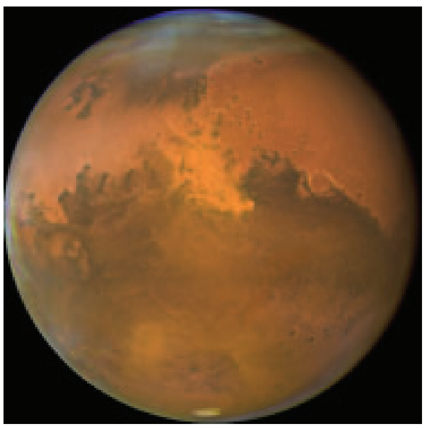
e) $-2(x - 3)^2 + 50 = 0$

f) $1.5x^2 - 26.7x + 2.4 = 0$

6. On Mars, if you hit a baseball, the height of the ball at time t would be modelled by the quadratic function $h(t) = -1.85t^2 + 20t + 1$, where t is in seconds and $h(t)$ is in metres.

a) When will the ball hit the ground?

b) How long will the ball be above 17 m?



7. A gardener wants to fence three sides of the yard in front of her house. She bought 60 m of fence and wants an area of about 400 m^2. The quadratic equation $f(x) = 60x - 2x^2$, where x is the width of the yard in metres and $f(x)$ is the area in square metres, gives the area that can be enclosed. Determine the dimensions that will give the desired area.

A

8. The height of an arrow shot on Neptune can be modelled by the quadratic function $h(t) = 2.3 + 50t - 5.57t^2$, where time, t, is in seconds and height, $h(t)$, is in metres. Use the quadratic formula to determine when the arrow will hit the surface.

9. The quadratic function $d(s) = 0.0056s^2 + 0.14s$ models the relationship between stopping distance, d, in metres and speed, s, in kilometres per hour in driving a car. What is the fastest you can drive and still be able to stop within 60 m?

10. Create five quadratic equations by selecting integer values for a, b, and c in $ax^2 + bx + c = 0$. Choose values so that

T

- 2 of your equations have two solutions
- 2 of your equations have one solution
- 1 of your equations has no solution

Use the quadratic formula to verify.

11. **a)** Determine the roots of $3(x - 4)^2 - 17 = 0$ to two decimal places by isolating $(x - 4)^2$ and then taking the square root of both sides.

C

b) Solve the equation $3(x - 4)^2 - 17 = 0$ by expanding $(x - 4)^2$ and then using the quadratic formula.

c) Which method was better for you? Explain.

Extending

12. Determine the intersection points of $y - 3x^2 - 4x - 9$ and $y = 2x$.

13. Determine the intersection points of $y = -2x^2 + 5x + 3$ and $y = x^2 + 2x - 5$.

14. Solve each of the following.

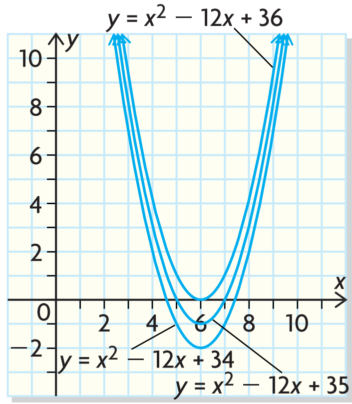
a) $x^3 + 5x^2 - 2x = 0$ **b)** $x^4 - 15x^2 + 54 = 0$

Curious Math

Exact Solution Patterns

The congruent parabolas that correspond to the equations below share the same axis of symmetry. Because they share a characteristic, they are called a family of quadratic equations.

The family of quadratic equations on the left has the solutions on the right.



$x^2 - 12x + 36 = 0$	$x = 6$		
$x^2 - 12x + 35 = 0$	$x = 6 + \sqrt{1}$	and	$x = 6 - \sqrt{1}$
$x^2 - 12x + 34 = 0$	$x = 6 + \sqrt{2}$	and	$x = 6 - \sqrt{2}$
$x^2 - 12x + 33 = 0$	$x = 6 + \sqrt{3}$	and	$x = 6 - \sqrt{3}$
$x^2 - 12x + 32 = 0$	$x = 6 + \sqrt{4}$	and	$x = 6 - \sqrt{4}$
$x^2 - 12x + 31 = 0$	$x = 6 + \sqrt{5}$	and	$x = 6 - \sqrt{5}$
$x^2 - 12x + 26 = 0$	$x = 6 + \sqrt{10}$	and	$x = 6 - \sqrt{10}$

1. How do the solutions relate to the original equations?
2. Using the above pattern, determine the exact solution of the following family of quadratic equations:

$$x^2 - 22x + 121 = 0$$
$$x^2 - 22x + 120 = 0$$
$$x^2 - 22x + 119 = 0$$
$$x^2 - 22x + 118 = 0$$
$$x^2 - 22x + 115 = 0$$
$$x^2 - 22x + 100 = 0$$
$$x^2 - 22x + 99 = 0$$
$$x^2 - 22x + 122 = 0$$

3. What do you need to start with to build your own family of quadratic equations?
4. Create your own family of quadratic equations from your base function.

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: What information about a parabola is easily determined from the vertex form of a quadratic function?

A: The vertex, axis of symmetry, direction of opening, domain, and range are easily determined.

EXAMPLE

$f(x) = 2(x-3)^2 + 7$	$g(x) = -3(x+5)^2 + 2$
vertex $(3, 7)$	vertex $(-5, 2)$
axis of symmetry: $x = 3$	axis of symmetry: $x = -5$
opens up	opens down
domain: $\{x \in \mathbf{R}\}$	domain: $\{x \in \mathbf{R}\}$
range: $\{y \in \mathbf{R} \mid y \geq 7\}$	range: $\{y \in \mathbf{R} \mid y \leq 2\}$

Study *Aid*

- See Lesson 4.1, Example 3.
- Try Mid-Chapter Review Question 2.

Q: How can you change a quadratic function from standard to vertex form and vice versa?

A: To change from standard to vertex form, you complete the square.

EXAMPLE

$g(x) = ax^2 + bx + c, a \neq 1, a \neq 0$

$g(x) = 2x^2 + 12x - 7$

$g(x) = 2(x^2 + 6x) - 7$

Factor the coefficient in front of the x^2-term out of the first two terms.

$g(x) = 2(x^2 + 6x + 9 - 9) - 7$

$g(x) = 2(x^2 + 6x + 9) - 18 - 7$

Add and subtract the square of half the coefficient of x. Only three terms are required to complete the square, so multiply the last term by the coefficient in front of the brackets.

$g(x) = 2(x+3)^2 - 25$

Factor the perfect trinomial square and collect like terms

To change from vertex to standard form, expand the function and simplify.

Study *Aid*

- See Lesson 4.1, Example 1, Kristen's Solution, and Lesson 4.3, Examples 2, 3, and 4.
- Try Mid-Chapter Review Questions 1, 4, and 5.

Q: What formula can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$?

A: The quadratic formula, which is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Study *Aid*

- See Lesson 4.3, Examples 3 and 4.
- Try Mid-Chapter Review Questions 7, 8, and 9.

PRACTICE Questions

Lesson 4.1

1. Write each function in standard form.

a) $f(x) = (x - 8)^2 + 4$

b) $g(x) = -(x - 3)^2 - 8$

c) $f(x) = 4(x - 5)^2 + 9$

d) $g(x) = -0.5(x - 4)^2 + 2$

2. State the vertex, axis of symmetry, maximum or minimum value, domain, and range of each function. Then graph each function.

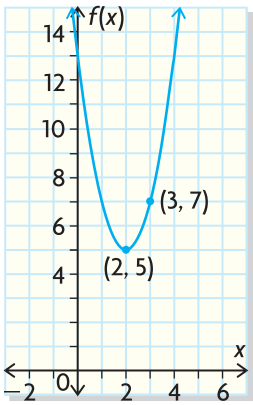
a) $f(x) = (x - 3)^2 + 6$

b) $f(x) = -(x + 5)^2 - 7$

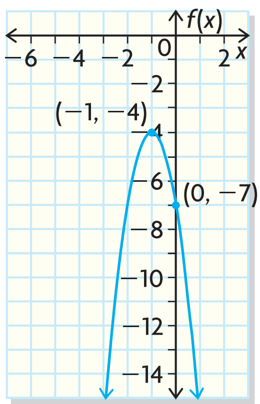
3. Which form of a quadratic function do you like using? Explain, using an example.

4. Determine the equation of each parabola.

a)



b)



Lesson 4.2

5. Write each function in vertex form, and then sketch its graph.

a) $f(x) = x^2 + 10x + 12$

b) $f(x) = 2x^2 + 12x - 3$

c) $f(x) = -x^2 - 8x - 10$

d) $g(x) = 2x^2 - 2x + 7.5$

6. The cost, $C(n)$, of operating a cement-mixing truck is modelled by the function $C(n) = 2.2n^2 - 66n + 700$, where n is the number of minutes the truck is running. What is the minimum cost of operating the truck?

7. A police officer has 400 m of yellow tape to seal off the area of a crime scene. What is the maximum area that can be enclosed?

Lesson 4.3

8. Solve using the quadratic formula.

a) $x^2 + 2x - 15 = 0$

b) $9x^2 - 6x + 1 = 0$

c) $2(x - 7)^2 - 6 = 0$

d) $x^2 + 7x = -24$

9. The height of a ball at a given time can be modelled with the quadratic function $h(t) = -5t^2 + 20t + 1$, where height, $h(t)$, is in metres and time, t, is in seconds. How long is the ball in the air?

10. A theatre company's profit can be modelled by the function $P(x) = -60x^2 + 700x - 1000$, where x is the price of a ticket in dollars. What is the break-even price of the tickets?

11. A model rocket is launched into the air. Its height, $h(t)$, in metres after t seconds is $h(t) = -5t^2 + 40t + 2$.

a) What is the height of the rocket after 2 s?

b) When does the rocket hit the ground?

c) When is the rocket at a height of 77 m?

4.4 Investigating the Nature of the Roots

YOU WILL NEED

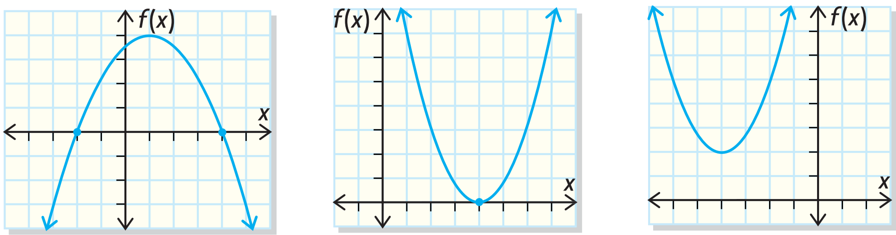
- graphing calculator (optional)

GOAL

Determine how many real roots a quadratic equation has without actually locating them.

INVESTIGATE the Math

Quadratic equations can have 2, 1, or 0 real solutions.



This is because, graphically, the zeros of any parabola $f(x) = ax^2 + bx + c$ are also the solutions of the corresponding quadratic equation $0 = ax^2 + bx + c$.

Sergio wonders whether he can tell how many zeros a quadratic function has without factoring or graphing.

? How can you predict the number of real solutions a quadratic equation has without graphing or determining the solution(s)?

A. Use the quadratic formula to determine the solution(s) of each quadratic equation.

a) $x^2 + 3x + 1 = 0$
b) $2x^2 - x + 1 = 0$
c) $x^2 + 6x + 9 = 0$

B. Create three more quadratic equations and determine their solutions.

C. Based on your results in part A, sketch a parabola that could represent each of the given functions. Confirm your sketch by graphing each function on a graphing calculator.

D. What part of the quadratic formula is directly related to the number of real solutions each quadratic equation has? Explain.

Reflecting

E. Use the graph of the corresponding function to determine when a quadratic equation has
a) two distinct real solutions
b) one real solution
c) no real solution

F. Use the quadratic formula to determine when a quadratic equation has
a) two real distinct solutions
b) one real solution
c) no real solution

G. Explain why the discriminant determines the number of real solutions of a quadratic equation.

discriminant
the expression $b^2 - 4ac$ in the quadratic formula

APPLY the Math

EXAMPLE 1 Connecting the number of real roots of a quadratic equation to the value of the discriminant

Use the discriminant to determine the number of roots of each quadratic equation.
a) $-2x^2 - 3x - 5 = 0$
b) $2x^2 - x - 6 = 0$
c) $x^2 - 10x = -25$

Dave's Solution

a) $b^2 - 4ac$

$(-3)^2 - 4(-2)(-5)$

$= 9 - 40$

$= -31$

No real roots.

I substituted $a = -2$, $b = -3$, and $c = -5$ into the discriminant.

Since the answer is less than 0, there are no real roots.

b) $b^2 - 4ac$

$(-1)^2 - 4(2)(-6)$

$= 1 + 48$

$= 49$

Two distinct real roots.

I substituted $a = 2$, $b = -1$, and $c = -6$ into the discriminant.

Since the answer is greater than 0, there are two distinct real roots.

c) $b^2 - 4ac$

$(-10)^2 - 4(1)(25)$

$= 100 - 100$

$= 0$

One real root.

I substituted $a = 1$, $b = -10$, and $c = 25$ into the discriminant.

Since the answer is equal to 0, there is one real root.

EXAMPLE 2 Connecting the number of zeros of a quadratic function to the value of the discriminant

Without drawing the graph, state whether the quadratic function intersects the x-axis at one point, two points, or not at all.

a) $f(x) = 5x^2 + 3x - 7$

b) $g(x) = 4x^2 - x + 3$

Andrew's Solution

a) $b^2 - 4ac$

$(3)^2 - 4(5)(-7)$

$= 9 + 140$

$= 149$

The parabola will intersect the x-axis at two points.

I substituted $a = 5$, $b = 3$, and $c = -7$ into the discriminant.

Since the answer is greater than 0, the graph will intersect the x-axis at two points.

b) $b^2 - 4ac$

$(-1)^2 - 4(4)(3)$

$= 1 - 48$

$= -47$

The parabola will not intersect the x-axis.

I substituted $a = 4$, $b = -1$, and $c = 3$ into the discriminant.

Since the answer is less than 0, the graph will not intersect the x-axis.

If a quadratic function is expressed in vertex form, the location of the vertex and the direction in which the parabola opens can be used to identify how many zeros it has.

EXAMPLE 3 Using reasoning to determine the number of zeros without graphing

Without drawing the graph, state whether the quadratic function intersects the *x*-axis at one point, two points, or not at all.

a) $f(x) = 2.3(x - 5)^2 + 4.5$

b) $g(x) = -3.7(x + 2)^2 + 3.5$

Talia's Solution

a) $f(x) = 2.3(x - 5)^2 + 4.5$ has no zeros.

The function is in vertex form. The vertex is (5, 4.5). The value of a is 2.3, a positive number. The parabola opens up. The vertex is above the x-axis. Since the parabola opens up, the graph will never cross the x-axis, so this function has no zeros.

b) $g(x) = -3.7(x + 2)^2 + 3.5$ has two zeros.

The function is in vertex form. The vertex is $(-2, 3.5)$. The value of a is -3.7, a negative number. The parabola opens down. The vertex is above the x-axis. Since the parabola opens down, the graph will cross the x-axis twice.

EXAMPLE 4 Solving a problem using the discriminant

For what value(s) of k does the equation $kx - 10 = 5x^2$ have

a) one real solution?

b) two distinct real solutions?

c) no real solution?

Kelly's Solution

a) $kx - 10 = 5x^2$

$-5x^2 + kx - 10 = 0$

I put the equation into standard form to see the discriminant better.

$b^2 - 4ac$

$= k^2 - 4(-5)(-10)$

I substituted $a = -5$, $b = k$, and $c = -10$ into the discriminant.

$= k^2 - 100$

$k^2 - 100 > 0$

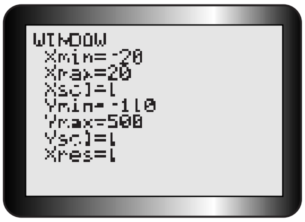
For there to be two distinct solutions, the discriminant must be greater than zero.

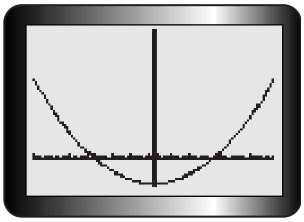
$k^2 - 100 = 0$

For there to be one solution, the discriminant must equal zero.

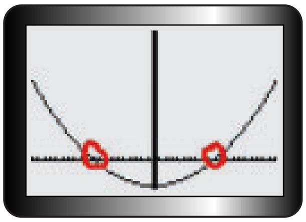
$k^2 - 100 < 0$

For there to be no solution, the discriminant must be less than zero.





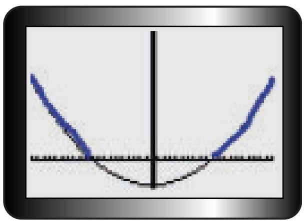
To get the values of k, I graphed $y = x^2 - 100$ using my graphing calculator and the WINDOW settings shown.



The zeros of the graph occur at -10 and 10, giving me the values that solve the equation $k^2 - 100 = 0$.

When $k = -10$ or $k = 10$, the function will have one real solution.

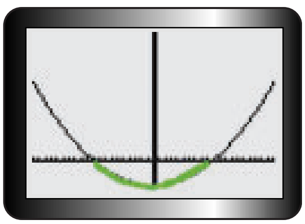
b)



$k^2 - 100 > 0$ when the y-values on the graph are positive. This occurs on the portions of the graph that lie above the x-axis.

The function will have two distinct real solutions when $k > 10$ and when $k < -10$.

c)



$k^2 - 100 < 0$ when the y-values on the graph are negative. This occurs on the portions of the graph that lie below the x-axis

The function will have no real solution when $-10 < k < 10$.

In Summary

Key Idea

- The value of the discriminant, $b^2 - 4ac$, tells you how many real solutions a quadratic equation has and how many x-intercepts the corresponding function has.

Need to Know

- For the quadratic equation $ax^2 + bx + c = 0$ and its corresponding function $f(x) = ax^2 + bx + c$, if
  - $b^2 - 4ac > 0$, then the equation has two distinct real solutions and the function has two x-intercepts
  - $b^2 - 4ac = 0$, then the equation has one real solution and the function has one x-intercept
  - $b^2 - 4ac < 0$, then the equation has no real solution and the function has no x-intercepts

CHECK *Your Understanding*

1. Write the discriminant. Do not evaluate.

a) $x^2 - 5x + 7 = 0$

b) $x^2 + 11x = 6x^2 - 17$

c) $3(x + 2)^2 - 19 = 0$

d) $(x + 3)(2x - 1) = 4(x + 2)$

2. Determine the number of real solutions of each quadratic equation. Do not solve.

a) $x^2 + 3x - 4 = 0$

b) $2x^2 - x + 5 = 0$

c) $4x^2 - 8x = 4x - 9$

d) $3(x + 5)^2 + 7 = 0$

e) $-2(x - 5)^2 + 3 = 0$

f) $4(x + 1)^2 = 0$

3. Andrew thinks that the quadratic function $f(x) = x^2 - 5x + 2$ does not intersect the x-axis because the discriminant is negative. Do you agree? Explain.

PRACTISING

4. **K** Determine whether each quadratic function intersects the x-axis at one point, two points, or not at all. Do not draw the graph.

a) $f(x) = 3x^2 + 6x - 1$

b) $g(x) = 4(x - 6)^2 + 2$

c) $f(x) = 9x^2 - 30x + 25$

d) $g(x) = -3(x - 5)^2 + 2$

e) $f(x) = 2x^2 + 3x + 5$

f) $g(x) = -3(x + 2)^2$

5. For what value(s) of k does the function $f(x) = kx^2 - 8x + k$ have no zeros?

6. For what value of m does $g(x) = 49x^2 - 28x + m$ have exactly one zero?

7. For what value of k does $8x^2 + 4x + k = 0$ have two distinct real solutions? one solution? no solution?

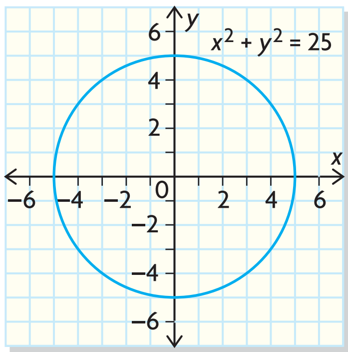
8. **a)** Explain how you would solve this problem: For what value of k does the function $f(x) = 3x^2 - 5x + k$ have only one zero?

b) Use your strategy to find the value of k.

9. The function $f(x) = x^2 + kx + k + 8$ touches the x-axis once. What value(s) could k be?

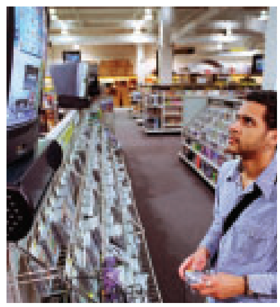
10. T For what values of k does the line $y = x + k$ pass through the circle defined by $x^2 + y^2 = 25$ at

a) 2 points? **b)** 1 point? **c)** 0 points?



11. Generate 10 quadratic equations by randomly selecting integer values for a, b, and c in $ax^2 + bx + c = 0$. Use the discriminant to identify how many real solutions each equation has.

12. A The profit, $P(x)$, of a video company, in thousands of dollars, is given by $P(x) = -5x^2 + 550x - 5000$, where x is the amount spent on advertising, in thousands of dollars. Can the company make a profit of $50 000? Explain.



13. C State two different ways to determine the number of zeros of the function $f(x) = 2(x + 1)^2 - 6$.

Extending

14. Can $p^2 - 13$ equal $-12p$? Explain your answer.

15. Using different values of k, determine the number of zeros of the function $f(x) = (k + 1)x^2 + 2kx + k - 1$.

4.5 Using Quadratic Function Models to Solve Problems

YOU WILL NEED

- graphing calculator (optional)

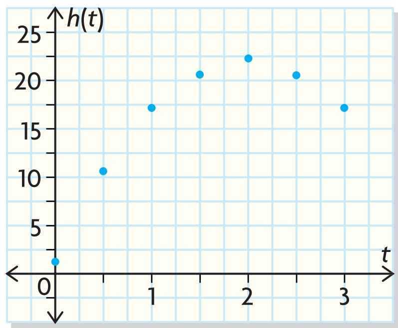
GOAL

Solve problems involving quadratic functions and equations arising from standard and vertex forms.

LEARN ABOUT the Math

The graph shows the height of a rock launched from a slingshot, where time, t, is in seconds and height, $h(t)$, is in metres.

Height of a Rock



? How can you use the vertex form of a quadratic function to model the height of the rock over time and to determine when the rock hits the ground?

EXAMPLE 1 Connecting the vertex to an algebraic model for height

a) Use the graph to determine an algebraic model.
b) Use the model to determine when the rock hits the ground.

Nadia's Solution

a)

$$y = a(x - h)^2 + k$$
$$h(t) = a(t - 2)^2 + 22$$
$$2 = a(0 - 2)^2 + 22$$
$$2 = a(-2)^2 + 22$$
$$2 = 4a + 22$$
$$2 - 22 = 4a$$

The vertex appears to be (2, 22), so I replaced h with 2 and k with 22.

Then I substituted (0, 2), a point on the parabola, to solve for a.

$$-20 = 4a$$

$$\frac{-20}{4} = a$$

$$-5 = a$$

The function that models the rock's height is $h(t) = -5(t - 2)^2 + 22$.

I put the value of a in the vertex form.

b) $-5(t - 2)^2 + 22 = 0$

The height of the ground is zero, so I set $h(t) = 0$ and solved for t.

$$-5(t - 2)^2 = -22$$

$$(t - 2)^2 = \frac{-22}{-5}$$

$$(t - 2)^2 = \frac{22}{5}$$

$$t - 2 = \pm\sqrt{\frac{22}{5}}$$

$$t = 2 \pm \sqrt{\frac{22}{5}}$$

$t \doteq -0.1$ and $t \doteq 4.1$

Time, t, must be positive, so the domain is $\{t \in \mathbf{R} \mid 0 \le t \le 4.1\}$. The rocket hits the ground at about 4.1 s.

Reflecting

A. Why was using the vertex form to determine the function a good strategy for solving this problem?

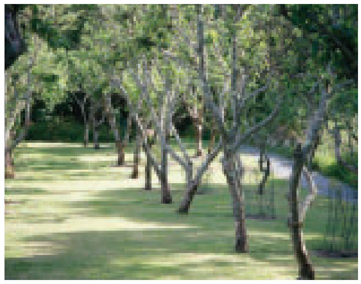
B. Why is determining the domain important when modelling quadratic functions?

C. What factors can affect the form of the quadratic function you choose to model a problem?

APPLY the Math

EXAMPLE 2 Selecting a strategy to determine when a quadratic function reaches its maximum value

Mr. McIntosh has 90 apple trees. He earns an annual revenue of \$120 per tree. If he plants more trees, they have less room to grow, resulting in fewer apples per tree. As a result, the annual revenue per tree is reduced by \$1 for each additional tree. The revenue, in dollars, is modelled by the function $R(x) = (90 + x)(120 - x)$, where x is the number of additional trees planted.



Regardless of the number of trees planted, the cost of maintaining each tree is \$8. The cost, in dollars, is modelled by the function $C(x) = 8(90 + x)$, where x is the number of additional trees planted. How many trees must Mr. McIntosh plant to maximize profit?

Talia's Solution

$P(x) = R(x) - C(x)$

To get the profit function, I subtracted the cost function from the revenue function.

$P(x) = (90 + x)(120 - x) - 8(90 + x)$

I expanded and then simplified the profit function.

$P(x) = 10\,800 - 90x + 120x - x^2 - 720 - 8x$

$P(x) = -x^2 + 22x + 10\,080$

If the function is written in vertex form, I can just read when the maximum happens and what it will be.

I completed the square to get the function in vertex form.

$P(x) = -(x^2 - 22x) + 10\,080$

I factored -1 from the first two terms. I added and subtracted the square of half of the coefficient of the x-term.

$P(x) = -(x^2 - 22x + 121 - 121) + 10\,080$

$P(x) = -(x^2 - 22x + 121) + 121 + 10\,080$

I grouped the three terms of the perfect square and I multiplied the last term by -1 to move it outside the brackets.

I factored the perfect square and added the constant terms together.

$P(x) = -(x - 11)^2 + 10\,201$

The vertex is (11, 10 201). The parabola opens down, so the maximum is 10 201 when $x = 11$.

Number of trees $= 90 + 11 = 101$

Mr. McIntosh needs to plant 11 extra trees, so he should have a total of 101 trees to maximize his revenue.

EXAMPLE 3 Selecting a strategy to determine when a quadratic function reaches its minimum value

The cost of running an assembly line is a function of the number of items produced per hour. The cost function is $C(x) = 0.28x^2 - 1.12x + 2$, where $C(x)$ is the cost per hour in thousands of dollars, and x is the number of items produced per hour in thousands. Determine the most economical production level.

Andrew's Solution

$$y = a(x - h)^2 + k$$
$$C(x) = 0.28x^2 - 1.12x + 2$$

"The most economical" means "the minimum cost." I wrote the function in vertex form so I could see the minimum value and when it happens.

$$C(x) = 0.28(x^2 - 4x) + 2$$

I factored 0.28 from the first two terms.

$$C(x) = 0.28(x^2 - 4x + 4 - 4) + 2$$

I added and subtracted the square of half of the coefficient of the *x*-term.

$$C(x) = 0.28(x^2 - 4x + 4) - 1.12 + 2$$

I grouped the first three terms of the perfect square and multiplied the last term by 0.28 to move it outside the brackets.

$$C(x) = 0.28(x - 2)^2 + 0.88$$

I factored the perfect square and added the constant terms together.

The most economical production level is 2000/h.

The vertex is (2, 0.88). The parabola opens up, so a minimum value of 0.88 happens when $x = 2$.

I multiplied by 1000 because x is in thousands.

EXAMPLE 4 Selecting a strategy to determine when a quadratic function reaches a given value

A bus company usually charges \$2 per ticket, but wants to raise the price by 10¢ per ticket. The revenue that could be generated is modelled by the function $R(x) = -40(x - 5)^2 + 25\,000$, where x is the number of 10¢ increases and the revenue, $R(x)$, is in dollars. What should the price of the tickets be if the company wants to earn \$21 000?

Rachel's Solution: Solving Algebraically

$$-40(x - 5)^2 + 25\,000 = 21\,000$$

I set the revenue function equal to 21 000.

$$-40(x - 5)^2 = 21\,000 - 25\,000$$
$$-40(x - 5)^2 = -4000$$

To isolate the squared term, I subtracted 25 000 from both sides and simplified.

$$(x - 5)^2 = 100$$
$$(x - 5) = \pm 10$$

I divided both sides by −40 and simplified.

Then I took the square root, which can be positive or negative, of both sides.

$$x - 5 = 10 \quad \text{or} \quad x - 5 = -10$$
$$x = 10 + 5 \quad \text{or} \quad x = -10 + 5$$
$$x = 15 \quad \text{or} \quad x = -5$$

I rewrote the resulting equations so that I could see the two answers.

The price of the ticket is $\$2 + 15(0.10) = \$3.50$.

I can only use the positive value, so there will be 15 ten-cent increases, resulting in a ticket price of \$3.50.

Joyce's Solution: Expanding the Equation and Using the Quadratic Formula

$$-40(x - 5)^2 + 25\,000 = 21\,000$$
$$-40(x - 5)(x - 5) + 25\,000 - 21\,000 = 0$$
$$-40(x^2 - 5x - 5x + 25) + 4000 = 0$$
$$-40(x^2 - 10x + 25) + 4000 = 0$$
$$-40x^2 + 400x - 1000 + 4000 = 0$$
$$-40x^2 + 400x + 3000 = 0$$

I set the revenue function equal to 21 000.

I put the equation into standard form so that I could use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(400) \pm \sqrt{(400)^2 - 4(-40)(3000)}}{2(-40)}$$

I substituted $a = -40$, $b = 400$, and $c = 3000$ into the quadratic formula and solved.

$$x = \frac{-400 \pm \sqrt{160\,000 + 480\,000}}{-80}$$

$$x = \frac{-400 \pm \sqrt{640\,000}}{-80}$$

$$x = \frac{-400 \pm 800}{-80}$$

$$x = \frac{-400 + 800}{-80} \quad \text{or} \quad x = \frac{-400 - 800}{-80}$$

$$x = \frac{400}{-80} \quad \text{or} \quad x = \frac{-1200}{-80}$$

$$x = -5 \quad \text{or} \quad x = 15$$

The price of the ticket will be $\$2 + 15(0.10) = \$3.50$.

I can only use the positive value for *x*. There will be 15 ten-cent increases, resulting in a ticket price of $3.50.

In Summary

Key Idea

- The vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, can be used to solve a variety of problems, such as determining the maximum or minimum value of the quadratic model. This value can then be used as needed to interpret the situation presented.

Need to Know

- If a problem requires you to determine the value of the independent variable, *x*, for a given value of the dependent variable, $f(x)$, for a quadratic function model, then substitute the number in $f(x)$. This will result in a quadratic equation that can be solved by graphing, factoring, or using the quadratic formula.

CHECK Your Understanding

1. The manager of a hardware store sells batteries for $5 a package. She wants to see how much money she will earn if she increases the price in 10¢ increments. A model of the price change is the revenue function $R(x) = -x^2 + 10x + 3000$, where x is the number of 10¢ increments and $R(x)$ is in dollars. Explain how to determine the maximum revenue.

2. Determine the maximum revenue generated by the manager in question 1.

3. A cliff diver dives from about 17 m above the water. The diver's height above the water, $h(t)$, in metres, after t seconds is modelled by $h(t) = -4.9t^2 + 1.5t + 17$. Explain how to determine when the diver is 5 m above the water.

4. Determine when the diver in question 3 is 5 m above the water.

PRACTISING

5. **K** The function $P(x) = -30x^2 + 360x + 785$ models the profit, $P(x)$, earned by a theatre owner on the basis of a ticket price, x. Both the profit and ticket price are in dollars. What is the maximum profit, and how much should the tickets cost?

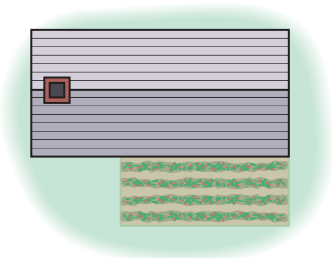
6. **A** The population of a town is modelled by the function $P(t) = 6t^2 + 110t + 4000$, where $P(t)$ is the population and t is the time in years since 2000.

a) What will the population be in 2020?

b) When will the population be 6000?

c) Will the population ever be 0? Explain your answer.

7. The profit of a shoe company is modelled by the quadratic function $P(x) = -5(x - 4)^2 + 45$, where x is the number of pairs of shoes produced, in thousands, and $P(x)$ is the profit, in thousands of dollars. How many thousands of pairs of shoes will the company need to sell to earn a profit?



8. Beth wants to plant a garden at the back of her house. She has 32 m of fencing. The area that can be enclosed is modelled by the function $A(x) = -2x^2 + 32x$, where x is the width of the garden in metres and $A(x)$ is the area in square metres. What is the maximum area that can be enclosed?

9. The stopping distance for a boat in calm water is modelled by the function $d(v) = 0.004v^2 + 0.2v + 6$, where $d(v)$ is in metres and v is in kilometres per hour.

a) What is the stopping distance if the speed is 10 km/h?

b) What is the initial speed of the boat if it takes 11.6 m to stop?

10. Mario wants to install a wooden deck around a rectangular swimming pool. The function $C(w) = 120w^2 + 1800w$ models the cost, where the cost, $C(w)$, is in dollars and width, w, is in metres. How wide will the deck be if he has \$4080 to spend?

11. The population of a rural town can be modelled by the function $P(x) = 3x^2 - 102x + 25\,000$, where x is the number of years since 2000. According to the model, when will the population be lowest?

12. A bowling alley has a \$5 cover charge on Friday nights. The manager is considering increasing the cover charge in 50¢ increments. The revenue is modelled by the function $R(x) = -12.5x^2 + 75x + 2000$, where revenue $R(x)$ is in dollars and x is the number of 50¢ increments.

a) What cover charge will yield the maximum revenue?
b) What will the cover charge be if the revenue is \$2000?

13. The height of a soccer ball kicked in the air is given by the quadratic equation $h(t) = -4.9(t - 2.1)^2 + 23$, where time, t, is in seconds and height, $h(t)$, is in metres.

a) What was the height of the ball when it was kicked?
b) What is the maximum height of the ball?
c) Is the ball still in the air after 6 s? Explain.
d) When is the ball at a height of 10 m?

14. The student council is deciding how much to charge for a ticket to the school dance to make the most money. The data collected are listed in the table.

Ticket Cost (\$)	3.50	4.00	5.00	6.00	6.50	7.00	8.00
Revenue (\$)	1260	1365	1485	1485	1440	1365	1125

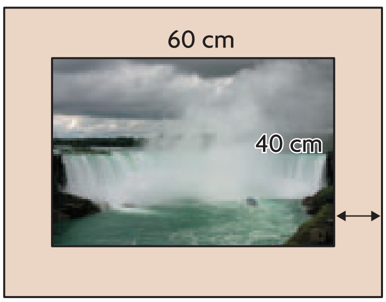
a) Determine the function, in vertex form, that represents these data.
b) What should the ticket price be to earn \$765 in revenue?

15. What are some advantages and disadvantages in using the vertex form to solve questions about quadratic functions?

Extending

16. A rectangle is 7 cm longer than it is wide. The diagonal is 13 cm. What are the rectangle's dimensions?

17. A photo framer wants to place a matte of uniform width all around a photo. The area of the matte should be equal to the area of the photo. The photo measures 40 cm by 60 cm. How wide should the matte be?



4.6 Using the Vertex Form to Create Quadratic Function Models from Data

YOU WILL NEED

- graphing calculator
- graph paper
- dynamic geometry software
- graphing software

GOAL

Determine the equation of a curve of good fit from data.

INVESTIGATE the Math

The table shows data on the percent of 15- to 19-year-old male Canadians who smoke.

Year	1981	1983	1985	1986	1989	1991	1994	1995	1996
Percent	43.4	39.6	26.7	25.2	22.6	22.6	27.3	28.5	29.1

? What is a function that will model the data?

A. Create a scatter plot of the data with an appropriate scale.

B. What shape best describes the graph? Draw a curve of good fit.

C. Estimate the coordinates of the vertex.

D. Use the vertex to write an equation in vertex form.

E. In what direction does the parabola open? What does this tell you?

F. Using one of the points in the table, calculate the a-value. Write the equation for the data in vertex form and in standard form.

G. Determine the domain and range of your model.

H. Using a graphing calculator and quadratic regression, determine a quadratic function that will model the data.

Tech *Support*

For help using quadratic regression to determine the equation of a curve, see Technical Appendix, B-10.

Reflecting

I. How does your model compare with the graphing calculator's model?

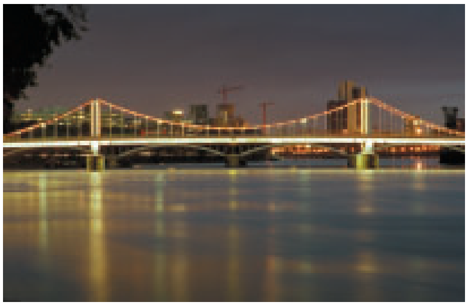
J. How does the vertex form of an equation help you determine an equation for a curve of good fit?

K. How will you know whether the equation is a good representation of your data?

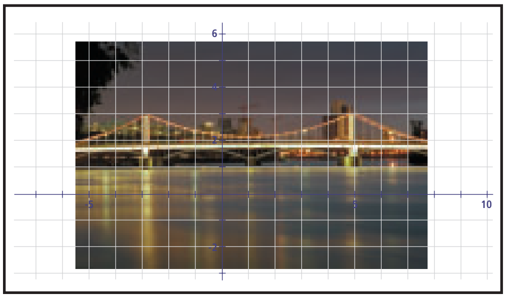
APPLY the Math

EXAMPLE 1 Selecting a strategy to determine the equation of a curve of good fit: dynamic geometry software

Determine a quadratic equation in vertex form that best represents the arch between the towers in the suspension bridge photo. Express your equation in standard form as well.



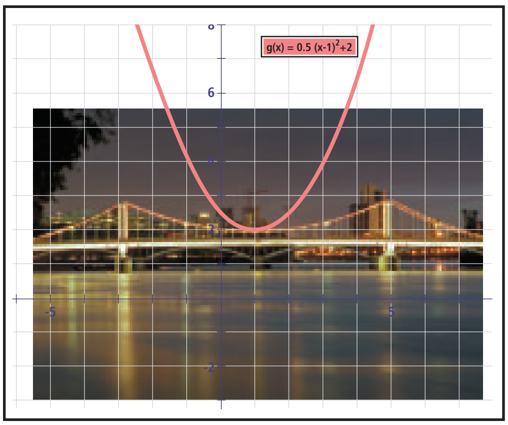
Veronica's Solution



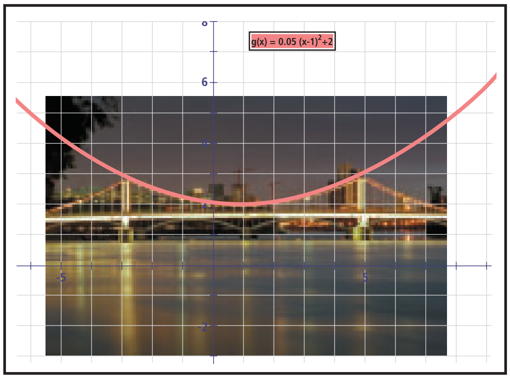
I imported the picture into dynamic geometry software and superimposed a grid over the picture.

$$y = a(x - 1)^2 + 2$$

It looks like the vertex is at (1, 2), so I used the vertex form for the function.



I guessed that 0.5 might work for the value of *a* since the shape is wide. I chose a positive value because the parabola opens up.



The value of *a* that I chose didn't work, so I tried something less than 0.1. I know that making the value of *a* closer to 0 creates a wider parabola. I tried 0.06. This fit better.

The parabola extends past the towers, so I need to restrict the domain. The towers appear to be located at $x = -3$ and $x = 5$. If so, the range will be restricted to between $y = 2$ and $y = 2.96$.

The equation is $y = 0.06(x - 1)^2 + 2$.

domain: $\{x \in \mathbf{R} \mid -3 \leq x \leq 5\}$

range: $\{y \in \mathbf{R} \mid 2 \leq y \leq 2.96\}$

$y = 0.06(x^2 - 2x + 1) + 2$

I expanded to get the standard form.

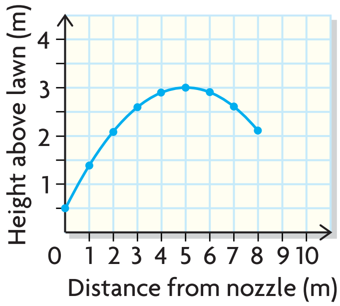
The standard form is
$y = 0.06x^2 - 0.12x + 2.06$.

EXAMPLE 2 Representing a quadratic function from data

A hose sprays a stream of water across a lawn. The table shows the approximate height of the stream above the lawn at various distances from the person holding the nozzle. Write an algebraic model in vertex form that relates the height of the water to the distance from the person. Check your answer using a graphing calculator.

Distance from Nozzle (m)	0	1	2	3	4	5	6	7	8
Height above Lawn (m)	0.5	1.4	2.1	2.6	2.9	3.0	2.9	2.5	1.9

Betty's Solution: Using the Vertex Form



I graphed the data from the table by hand.

It looks quadratic. I estimated the vertex as (5, 3) and substituted these numbers into the vertex form.

$$y = a(x - 5)^2 + 3$$
$$0.5 = a(0 - 5)^2 + 3$$
$$0.5 = a(-5)^2 + 3$$
$$0.5 = a(25) + 3$$
$$0.5 - 3 = 25a$$
$$-2.5 = 25a$$
$$-0.1 = a$$

The graph must pass through one of the pairs of data from the table. I chose the point 0, 0.5) to get *a*.

$$y = -0.1(x - 5)^2 + 3$$
$$y = -0.1(x - 5)(x - 5) + 3$$
$$y = -0.1(x^2 - 10x + 25) + 3$$
$$y = -0.1x^2 + 1x - 2.5 + 3$$
$$y = -0.1x^2 + 1x + 0.5$$

I substituted *a* into the original equation and expanded it into standard form.

I checked to see how close my equation was to the calculator's.

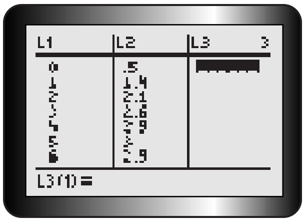
I did well, but the graphing calculator was much faster than my hand calculation.

From the calculator,
$y = -0.1068x^2 + 1.0362x - 0.4763$;
from my scatter plot,
$y = -0.1x^2 + x + 0.5$.

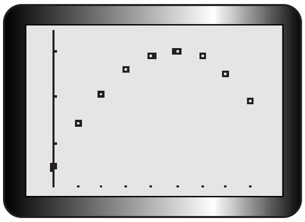
Tech Support

For help creating a scatter plot and determining the equation of the curve of best fit, see Technical Appendix, B-10.

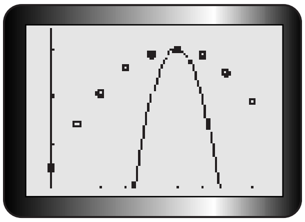
Sheila's Solution: Using the Vertex Form and a Graphing Calculator



I entered the data into list 1 and list 2 in my graphing calculator and created a scatter plot.

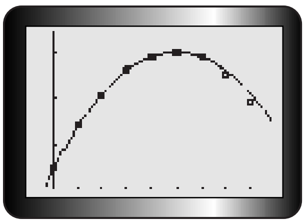


The graph looks quadratic. I used a quadratic equation in the form $y = a(x - h)^2 + k$. The value of a had to be negative because the parabola opens down. The vertex of the function looks like it might be about $(5, 3)$.



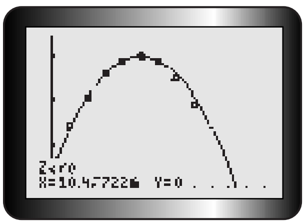
I tried $y = -(x - 5)^2 + 3$.

It wasn't right: The graph wasn't wide enough, and it passed through the vertex only and not through any of the other points of the scatter plot.



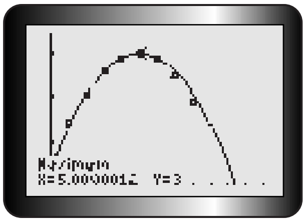
I tried smaller values of a to make the parabola wider. -0.1 gave me a pretty good fit.

So $y = -0.1(x - 5)^2 + 3$ is my equation.



Since the domain represents the time the water is in the air, the smallest x-value will be zero, the largest when the water hits the lawn.

domain: $\{x \in \mathbf{R} \mid 0 \leq x \leq 10.5\}$



The range represents the water's height, which could be anywhere from 0 m above the ground to the maximum height of 3 m above the ground.

range: $\{y \in \mathbf{R} \mid 0 \leq y \leq 3\}$

Tech Support

For help creating a scatter plot, inserting a function, and using sliders in Fathom, see Technical Appendix, B-23.

EXAMPLE 3 Selecting a strategy to determine the equation of a curve of good fit: Graphing software

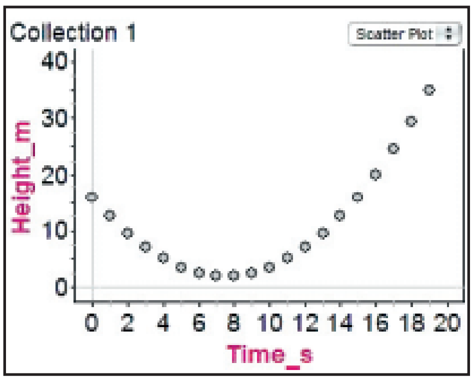
A plastic glider is launched from a hilltop. The height of the glider above the ground at a given time is recorded in the table. When will it reach a height of 45 m?

Collection 1

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Height (m)	16.0	12.5	9.5	7.0	5.0	3.5	2.5	2.0	2.0	2.5	3.5

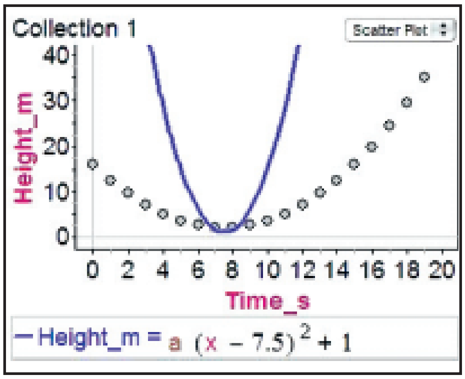
Time (s)	11	12	13	14	15	16	17	18	19	20
Height (m)	5.0	7.0	9.5	12.5	16.0	20.0	24.5	29.5	35.0	41.0

Griffen's Solution



From the trend in the table of values, I predicted that the glider would reach a height of 45 m just after 20 s.

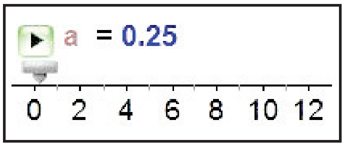
To get a more precise answer, I used graphing software to create a scatter plot. I observed the shape of the plot.



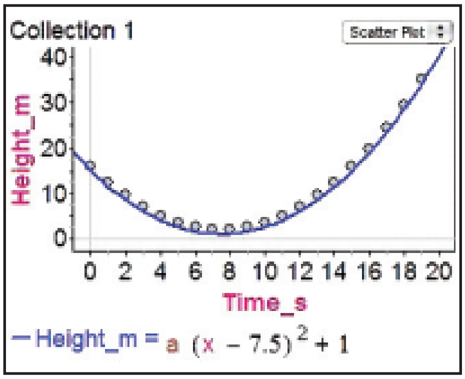
It looks quadratic because the curve decreases, then increases, forming a parabola. The minimum height occurs when $x = 7.5$ s and appears to be 1.

I substituted the vertex (7.5, 1) into the vertex form of the equation and got $f(x) = a(x - 7.5)^2 + 1$.

I added a function to my graph and used a slider to get the value of *a*. I tried to get a good fit by changing the value of *a*.



When $a = 0.25$, the fit is pretty good.



So the equation that models the data is $f(x) = 0.25(x - 7.5)^2 + 1$.

The domain represents the time the glider is in the air. So time must begin at $x = 0$.

The range is the height reached by the glider. The lowest height is 1 m.

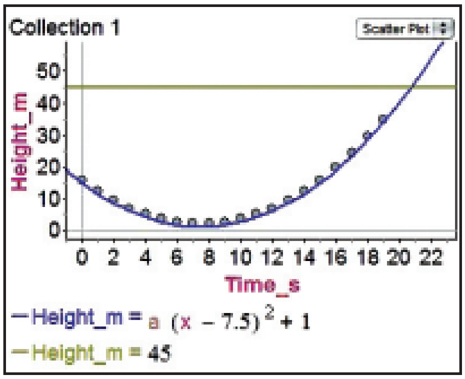
domain: $\{x \in \mathbf{R} \mid x \geq 0\}$

range: $\{y \in \mathbf{R} \mid y \geq 1\}$

$$0.25(x - 7.5)^2 + 1 = 45$$
$$0.25(x - 7.5)^2 = 44$$
$$(x - 7.5)^2 = 176$$
$$x - 7.5 = \pm\sqrt{176}$$
$$x = 7.5 \pm \sqrt{176}$$
$$x \doteq 7.5 + 13.27,\ x \geq 0$$
$$x = 20.77$$

To find when the glider reaches a height of 45 m, I substituted 45 in $f(x)$ and used inverse operations to solve the resulting quadratic equation.

Since *x* represents time, I can only use positive values.



I checked my answer by graphing.

In Summary

Key Ideas

- If a scatter plot has a parabolic shape and its curve of good fit passes through or near the vertex, then the vertex form of the quadratic function can determine an algebraic model of the relationship.
- Once the algebraic model has been determined, it can be used to solve problems involving the relationship.

Need to Know

- Some graphing calculators and graphing and data programs can determine an algebraic model of a scatter plot by regression. This produces the "best" possible model for the situation. If the data form a nonlinear pattern, the graph produced by the regression equation is called the "curve of best fit."
- Curves of good fit are useful for interpolating. They are not necessarily useful for extrapolating because they assume that the trend in the data will continue. Many factors can affect the relationship between the independent and dependent variable and change the trend, resulting in a different curve and a different algebraic model.
- It is often necessary to restrict the domain and range of the function to represent a realistic situation. For example, when a ball is in the air, the domain is between zero and the time the ball hits the ground; the range is between zero and the maximum height of the ball.

CHECK Your Understanding

1. Create a scatter plot, and decide whether the data appear to be quadratic. Justify your decision.

a)

Time (s)	0	1	2	3	4	5
Number	1800	5400	16 200	48 600	145 800	437 400

b)

Distance (m)	0	1	2	3	4	5	6	7	8
Height (m)	3	1	1	3	7	13	21	31	43

2. The following are data on the percent of 15- to 19-year-old female Canadians who smoke.

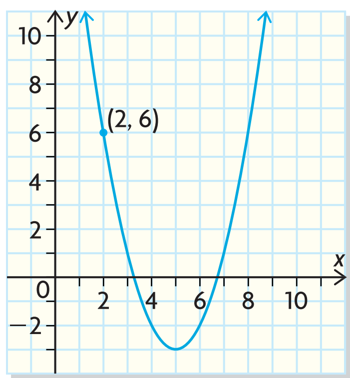
Year	1981	1983	1985	1986	1989	1991	1994	1995	1996
Percent	41.7	40.5	27.7	27	23.5	25.6	28.9	29.5	31

a) Use your graphing calculator to create a scatter plot.
b) Estimate the coordinates of the vertex.
c) Is the parabola opening up or down? Use your answer to estimate a value for a.
d) Adjust your estimate of a until you are satisfied with the fit.
e) Write the function that defines your curve of good fit.
f) State any restrictions on the domain and range of your model.

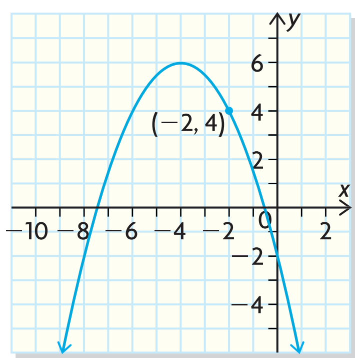
PRACTISING

3. Determine the equation of each parabola.

K **a)**



b)



4. Write the standard form of the quadratic equation.

	Vertex	***y*-intercept**
a)	(2, 3)	11
b)	(−1, 5)	3
c)	(3, −7)	−43
d)	(−2, −5)	19

5. A car skids in an accident. The investigating police officer knows that the distance a car skids depends on the speed of the car just before the brakes are applied.

A

Speed (km/h)	0	10	20	30	40	50	60	70	80	90	100
Length of Skid (m)	0.0	0.7	2.8	6.4	11.4	17.8	25.7	35.0	45.7	57.8	71.4

a) Create a scatter plot of the data in the table, and draw a curve of good fit.
b) Determine an equation of the curve of good fit. Assume that there is only one zero, located at the origin.
c) Use the curve to determine the length of the skid mark to the nearest tenth of a kilometre if the initial speed was 120 km/h.
d) State any restrictions on the domain and range of your model.

6. The amount of gas used by a car per kilometre depends on the car's speed.

Speed (km/h)	20	40	60	80	100	120
Cost of Gas (¢/km)	19.1	17.8	17.1	17.1	17.8	19.1

a) Use the data in the table to determine an equation for a curve of good fit.
b) Use the curve to determine the cost of gas if the car's speed is 140 km/h.
c) State any restrictions on the domain and range of your model.

7. The number of new cars sold in Canada from 1982 to 1992 is shown in the table.

Year	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
New Cars Sold (000s)	718	841	971	1135	1102	1061	1056	985	885	873	798

a) Determine the equation of a curve of good fit.
b) How well does the equation predict car sales after 2000?
c) State any restrictions on the domain and range of your model.

8. The height of an arrow shot by an archer is given in the table. Determine the equation for a curve of good fit. State any restrictions on the domain and range of your model. Use it to predict when the arrow will hit the ground.

Time (s)	0	0.5	1.0	1.5	2.0	2.5
Height (m)	0.5	9.2	15.5	19.3	20.5	19.3

9. A farming cooperative collected data on the effect of different amounts of fertilizer, x, in hundreds of kilograms per hectare (kg/ha) on the yield of carrots, y, in tonnes (t).

Fertilizer (hundreds kg/ha)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Yield (t)	0.15	0.45	0.65	0.90	0.95	1.10	1.05	0.90	0.80

a) Write an equation to predict the yield of carrots based on fertilizer used.
b) State any restrictions on the domain and range of your model.
c) How much fertilizer is used for a yield of 0.75 t?

Average Miles per Gallon	Top Speed (miles/h)
96.0	17.5
56.0	97.0
97.0	20.0
105.0	20.0
45.4	97.0
38.8	111.0
35.4	111.0
121.0	45.0
18.1	165.0
17.0	147.0
16.7	157.0
130.0	55.0

10. The data in the table at the left show the average miles per gallon for various cars and their top speeds in miles per hour.
a) Determine the equation of the curve of good fit in vertex form.
b) Use your equation from part (a) to estimate the top speed reached by a car that gets 150 miles per gallon. Does this make sense?

11. T A quadratic function passes through the points $(-2, 6)$, $(0, 6)$, and $(2, 22)$. Determine its equation algebraically without using quadratic regression.

12. C If data appear to be quadratic, explain how an equation of the curve of good fit could be obtained even if the vertex for the function does not appear in the data.

Extending

13. A sprinkler waters a circular area of lawn of radius 9 m. Another sprinkler waters an area that is 100% larger. What is the radius of lawn reached by the second sprinkler?

14. Determine two numbers that add to 39 and multiply to 360. Use a method other than guess and check.

15. A school has decided to sell T-shirts as a fundraiser. Research shows that 800 students will buy one if they cost \$5 each. For every 50¢ increase in the price, 20 fewer students will buy T-shirts. What is the maximum revenue, and for what price should the shirts sell?

4 Chapter Review

FREQUENTLY ASKED Questions

Q: How can you determine the number of zeros and the number of solutions of a quadratic equation?

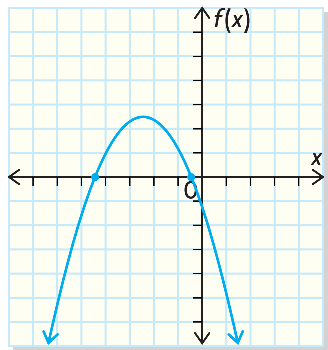
A1: If the function $f(x) = ax^2 + bx + c$ or the equation $0 = ax^2 + bx + c$ is in standard form, substitute the values of a, b, and c into the discriminant, $b^2 - 4ac$, and evaluate.

- If $b^2 - 4ac > 0$, there are two distinct zeros/real solutions.
- If $b^2 - 4ac = 0$, there is one zero/real solution.
- If $b^2 - 4ac < 0$, there are no zeros/real solutions.

A2: Write the function in vertex form. Then you can identify the vertex and whether the parabola opens up or down. For example, if the vertex is above the x-axis and opens down, there will be two distinct zeros.

Study *Aid*

- See Lesson 4.4, Examples 1 and 2.
- Try Chapter Review Question 7.



Q: What strategies can you use to solve problems involving quadratic functions and equations?

A: To determine the maximum or minimum value, you can use a graphing calculator or put the function into vertex form by completing the square.

To solve a quadratic equation:

- A table of values might work but is time-consuming.
- Graphing by hand is also time-consuming.
- Using a graphing calculator is faster and more reliable, especially if the answer is not an integer.
- Factoring works only some of the time and becomes difficult when the numbers in the equation are large.
- The quadratic formula always works, but the quadratic equation must be in standard form.

Study *Aid*

- See Lesson 4.5, Examples 1 to 4.
- Try Chapter Review Questions 9,10, and 11.

Q: What strategies can you use to determine the equation of a curve of good fit of a quadratic function in vertex form from data?

A: Graph the data either by hand or using a graphing calculator to see if the curve looks quadratic. Locate or estimate the coordinates of the vertex. Replace h and k in the vertex form of the quadratic function, $f(x) = a(x - h)^2 + k$. Pick a point that is on the curve to determine the value of a. Verify that your equation matches the data.

Study *Aid*

- See Lesson 4.6, Example 2.
- Try Chapter Review Questions 2, 12, and 13.

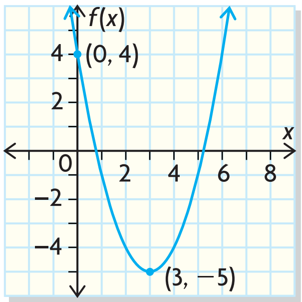
PRACTICE Questions

Lesson 4.1

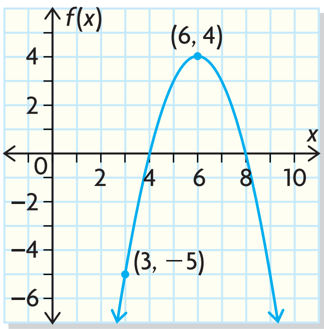
1. Write in standard form.
   a) $f(x) = (x + 3)^2 - 7$
   b) $f(x) = -(x + 7)^2 + 3$
   c) $f(x) = 2(x - 1)^2 + 5$
   d) $f(x) = -3(x - 2)^2 - 4$

2. Write the equation of each graph in vertex form.

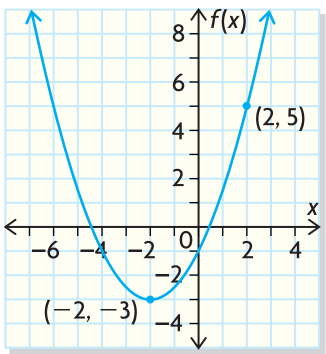
a)



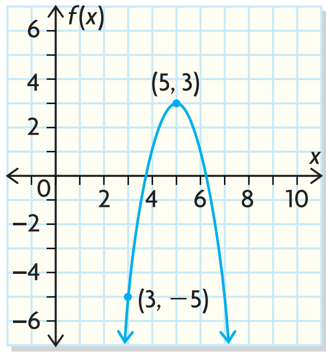
b)



c)



d)



Lesson 4.2

3. Write in vertex form by completing the square.
   a) $f(x) = x^2 + 2x - 15$
   b) $f(x) = -x^2 + 8x - 7$
   c) $f(x) = 2x^2 + 20x + 16$
   d) $f(x) = 3x^2 + 12x + 19$
   e) $f(x) = \frac{1}{2}x^2 - 6x + 26$
   f) $f(x) = 2x^2 + 2x + 4$

4. Determine the vertex, the axis of symmetry, the direction the parabola opens, and the number of zeros for each quadratic function. Sketch a graph of each.
   a) $f(x) = 3(x - 5)^2 - 2$
   b) $g(x) = -2(x + 3)^2 - 1$
   c) $f(x) = 2x^2 + 4x + 7$
   d) $g(x) = -x^2 + 16x - 64$

Lesson 4.3

5. Use the quadratic formula to determine the solutions.
   a) $2x^2 - 15x - 8 = 0$
   b) $3x^2 + x + 7 = 0$
   c) $9x^2 = 6x - 1$
   d) $2.5x^2 = -3.1x + 7$

6. A T-ball player hits a ball from a tee that is 0.6 m tall. The height of the ball at a given time is modelled by the function $h(t) = -4.9t^2 + 7t + 0.6$, where height, $h(t)$, is in metres and time, t, is in seconds.
   a) What will the height be after 1 s?
   b) When will the ball hit the ground?

Lesson 4.4

7. Without solving, determine the number of real solutions of each equation.
   a) $x^2 - 5x + 9 = 0$
   b) $3x^2 - 5x - 9 = 0$
   c) $16x^2 - 8x + 1 = 0$

8. For the function $f(x) = kx^2 + 8x + 5$, what value(s) of k will have
   a) two distinct real solutions?
   b) one real solution?
   c) no real solution?

Lesson 4.5

9. The daily production cost, C, of a special-edition toy car is given by the function $C(t) = 0.2t^2 - 10t + 650$, where $C(t)$ is in dollars and t is the number of cars made.
   a) How many cars must be made to minimize the production cost?
   b) Using the number of cars from part (a), determine the cost.

10. The function $A(w) = 576w - 2w^2$ models the area of a pasture enclosed by a rectangular fence, where w is width in metres.
   a) What is the maximum area that can be enclosed?
   b) Determine the area that can be enclosed using a width of 20 m.
   c) Determine the width of the rectangular pasture that has an area of 18 144 m^2.

Lesson 4.6

11. The vertical height of an arrow at a given time, t, is shown in the table.

Time (s)	0	0.5	1.0	1.5	2.0	2.5
Height (m)	0.5	6.3	9.6	10.5	8.9	4.9

   a) Determine an equation of a curve of good fit in vertex form.
   b) State any restrictions on the domain and range of your function.
   c) Use your equation from part (a) to determine when the arrow will hit a target that is 2 m above the ground.

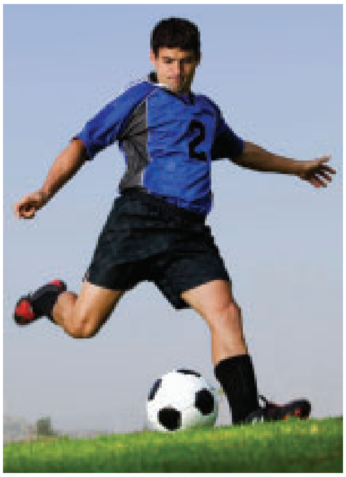
12. The table shows how many injuries resulted from motor vehicle accidents in Canada from 1984 to 1998.

Year	Injuries
1984	237 455
1986	264 481
1988	278 618
1990	262 680
1992	249 821
1994	245 110
1996	230 890
1998	217 614

   a) Create a scatter plot, and draw a curve of good fit.
   b) Determine an equation of a curve of good fit.
   c) Check the accuracy of your model using quadratic regression.
   d) Use one of your models to predict how many accidents will result in injury in 2002. Explain why you chose the model you did.
   e) According to the quadratic regression model, when were accidents that resulted in injury at their maximum levels?

4 Chapter Self-Test

1. Write in standard form.
   a) $f(x) = (x + 3)^2 - 7$
   b) $f(x) = -3(x + 5)^2 + 2$
2. Write in vertex form by completing the square.
   a) $f(x) = x^2 - 10x + 33$
   b) $f(x) = -5x^2 + 20x - 12$
3. Determine the vertex, the equation of the axis of symmetry, and the maximum or minimum value for each function. Sketch a graph of each.
   a) $f(x) = 2(x - 8)^2 + 3$
   b) $f(x) = -3x^2 + 42x - 141$
4. Can all quadratic equations be solved by the quadratic formula? Explain.
5. Solve using the quadratic formula.
   a) $x^2 - 6x - 8 = 0$
   b) $3x^2 + x = -9$
6. Use the discriminant to determine the number of real solutions of each quadratic equation.
   a) $5x^2 - 4x + 11 = 0$
   b) $x^2 = 8x - 16$



7. The height of a soccer ball is modelled by $h(t) = -4.9t^2 + 19.6t + 0.5$, where height, $h(t)$, is in metres and time, t, is in seconds.
   a) What is the maximum height the ball reaches?
   b) What is the height of the ball after 1 s?
8. The profit, $P(t)$, made at a fair depends on the price of the ticket, t. The profit is modelled by the function $P(t) = -37t^2 + 1776t - 7500$.
   a) What is the maximum profit?
   b) What is the price of a ticket that gives the maximum profit?
9. The price of an ice cream cone and the revenue generated for each price are listed in the table.
   a) What is an equation for a curve of good fit?

Price ($)	0.50	1.00	1.50	2.00	2.50	3.00	3.50
Revenue ($)	2092	3340	4060	4180	3700	2620	940

   b) How do you know whether it is a good fit?
   c) State any restrictions on the domain and range of your model.
   d) Use your equation to calculate the revenue if the price of an ice cream cone is $2.25.
10. Why might someone choose to write a curve of good fit in vertex form rather than factored form?

4 Chapter Task

Water Fountain

The water flowing from a fountain forms a parabola. If you adjust the pressure, the shape of the flow changes.

YOU WILL NEED

- pipe cleaners or flexible wire, such as copper
- graph paper

? How does the change in pressure affect the equation of a curve of good fit?

A. Turn the water in the fountain on at about half pressure. Shape a piece of wire so that it matches the shape produced by the flow of water.

B. Measure the height from the bowl to the point where the water leaves the spout of the fountain.

C. On graph paper mark a point on the y-axis above the origin that corresponds to the height of the fountain's spout. Using this point as the starting point of your graph, copy the shape of your wire onto the graph paper.

D. Repeat the process for two different water pressures. In the first trial, increase the pressure over the original pressure. In the second trial, decrease the pressure.

E. Determine a curve of good fit, in vertex form, for all of your trials.

F. In a report, indicate
- how you manually determined the vertex form of the equation that fits your data
- the characteristics of each of your curves of good fit (vertex, axis of symmetry, domain, and range)
- how well you think the equation fits your data
- how the change in water pressure affected the equation of the curve of good fit

Task Checklist

- ✔ Did you show all your steps?
- ✔ Did you include a graph?
- ✔ Did you support your choice of data?
- ✔ Did you explain your thinking clearly?