Appendix A

Review of Essential Skills and Knowledge

A–1 Operations with Integers

Set of integers $\mathbf{I} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Addition

To add two integers,

- if the signs are the same, then the sum has the same sign as well: $(-12) + (-5) = -17$
- if the signs are different, then the sum takes the sign of the larger number: $18 + (-5) = 13$

Subtraction

Add the opposite:

$$\begin{aligned} -15 - (-8) &= -15 + 8 \\ &= -7 \end{aligned}$$

Multiplication and Division

To multiply or divide two integers,

- if the two integers have the same sign, then the answer is positive: $6 \times 8 = 48$, $(-36) \div (-9) = 4$
- if the two integers have different signs, then the answer is negative: $(-5) \times 9 = -45$, $54 \div (-6) = -9$

More Than One Operation

Follow the order of operations.

B	Brackets	
E	Exponents	
D **M**	Division Multiplication	} from left to right
A **S**	Addition Subtraction	} from left to right

EXAMPLE

Evaluate.

a) $-10 + (-12)$

b) $(-12) + 7$

c) $(-11) + (-4) + 12 + (-7) + 18$

d) $(-6) \times 9 \div 3$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$

Solution

a) $-10 + (-12) = -22$

b) $(-12) + 7 = -5$

c) $(-11) + (-4) + 12 + (-7) + 18$
$= (-22) + 30$
$= 8$

d) $(-6) \times 9 \div 3$
$= -54 \div 3$
$= -18$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$

$= \dfrac{20 + 4}{8 \div (-2)}$

$= \dfrac{24}{-4}$

$= -6$

Practising

1. Evaluate.

a) $6 + (-3)$
b) $12 - (-13)$
c) $-17 - 7$
d) $(-23) + 9 - (-4)$
e) $24 - 36 - (-6)$
f) $32 + (-10) + (-12) - 18 - (-14)$

2. Which choice would make each statement true: $>$, $<$, or $=$?

a) $-5 - 4 - 3 + 3 \blacksquare -4 - 3 - 1 - (-2)$
b) $4 - 6 + 6 - 8 \blacksquare -3 - 5 - (-7) - 4$
c) $8 - 6 - (-4) - 5 \blacksquare 5 - 13 - 7 - (-8)$
d) $5 - 13 + 7 - 2 \blacksquare 4 - 5 - (-3) - 5$

3. Evaluate.

a) $(-11) \times (-5)$
b) $(-3)(5)(-4)$
c) $35 \div (-5)$
d) $(-72) \div (-9)$
e) $(5)(-9) \div (-3)(7)$
f) $56 \div [(8)(7)] \div 49$

4. Evaluate.

a) $(-3)^2 - (-2)^2$
b) $(-5)^2 - (-7) + (-12)$
c) $-4 + 20 \div (-4)$
d) $-3(-4) + 8^2$
e) $(-16) - [(-8) \div 2]$
f) $8 \div (-4) + 4 \div (-2)^2$

5. Evaluate.

a) $\dfrac{-12 - 3}{-3 - 2}$

b) $\dfrac{-18 + 6}{(-3)(-4)}$

c) $\dfrac{(-16 + 4) \div 2}{8 \div (-8) + 4}$

d) $\dfrac{-5 + (-3)(-6)}{(-2)^2 + (-3)^2}$

A–2 Operations with Rational Numbers

Set of rational numbers $\mathbf{Q} = \left\{ \frac{a}{b} \middle| \, a, b \in I, b \neq 0 \right\}$

Addition and Subtraction
To add or subtract rational numbers, you need to find a common denominator.

Division
To divide by a rational number, multiply by the reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$
$$= \frac{ad}{bc}$$

Multiplication

$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, but first reduce to lowest terms where possible.

More Than One Operation
Follow the order of operations.

EXAMPLE 1

Simplify $\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10}$.

Solution

$$\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10} = \frac{-4}{10} + \frac{-15}{10} - \frac{3}{10}$$
$$= \frac{-4 - 15 - 3}{10}$$
$$= \frac{-22}{10}$$
$$= -\frac{11}{5} \text{ or } -2\frac{1}{5}$$

EXAMPLE 2

Simplify $\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7}$.

Solution

$$\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7} = \frac{3}{4} \times \frac{-4}{5} \times \frac{7}{-3}$$
$$= \frac{\cancel{3}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{-4}^{-1}}{5} \times \frac{7}{\cancel{-3}_{-1}}$$
$$= \frac{7}{5} \text{ or } 1\frac{2}{5}$$

Practising

1. Evaluate.

a) $\frac{1}{4} + \frac{-3}{4}$

b) $\frac{1}{2} - \frac{-2}{3}$

c) $\frac{-1}{4} - 1\frac{1}{3}$

d) $-8\frac{1}{4} - \frac{-1}{-3}$

e) $\frac{-3}{5} + \frac{-3}{4} - \frac{7}{10}$

f) $\frac{2}{3} - \frac{-1}{2} - \frac{1}{6}$

2. Evaluate.

a) $\frac{4}{5} \times \frac{-20}{25}$

b) $\frac{3}{-2} \times \frac{6}{5}$

c) $\left(\frac{-1}{3}\right)\left(\frac{2}{-5}\right)$

d) $\left(\frac{9}{4}\right)\left(\frac{-2}{-3}\right)$

e) $\left(-1\frac{1}{10}\right)\left(3\frac{1}{11}\right)$

f) $-4\frac{1}{6} \times \left(-7\frac{3}{4}\right)$

3. Evaluate.

a) $\frac{-4}{3} \div \frac{2}{-3}$

b) $-7\frac{1}{8} \div \frac{3}{2}$

c) $\frac{-2}{3} \div \frac{-3}{8}$

d) $\frac{-3}{-2} \div \left(\frac{-1}{3}\right)$

e) $-6 \div \left(\frac{-4}{5}\right)$

f) $\left(-2\frac{1}{3}\right) \div \left(-3\frac{1}{2}\right)$

4. Simplify.

a) $\frac{-2}{5} - \left(\frac{-1}{10} + \frac{1}{-2}\right)$

b) $\frac{-3}{5}\left(\frac{-3}{4} - \frac{-1}{4}\right)$

c) $\left(\frac{3}{5}\right)\left(\frac{1}{-6}\right)\left(\frac{-2}{3}\right)$

d) $\left(\frac{-2}{3}\right)^2\left(\frac{1}{-2}\right)^3$

e) $\left(\frac{-2}{5} + \frac{1}{-2}\right) \div \left(\frac{5}{-8} - \frac{-1}{2}\right)$

f) $\dfrac{\frac{-4}{5} - \frac{-3}{5}}{\frac{1}{3} - \frac{-1}{5}}$

A–3 Exponent Laws

3^4 (exponent, base) and a^n are called powers

4 factors of 3
$3^4 = (3)(3)(3)(3)$

n factors of a
$a^n = (a)(a)(a)\ldots(a)$

Operations with powers follow a set of procedures or rules.

Rule	Description	Algebraic Expression	Example
Multiplication	When the bases are the same, keep the base the same and add exponents.	$(a^m)(a^n) = a^{m+n}$	$(5^4)(5^{-3}) = 5^{4+(-3)} = 5^{4-3} = 5^1 = 5$
Division	When the bases are the same, keep the base the same and subtract exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{4^6}{4^{-2}} = 4^{6-(-2)} = 4^{6+2} = 4^8$
Power of a Power	Keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^{(2)(4)} = 3^8$

EXAMPLE

Simplify and evaluate.
$3(3^7) \div (3^3)^2$

Solution

$$\begin{aligned}3(3^7) \div (3^3)^2 &= 3^{1+7} \div 3^{3\times 2}\\ &= 3^8 \div 3^6\\ &= 3^{8-6}\\ &= 3^2\\ &= 9\end{aligned}$$

Practising

1. Evaluate to three decimal places where necessary.

a) 4^2
b) 5^0
c) 3^2
d) -3^2
e) $(-5)^3$
f) $\left(\frac{1}{2}\right)^3$

2. Evaluate.

a) $3^0 + 5^0$
b) $2^2 + 3^3$
c) $5^2 - 4^2$
d) $\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^2$
e) $-2^5 + 2^4$
f) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2$

3. Evaluate to an exact answer.

a) $\frac{9^8}{9^7}$
b) $\frac{2(5^5)}{5^3}$
c) $(4^5)(4^2)^3$
d) $\frac{(3^2)(3^3)}{(3^4)^2}$

4. Simplify.

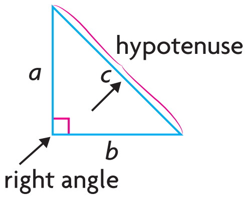
a) $(x)^5(x)^3$
b) $(m)^2(m)^4(m)^3$
c) $(y)^5(y)^2$
d) $(a^b)^c$
e) $\frac{(x^5)(x^3)}{x^2}$
f) $\left(\frac{x^4}{y^3}\right)^3$

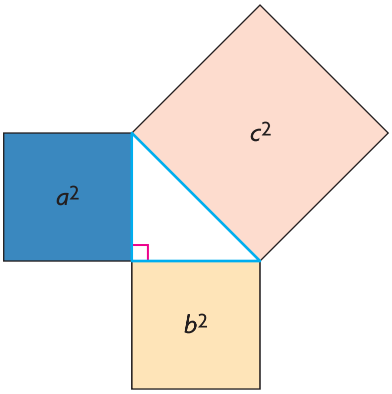
5. Simplify.

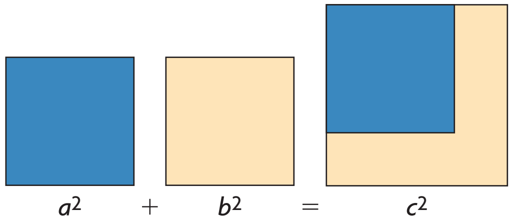
a) $(x^2y^4)(x^3y^2)$
b) $(-2m^3)^2(3m^2)^3$
c) $\frac{(5x^2)^2}{(5x^2)^0}$
d) $(4u^3v^2)^2 \div (-2u^2v^3)^2$

A–4 The Pythagorean Theorem

The three sides of a right triangle are related to each other in a unique way. Every right triangle has a longest side, called the **hypotenuse**, which is always opposite the right angle. One of the important relationships in mathematics is known as the **Pythagorean theorem**. It states that the area of the square of the hypotenuse is equal to the sum of the areas of the squares of the other two sides.



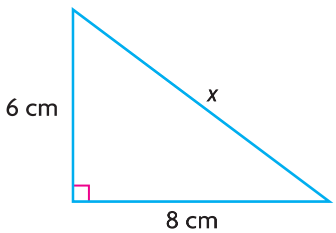




Practising

1. For each right triangle, write the equation for the Pythagorean theorem.

a)



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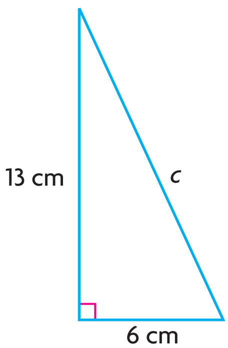

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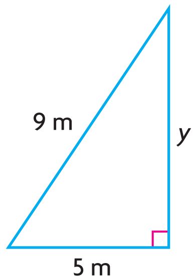

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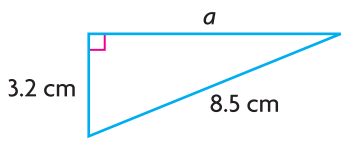


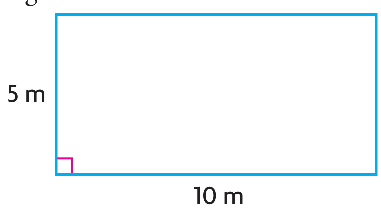
2. Calculate the length of the unknown side of each triangle in question 1. Round all answers to one decimal place.

3. Find the value of each unknown measure to the nearest hundredth.

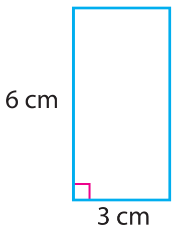
a) $a^2 = 5^2 + 13^2$
b) $10^2 = 8^2 + m^2$
c) $26^2 = b^2 + 12^2$
d) $2.3^2 + 4.7^2 = c^2$

4. Determine the length of the diagonals of each rectangle to the nearest tenth.

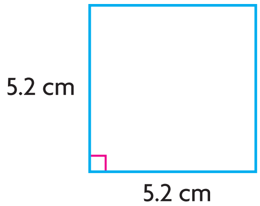
a)



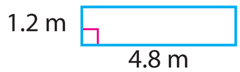
b)



c)



d)



5. An isosceles triangle has a hypotenuse 15 cm long. Determine the length of the two equal sides.

6. An apartment building casts a shadow. From the tip of the shadow to the top of the building is 100 m. The tip of the shadow is 72 m from the base of the building. How tall is the building?

A–5 Evaluating Algebraic Expressions and Formulas

To evaluate algebraic expressions and formulas, substitute the given numbers for the variables. Then follow the order of operations to calculate the answer.

EXAMPLE 1

Find the value of $2x^2 - y$ if $x = -2$ and $y = 3$.

Solution

$$\begin{aligned} 2x^2 - y &= 2(-2)^2 - 3 \\ &= 2(4) - 3 \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

EXAMPLE 2

The formula for finding the volume of a cylinder is $V = \pi r^2 h$. Find the volume of a cylinder with a radius of 2.5 cm and a height of 7.5 cm.

Solution

$$\begin{aligned} V &= \pi r^2 h \\ &\doteq (3.14)(2.5)^2(7.5) \\ &= (3.14)(6.25)(7.5) \\ &\doteq 147 \text{ cm}^3 \end{aligned}$$

Practising

1. Find the value of each expression for $x = -5$ and $y = -4$.

a) $-4x - 2y$

b) $-3x - 2y^2$

c) $(3x - 4y)^2$

d) $\left(\frac{x}{y}\right) - \left(\frac{y}{x}\right)$

2. If $x = -\frac{1}{2}$ and $y = \frac{2}{3}$, find the value of each expression.

a) $x + y$

b) $x + 2y$

c) $3x - 2y$

d) $\frac{1}{2}x - \frac{1}{2}y$

3. a) The formula for the area of a triangle is $A = \frac{1}{2}bh$. Find the area of a triangle when $b = 13.5$ cm and $h = 12.2$ cm.

b) The area of a circle is found using the formula $A = \pi r^2$. Find the area of a circle with a radius of 4.3 m.

c) The hypotenuse of a right triangle, c, is found using the formula $c = \sqrt{a^2 + b^2}$. Find the length of the hypotenuse when $a = 6$ m and $b = 8$ m.

d) A sphere's volume is calculated using the formula $V = \frac{4}{3}\pi r^3$. Determine the volume of a sphere with a radius of 10.5 cm.

A–6 Finding Intercepts of Linear Relations

A linear relation of the general form $Ax + By + C = 0$ has an x-intercept and a y-intercept—the points where the line $Ax + By + C = 0$ crosses the x-axis and y-axis, respectively.

EXAMPLE 1 $Ax + By + C = 0$ FORM

Determine the x- and y-intercepts of the linear relation $2x + y - 6 = 0$.

Solution

The x-intercept is where the relation crosses the x-axis. The x-axis has equation $y = 0$, so substitute $y = 0$ into $2x + y - 6 = 0$:

$$\begin{aligned} 2x + y - 6 &= 0 \\ 2x + 0 - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

To find the y-intercept, substitute $x = 0$ into $2x + y - 6 = 0$:

$$\begin{aligned} 2x + y - 6 &= 0 \\ 2(0) + y - 6 &= 0 \\ y &= 6 \end{aligned}$$

The x-intercept is at $(3, 0)$ and the y-intercept is at $(0, 6)$.

EXAMPLE 2 ANY FORM

Determine the x- and y-intercepts of the linear relation $3y = 18 - 2x$.

Solution

To find the x- and y-intercepts, substitute $y = 0$ and $x = 0$, respectively.

$$\begin{aligned} 3y &= 18 - 2x \qquad & 3y &= 18 - 2x \\ 3(0) &= 18 - 2x \qquad & 3y &= 18 - 2(0) \\ 2x &= 18 \qquad & 3y &= 18 \\ x &= 9 \qquad & y &= 6 \end{aligned}$$

The x-intercept is at $(9, 0)$ and the y-intercept is at $(0, 6)$.

A special case is when the linear relation is a horizontal or vertical line.

EXAMPLE 3

Find either the x- or the y-intercept of each linear relation.

a) $2x = -14$

b) $3y + 48 = 0$

Solution

a) $2x = -14$ is a vertical line, so it has no y-intercept. To find the x-intercept, solve for x.

$$2x = -14$$
$$x = -7$$

The x-intercept is at $(-7, 0)$.

b) $3y + 48 = 0$ is a horizontal line, so it has no x-intercept. To find the y-intercept, solve for y.

$$3y + 48 = 0$$
$$3y = -48$$
$$y = -16$$

The y-intercept is at $(0, -16)$.

Practising

1. Determine the x- and y-intercepts of each linear relation.

a) $x + 3y - 3 = 0$
b) $2x - y + 14 = 0$
c) $-x + 2y + 6 = 0$
d) $5x + 3y - 15 = 0$
e) $10x - 10y + 100 = 0$
f) $-2x + 5y - 15 = 0$

2. Determine the x- and y-intercepts of each linear relation.

a) $x = y + 7$
b) $3y = 2x + 6$
c) $y = 4x + 12$
d) $3x = 5y - 30$
e) $2y - x = 7$
f) $12 = 6x - 5y$

3. Find either the x- or the y-intercept of each linear relation.

a) $x = 13$
b) $y = -6$
c) $2x = 14$
d) $3x + 30 = 0$
e) $4y = -6$
f) $24 - 3y = 0$

4. A ladder resting against a wall is modelled by the linear relation $2y + 9x = 13.5$. The x-axis represents the ground and the y-axis represents the wall.

a) Determine the intercepts of the relation.
b) Using the intercept points, graph the relation.
c) What can you conclude about the foot and the top of the ladder?

A–7 Graphing Linear Relationships

The graph of a linear relationship ($Ax + By + C = 0$) is a straight line. The graph can be drawn if at least two ordered pairs of the relationship are known. This information can be determined in several different ways.

EXAMPLE 1 Table of Values

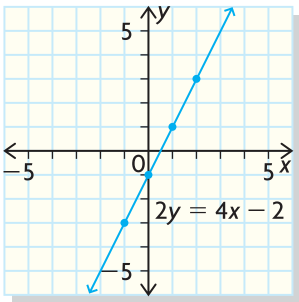
Sketch the graph of $2y = 4x - 2$.

Solution

A table of values can be created. Express the equation in the form $y = mx + b$.

$$\frac{2y}{2} = \frac{4x - 2}{2}$$
$$y = 2x - 1$$

x	y
-1	$2(-1) - 1 = -3$
0	$2(0) - 1 = -1$
1	$2(1) - 1 = 1$
2	$2(2) - 1 = 3$



EXAMPLE 2 Using Intercepts

Sketch the graph of $2x + 4y = 8$.

Solution

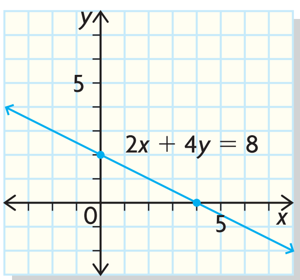
The intercepts of the line can be found.
For the x-intercept, let $y = 0$.

$$2x + 4(0) = 8$$
$$2x = 8$$
$$x = 4$$

For the y-intercept, let $x = 0$.

$$2(0) + 4y = 8$$
$$4y = 8$$
$$y = 2$$

x	y
4	0
0	2



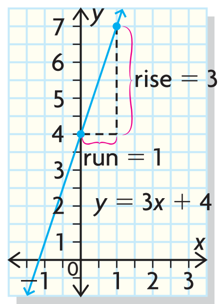
EXAMPLE 3 Using the Slope and y-intercept

Sketch the graph of $y = 3x + 4$.

Solution

When the equation is in the form $y = mx + b$, m is the slope and b is the y-intercept.

For $y = 3x + 4$, the line has a slope of 3 and a y-intercept of 4.



Practising

1. Express each equation in the form $y = mx + b$.

a) $3y = 6x + 9$ **b)** $2x - 4y = 8$ **c)** $3x + 6y - 12 = 0$ **d)** $5x = y - 9$

2. Graph each equation, using a table of values where $x \in \{-2, -1, 0, 1, 2\}$.

a) $y = 3x - 1$ **b)** $y = \frac{1}{2}x + 4$ **c)** $2x + 3y = 6$ **d)** $y = 4$

3. Determine the x- and y-intercepts of each equation.

a) $x + y = 10$ **b)** $2x + 4y = 16$ **c)** $50 - 10x - y = 0$ **d)** $\frac{x}{2} + \frac{y}{4} = 1$

4. Graph each equation by determining the intercepts.

a) $x + y = 4$ **b)** $x - y = 3$ **c)** $2x + 5y = 10$ **d)** $3x - 4y = 12$

5. Graph each equation, using the slope and y-intercept.

a) $y = 2x + 3$ **b)** $y = \frac{2}{3}x + 1$ **c)** $y = -\frac{3}{4}x - 2$ **d)** $2y = x + 6$

6. Graph each equation. Use the most suitable method.

a) $y = 5x + 2$ **b)** $3x - y = 6$ **c)** $y = -\frac{2}{3}x + 4$ **d)** $4x = 20 - 5y$

A–8 Graphing Quadratic Relations

The graph of a quadratic relation is called a **parabola**. All parabolas have a vertex (the lowest or highest point of the curve), an axis of symmetry (the vertical line through the vertex), and a y-intercept. However, a parabola may have 0, 1, or 2 x-intercepts, or **zeros**. To graph a quadratic relation, begin by creating a table of values.

EXAMPLE 1

Graph each quadratic relation. Use your graph to determine

i) the vertex of the parabola
ii) the axis of symmetry
iii) the y-intercept
iv) the x-intercept(s), if any

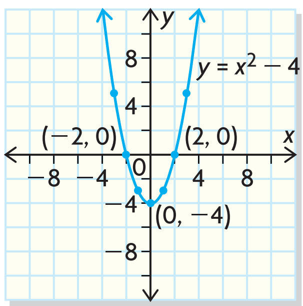
a) $y = x^2 - 4$ **b)** $y = -2x^2 + 4x - 2$

Solution

a) Create a table of values.

x	y
-3	$(-3)^2 - 4 = 5$
-2	$(-2)^2 - 4 = 0$
-1	$(-1)^2 - 4 = -3$
0	$(0)^2 - 4 = -4$
1	$(1)^2 - 4 = -3$
2	$(2)^2 - 4 = 0$
3	$(3)^2 - 4 = 5$

Use these values to plot the graph.

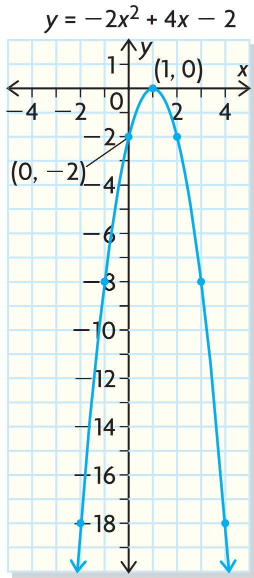


i) From the graph, the vertex is the point $(0, -4)$ (a minimum for this relation).
ii) The axis of symmetry (the y-axis, in this case) passes through $(0, -4)$, so its equation is $x = 0$.
iii) The y-intercept is at $(0, -4)$.
iv) There are two x-intercepts, at $(-2, 0)$ and $(2, 0)$.

b) Create a table of values.

x	y
-2	$-2(-2)^2 + 4(-2) - 2 = -18$
-1	$-2(-1)^2 + 4(-1) - 2 = -8$
0	$-2(0)^2 + 4(0) - 2 = -2$
1	$-2(1)^2 + 4(1) - 2 = 0$
2	$-2(2)^2 + 4(2) - 2 = -2$
3	$-2(3)^2 + 4(3) - 2 = -8$
4	$-2(4)^2 + 4(4) - 2 = -18$

Use these values to plot the graph.



i) The vertex is the point $(1, 0)$ (a maximum for this relation).

ii) The axis of symmetry passes through $(1, 0)$, so its equation is $x = 1$.

iii) The y-intercept is at $(0, -2)$.

iv) There is one x-intercept, at $(1, 0)$.

Practising

1. Graph each quadratic relation. Use your graph to determine

i) the vertex of the parabola

ii) the axis of symmetry

iii) the y-intercept

iv) the x-intercept(s), if any

a) $y = x^2 - 1$

b) $y = 3x^2$

c) $y = 2x^2 + 2$

d) $y = x^2 + 4x + 4$

e) $y = 8 - 2x^2$

f) $y = (x - 2)^2 + 3$

g) $y = (x + 3)(x - 2)$

h) $y = 4 - 2(x + 1)^2$

2. A stone is thrown from a cliff. Its motion is described by $y = -5x^2 + 10x + 20$. In this quadratic relation, y is the stone's height above the sea (in metres) and x is the horizontal distance (in metres) the stone has travelled.

a) Graph the quadratic relation.

b) Locate the vertex. What is the stone's maximum height, and how far is it horizontally from the edge of the cliff at this point?

c) What information does the y-intercept give you about the cliff?

d) Locate the x-intercepts. Is the intercept with x negative meaningful? Why or why not? What is the meaning of the other x-intercept?

A–9 Expanding and Simplifying Algebraic Expressions

Type	Description	Example
Collecting Like Terms $2a + 3a = 5a$	Add or subtract the coefficients of the terms that have the same variables and exponents.	$3a - 2b - 5a + b$ $= 3a - 5a - 2b + b$ $= -2a - b$
Distributive Property $a(b + c) = ab + ac$	Multiply each term of the binomial by the monomial.	$-4a(2a - 3b)$ $= -8a^2 + 12ab$
Product of Two Binomials $(a + b)(c + d)$ $= ac + ad + bc + bd$	Multiply the first term of the first binomial by the second binomial, and then multiply the second term of the first binomial by the second binomial. Collect like terms if possible.	$(2x^2 - 3)(5x^2 + 2)$ $= 10x^4 + 4x^2 - 15x^2 - 6$ $= 10x^4 - 11x^2 - 6$

Practising

1. Simplify.
a) $3x + 2y - 5x - 7y$
b) $5x^2 - 4x^3 + 6x^2$
c) $(4x - 5y) - (6x + 3y) - (7x + 2y)$
d) $m^2n + p - (2p - 3m^2n)$

2. Expand.
a) $3(2x + 5y - 2)$
b) $5x(x^2 - x + y)$
c) $m^2(3m^2 - 2n)$
d) $x^5y^3(4x^2y^4 - 2xy^5)$

3. Expand and simplify.
a) $3x(x + 2) + 5x(x - 2)$
b) $-7h(2h + 5) - 4h(5h - 3)$
c) $2m^2n(m^3 - n) - 5m^2n(3m^3 + 4n)$
d) $-3xy^3(5x + 2y + 1) + 2xy^3(-3y - 2 + 7x)$

4. Expand and simplify.
a) $(3x - 2)(4x + 5)$
b) $(7 - 3y)(2 + 4y)$
c) $(5x - 7y)(4x + y)$
d) $(3x^3 - 4y^2)(5x^3 + 2y^2)$

A–10 Factoring Algebraic Expressions

Common Factors
Sometimes, the terms in an algebraic expression have a common factor.

EXAMPLE 1

Factor the expression $3x + 12xy$.

Solution

Divide out a common factor from each term.

$3x + 12xy$
$= 3x(1) + 3x(4y)$
$= 3x(1 + 4y)$

Factoring Trinomials
$ax^2 + bx + c$, when $a = 1$:
Write as the product of two binomials. Determine two numbers whose sum is b and whose product is c.

EXAMPLE 2

Factor the expression $x^2 - 5x + 6$.

Solution

Find two numbers whose product is 6 and whose sum is -5.

$6 = (-2)(-3)$
and $-5 = (-2) + (-3)$
$x^2 - 5x + 6$
$= (x - 2)(x - 3)$

A special form of algebraic expression is called the difference of two squares; for example, the expression $x^2 - 49$.

EXAMPLE 3

Factor each expression.

a) $x^2 - 49$ **b)** $9y^2 - 16x^2$

Solution

a) $x^2 - 49$
$= x^2 - 7^2$
$= (x + 7)(x - 7)$
Check:
$(x + 7)(x - 7)$
$= x^2 + 7x - 7x + 49$ Eliminate middle terms.
$= x^2 - 49$

b) $9y^2 - 16x^2$
$= (3y)^2 - (4x)^2$
$= (3y + 4x)(3y - 4x)$

Sometimes you may need to use several strategies to factor an expression.

EXAMPLE 4

Factor $2x^2 - 10x - 48$.

Solution

$2x^2 - 10x - 48$	Divide out the common factor of 2.
$= 2(x^2 - 5x - 24)$	Factor the trinomial.
$= 2(x - 8)(x + 3)$	

Practising

1. Factor each expression.
- **a)** $ab + 2a$
- **b)** $4x + 6$
- **c)** $3y - 9xy$
- **d)** $7a + a^2$
- **e)** $77b^3 + 55b^2$
- **f)** $21a + 6ab - 15a^2$

2. Factor each trinomial.
- **a)** $x^2 - 2x + 1$
- **b)** $a^2 + 3a + 2$
- **c)** $x^2 - 3x - 28$
- **d)** $z^2 - 2z - 8$
- **e)** $3a^2 + 6a + 3$
- **f)** $5x^2 - 10x - 15$

3. Which expressions are differences of two squares?
- **a)** $x^2 - 9$
- **b)** $2x^2 - 27$
- **c)** $9a^2 - 25b^2$
- **d)** $25 - 3x^2$
- **e)** $z^2 - 441$
- **f)** $16xy - z^2$

4. Factor these expressions, if possible.
- **a)** $x^2 - 81$
- **b)** $4 - 18z^2$
- **c)** $4a^2 - 1$
- **d)** $16x^2 - 16$
- **e)** $369 - 4x^2$
- **f)** $400 - 16xy$

A–11 Solving Linear Equations Algebraically

To solve a linear equation, first eliminate any fractions by multiplying each term in the equation by the lowest common denominator. Eliminate any brackets by using the distributive property, and then isolate the variable. A linear equation has only one solution.

EXAMPLE 1

Solve $-3(x + 2) - 3x = 4(2 - 5x)$.

Solution

$$\begin{aligned} -3(x+2) - 3x &= 4(2 - 5x) \\ -3x - 6 - 3x &= 8 - 20x \\ -3x - 3x + 20x &= 8 + 6 \\ 14x &= 14 \\ x &= \frac{14}{14} \\ x &= 1 \end{aligned}$$

EXAMPLE 2

Solve $\frac{y-7}{3} = \frac{y-2}{4}$.

Solution

$$\begin{aligned} \frac{y-7}{3} &= \frac{y-2}{4} \\ 12\left(\frac{y-7}{3}\right) &= 12\left(\frac{y-2}{4}\right) \\ 4(y-7) &= 3(y-2) \\ 4y - 28 &= 3y - 6 \\ 4y - 3y &= -6 + 28 \\ y &= 22 \end{aligned}$$

Practising

1. Solve.

a) $6x - 8 = 4x + 10$
b) $2x + 7.8 = 9.4$
c) $13 = 5m - 2$
d) $13.5 - 2m = 5m + 41.5$
e) $8(y - 1) = 4(y + 4)$
f) $4(5 - r) = 3(2r - 1)$

2. Determine the root of each equation.

a) $\frac{x}{5} = 20$
b) $\frac{2}{5}x = 8$
c) $4 = \frac{3}{2}m + 3$
d) $\frac{5}{7}y = 3 + 12$
e) $3y - \frac{1}{2} = \frac{2}{3}$
f) $4 - \frac{m}{3} = 5 + \frac{m}{2}$

3. a) What is the height of a triangle with an area of 15 cm^2 and a base of 5 cm?

b) A rectangular lot has a perimeter of 58 m and is 13 m wide. How long is the lot?

4. At the December concert, 209 tickets were sold. There were 23 more student tickets sold than twice the number of adult tickets. How many of each were sold?

A–12 Pattern Recognition and Difference Tables

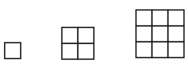
When the independent variable in a relation changes by a steady amount, the dependent variable often changes according to a pattern.

<table>
<tr><th>Type of Pattern</th><th>Description</th><th>Example</th></tr>
<tr><td>Linear</td><td>The first differences between dependent variables are constant.</td><td>
<table>
<tr><th>Number of Books (× 1000)</th><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><th>Cost of Books ($) (× 1000)</th><td>60</td><td>70</td><td>80</td><td>90</td><td>100</td></tr>
<tr><th>First Differences (× 1000)</th><td>10</td><td>10</td><td>10</td><td>10</td><td></td></tr>
</table>
In a linear relation, each first difference is always the same. Here, each first difference is $10 000.</td></tr>
<tr><td>Quadratic</td><td>The first differences between dependent variables change, but the second differences are constant.</td><td>
<table>
<tr><th>Time (s)</th><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><th>Height of Ball (m)</th><td>5</td><td>30.1</td><td>45.4</td><td>50.9</td><td>46.6</td><td>32.5</td></tr>
<tr><th>First Differences</th><td>25.1</td><td>15.3</td><td>5.5</td><td>−4.3</td><td>−14</td><td></td></tr>
<tr><th>Second Differences</th><td>−9.8</td><td>−9.8</td><td>−9.8</td><td>−9.8</td><td></td><td></td></tr>
</table>
In this case, the first differences are not the same, so the relation is nonlinear. In a quadratic relation, each second difference is always the same. Here, each second difference is −9.8.</td></tr>
</table>

Practising

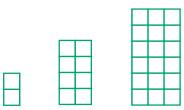
1. Examine each pattern.

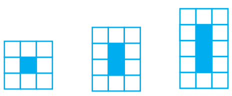
a) 

b) 

c) 

d) 

e) 

f) 

i) Draw the next diagram.
ii) Write the four numbers that represent the four diagrams, in terms of the number of squares or triangles.
iii) Determine the fifth number.

2. Suppose each diagram in question 1 were made from toothpicks.
i) What numbers represent each pattern of four diagrams, in terms of the number of toothpicks?
ii) How many toothpicks will be in the fifth diagram?

3. Examine the data in each table.

i) Is the pattern of the dependent variable linear or quadratic?

ii) State the next value of y.

a)

x	3	4	5	6
y	9	11	13	15

b)

x	2	4	6	8
y	−13	−49	−109	−193

c)

x	0	1	2	3
y	1	0.5	0.1	−0.2

d)

x	0	2	4	6
y	1	142	283	424

4. Neville has 49 m of fencing to build a dog run. Here are some dimensions he considered to maximize the area. Describe the type of pattern displayed by **(a)** the width and **(b)** the area.

Length (m)	1	2	3	4	5
Width (m)	23.5	22.5	21.5	20.5	19.5
Area (m^2)	23.5	45.0	64.5	82.0	97.5

5. What pattern does the change in the cost of renting a pickup truck show?

Distance (km)	0	100	200	300	400
Rental Cost ($)	35.00	41.00	47.00	53.00	59.00

6. Predict the next number in each pattern.

a) −3, 5, −7, 9, −11, ■

b) −4, −16, −36, −64, ■

c) $\frac{1}{8}, \frac{11}{24}, \frac{19}{24}, \frac{27}{24}, \frac{35}{24},$ ■

d) 0.3, 0.6, 1.2, 2.4, ■

A–13 Creating Scatter Plots and Lines or Curves of Good Fit

A **scatter plot** is a graph that shows the relationship between two sets of numeric data. The points in a scatter plot often show a general pattern, or **trend**. A line that approximates a trend for the data in a scatter plot is called a **line of best fit**. A line of best fit passes through as many points as possible, with the remaining points grouped equally above and below the line.

Data that have a **positive correlation** have a pattern that slopes up and to the right. Data that have a **negative correlation** have a pattern that slopes down and to the right. If the points nearly form a line, then the correlation is strong. If the points are dispersed, but still form some linear pattern, then the correlation is weak.

EXAMPLE 1

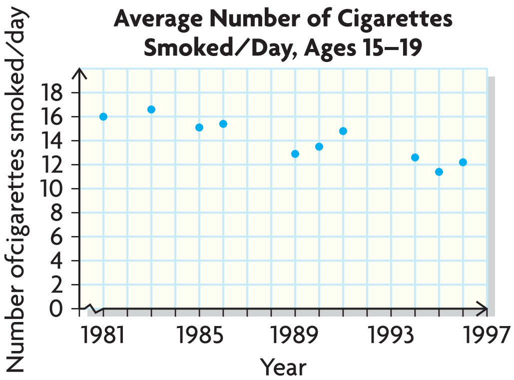
a) Make a scatter plot of the data and describe the kind of correlation the scatter plot shows.
b) Draw the line of best fit.

Long-Term Trends in Average Number of Cigarettes Smoked per Day by Smokers Aged 15–19

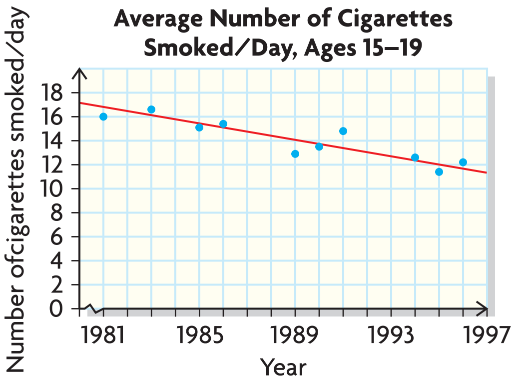
Year	1981	1983	1985	1986	1989	1990	1991	1994	1995	1996
Number per Day	16.0	16.6	15.1	15.4	12.9	13.5	14.8	12.6	11.4	12.2

Solution

a)



b)



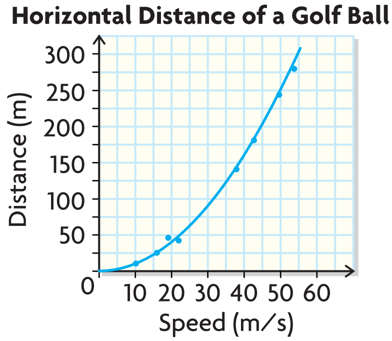
The scatter plot shows a negative correlation.

EXAMPLE 2

A professional golfer is taking part in a scientific investigation. Each time she drives the ball from the tee, a motion sensor records the initial speed of the ball. The final horizontal distance of the ball from the tee is also recorded. Here are the results:

Speed (m/s)	10	16	19	22	38	43	50	54
Distance (m)	10	25	47	43	142	182	244	280

Draw the line or curve of good fit.



Solution

The scatter plot shows that a line of best fit does not fit the data as well as an upward-sloping curve does. Therefore, sketch a curve of good fit.

Practising

1. For each set of data,

i) create a scatter plot and draw the line of best fit

ii) describe the type of correlation the trend in the data displays

a) Population of the Hamilton–Wentworth, Ontario, Region

Year	1966	1976	1986	1996	1998
Population	449 116	529 371	557 029	624 360	618 658

b) Percent of Canadians with Less than Grade 9 Education

Year	1976	1981	1986	1991	1996
Percent of the Population	25.4	20.7	17.7	14.3	12.4

2. In an experiment for a physics project, marbles are rolled up a ramp. A motion sensor detects the speed of the marble at the start of the ramp, and the final height of the marble is recorded. However, the motion sensor may not be measuring accurately. Here are the data:

Speed (m/s)	1.2	2.1	2.8	3.3	4.0	4.5	5.1	5.6
Final Height (m)	0.07	0.21	0.38	0.49	0.86	1.02	1.36	1.51

a) Draw a curve of good fit for the data.

b) How consistent are the motion sensor's measurements? Explain.

A–14 Interpolating and Extrapolating

A graph can be used to make predictions about values not actually recorded and plotted. When the prediction involves a point within the range of the values of the independent variable, this is called **interpolating**. When the value of the independent variable falls outside the range of recorded data, it is called **extrapolating**. With a scatter plot, estimates are more reliable if the data show a strong positive or negative correlation.

EXAMPLE 1

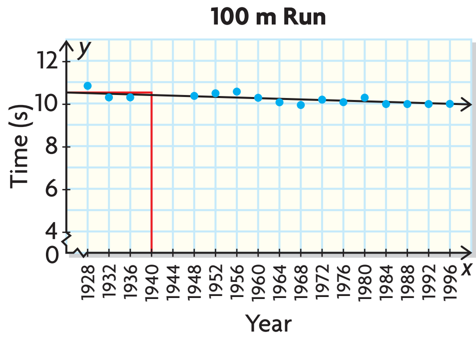
The Summer Olympics were cancelled in 1940 and 1944 because of World War II. Estimate what the 100 m run winning times might have been in these years if the Olympics had been held as scheduled.

Winning Times of 100 m Run

Year	Name (Country)	Time (s)
1928	Williams (Canada)	10.80
1932	Tolan (U.S.)	10.30
1936	Owens (U.S.)	10.30
1948	Dillard (U.S.)	10.30
1952	Remigino (U.S.)	10.40
1956	Morrow (U.S.)	10.50
1960	Hary (Germany)	10.20
1964	Hayes (U.S.)	10.00
1968	Hines (U.S.)	9.95
1972	Borzov (U.S.S.R.)	10.14
1976	Crawford (Trinidad)	10.06
1980	Wells (Great Britain)	10.25
1984	Lewis (U.S.)	9.99
1988	Lewis (U.S.)	9.92
1992	Christie (Great Britain)	9.96
1996	Bailey (Canada)	9.84

Solution

Draw a scatter plot and find the line of best fit.



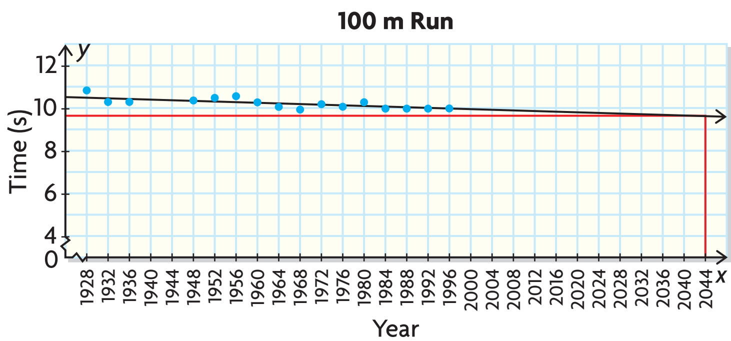
Find 1940 on the *x*-axis. Follow the vertical line for 1940 up until it meets the line of best fit. This occurs at about 10.8 s. For 1944, a reasonable estimate would be about 10.7 s.

EXAMPLE 2

Use the graph from Example 1 to predict what the winning time might be in 2044.

Solution

Extend the *x*-axis to 2044. Then extend the line of best fit to the vertical line through 2044.

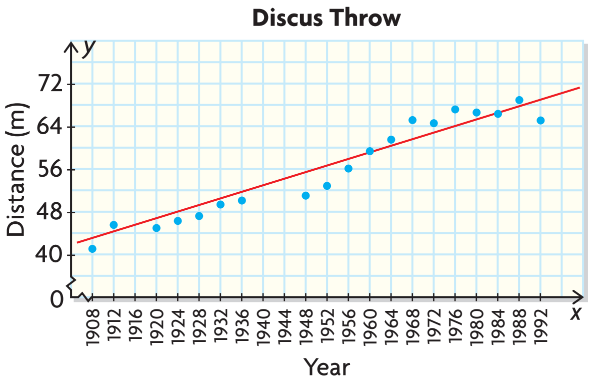


The vertical line for 2044 crosses the line of best fit at about 9.5, so the winning time in 2044 might be about 9.5 s.

It would be difficult to forecast much further into the future, since the winning times cannot continue to decline indefinitely. For example, a runner would likely never be able to run 100 m in 1 s and would certainly never run it in less than 0 s.

Practising

1. The scatter plot shows the gold medal throws in the discus competition in the Summer Olympics for 1908 to 1992.



a) Estimate the winning distance for 1940 and 1944.

b) Estimate the winning distance for 1996 and 2000.

2. As an object falls freely toward the ground, it accelerates at a steady rate due to gravity. The data show the speed, or velocity, an object would reach at one-second intervals during its fall.

Time from Start (s)	Velocity (m/s)
0	0
1	9.8
2	19.6
3	29.4
4	39.2
5	49.0

a) Graph the data.

b) Determine the object's velocity at 2.5 s, 3.5 s, and 4.75 s.

c) Find the object's velocity at 6 s, 9 s, and 10 s.

3. Explain why values you find by extrapolation are less reliable than those found by interpolation.

4. A school principal wants to know if there is a relationship between attendance and marks. You have been hired to collect data and analyze the results. You start by taking a sample of 12 students.

Days Absent	0	3	4	2	0	6	4	1	3	7	8	4
Average (%)	93	79	81	87	87	75	77	90	77	72	61	80

a) Create a scatter plot. Draw the line of best fit.

b) What appears to be the average decrease in marks for 1 day's absence?

c) Predict the average of a student who is absent for 6 days.

d) About how many days could a student likely miss before getting an average below 50%?

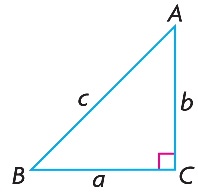
5. A series of football punts is studied in an experiment. The initial speed of the football and the length of the punt are recorded.

Speed (m/s)	10	17	18	21	25
Distance (m)	10	28	32	43	61

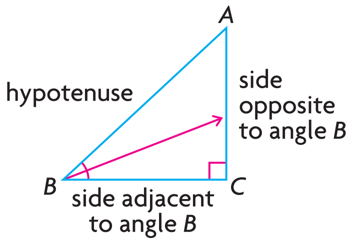
Use a curve of best fit to estimate the length of a punt with an initial speed of 29 m/s and the initial speed of a punt with a length of 55 m.

A–15 Trigonometry of Right Triangles

By the Pythagorean relationship, $a^2 + b^2 = c^2$ for any right triangle, where c is the length of the hypotenuse and a and b are the lengths of the other two sides.



In any right triangle, there are three primary trigonometric ratios that associate the measure of an angle with the ratio of two sides. For example, for $\angle ABC$, in Figure 1,



For $\angle B$

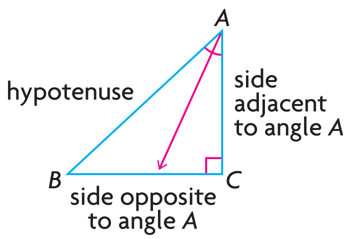
$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 1

Similarly, in Figure 2,



For $\angle A$

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

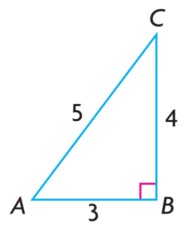
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 2

Note how the opposite and adjacent sides change in Figures 1 and 2 with angles A and B.

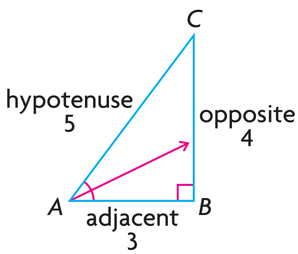
EXAMPLE 1

State the primary trigonometric ratios of $\angle A$.



Solution

Sketch the triangle. Then label the opposite side, the adjacent side, and the hypotenuse.



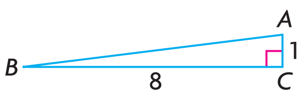
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$= \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$= \frac{3}{5}$$
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{4}{3}$$

EXAMPLE 2

A ramp must have a rise of one unit for every eight units of run. What is the angle of inclination of the ramp?

Solution

The slope of the ramp is $\frac{\text{rise}}{\text{run}} = \frac{1}{8}$. Draw a labelled sketch.



Calculate the measure of $\angle B$ to determine the angle of inclination.

The trigonometric ratio that associates $\angle B$ with the opposite and adjacent sides is the tangent. Therefore,

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

$$B = \frac{1}{8}$$

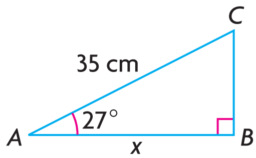
$$B = \tan^{-1}\left(\frac{1}{8}\right)$$

$$B \doteq 7°$$

The angle of inclination is about 7°.

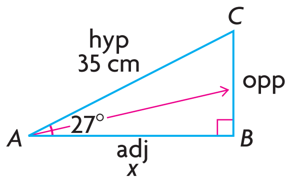
EXAMPLE 3

Determine x to the nearest centimetre.



Solution

Label the sketch. The cosine ratio associates $\angle A$ with the adjacent side and the hypotenuse.



Then,

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 27° = \frac{x}{35}$$

$$x = 35 \cos 27°$$

$$x \doteq 31$$

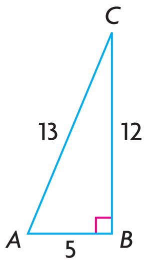
So x is about 31 cm.

Practising

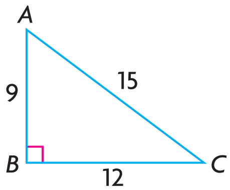
1. A rectangular lot is 15 m by 22 m. How long is the diagonal, to the nearest metre?

2. State the primary trigonometric ratios for $\angle A$.

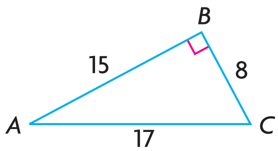
a)



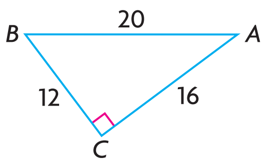
b)



c)



d)



3. Solve for x to one decimal place.

a) $\sin 39° = \frac{x}{7}$

b) $\cos 65° = \frac{x}{16}$

c) $\tan 15° = \frac{x}{22}$

d) $\tan 49° = \frac{31}{x}$

4. Solve for $\angle A$ to the nearest degree.

a) $\sin A = \frac{5}{8}$

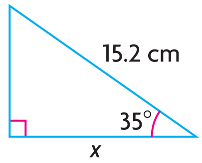
b) $\cos A = \frac{13}{22}$

c) $\tan B = \frac{19}{22}$

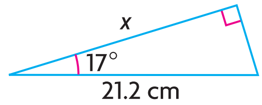
d) $\cos B = \frac{3}{7}$

5. Determine x to one decimal place.

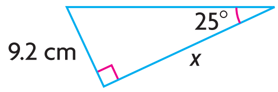
a)



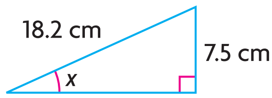
b)



c)



d)



6. In $\triangle ABC$, $\angle B = 90°$ and $AC = 13$ cm. Determine

a) BC if $\angle C = 17°$

b) AB if $\angle C = 26°$

c) $\angle A$ if $BC = 6$ cm

d) $\angle C$ if $BC = 9$ cm

7. A tree casts a shadow 9.3 m long when the angle of the sun is 43°. How tall is the tree?

8. Janine stands 30.0 m from the base of a communications tower. The angle of elevation from her eyes to the top of the tower is 70°. How high is the tower if her eyes are 1.8 m above the ground?

9. A surveillance camera is mounted on the top of a building that is 80 m tall. The angle of elevation from the camera to the top of another building is 42°. The angle of depression from the camera to the same building is 32°. How tall is the other building?