Functions 11

Teacher Notes

Course Notes

Chapter 5 – Trigonometric Ratios

TRIGONOMETRY IS MORE THAN TRIANGLES

We will learn

- *The three reciprocal trigonometric ratios*
- *To relate the six trigonometric ratios to the unit circle*
- *To solve problems using trig ratios, properties of triangles, and the sine/cosine laws*
- *How to prove trigonometric identities*

Chapter 5 – Trigonometric Equations

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

Section 5.1

Pg. 280 – 282 # 3, 4, 5i,ii,iv, 6, 7, 8a, 12, 14, 15

Section 5.6

Pg. 318 – 320 #4, 5bd, 7 (If only you had a side of the right angle triangle…),
#9 (recall the meaning of angle of depression??), 10, 13

Section 5.7

Page 326 - 327 #4ad (do you "need" to use the cosine law?), 6, 8 – 10

Section 5.8

Pg. 332 – 334 #3ac, 4a, 6, 9, Bonus: 7 (this one is tricky!!!)

Unit Test Part 1 & HW Part 1 Due

Section 5.2

Pg 286 – 288 #3 – 9, 11, 13

Section 5.3/5.4

Pg. 299 – 301 #1 – 3 (*For #3, **READ** example 3, pg. 296*), 5, 6 (*see example 5.3.4 above*),
#8 – 10, 12

If you struggle with this stuff…ASK QUESTIONS in EDSBY!!! (and in class too!)

Section 5.5

Handout

Unit Test Part 2 & HW Part 2 Due

Chapter 5 – Trigonometric Ratios

5.1 – Trigonometric Ratios of Acute Angles

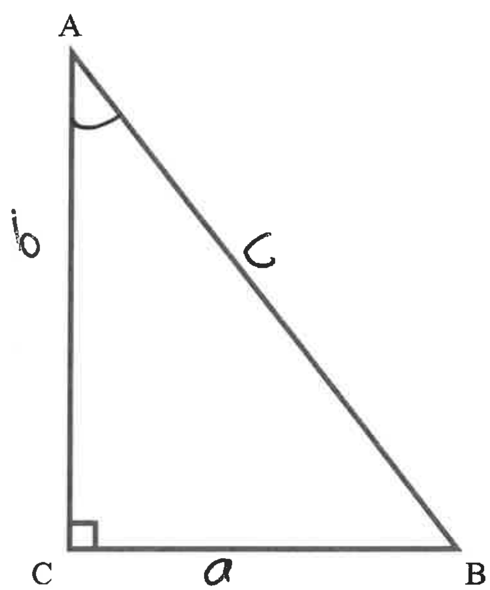
Learning Goal: We are learning to evaluate reciprocal trigonometric ratios.

Recall from Grade 10 the mnemonic

SOH CAH TOA

We use SOH CAH TOA to calculate the so-called "trig ratios" for a **right angle triangle.**

Consider the triangle:



The Trigonometric Ratios For ∠A

Primary Trig Ratios

$$\sin(A) = \frac{a}{c} \quad \left(\frac{opp}{hyp}\right)$$

$$\cos(A) = \frac{b}{c} \quad \left(\frac{adj}{hyp}\right)$$

$$\tan(A) = \frac{a}{b} \quad \left(\frac{opp}{adj}\right)$$

Reciprocal Trig Ratios

$$\frac{1}{\sin(A)} = \text{cosecant}(A) = \csc(A) = \frac{c}{a} = \frac{hyp}{opp}$$

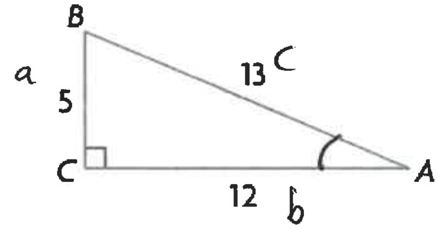
$$\frac{1}{\cos(A)} = \text{secant}(A) = \sec(A) = \frac{c}{b} = \frac{hyp}{adj}$$

$$\frac{1}{\tan(A)} = \text{cotangent}(A) = \cot(A) = \frac{b}{a_2} = \frac{adj}{opp}$$

Example 5.1.1

From your text, Pg. 280 #1

Given $\triangle ABC$, state the six trigonometric ratios for $\angle A$.



$\sin(A) = \frac{5}{13}$ $\quad$ $\csc(A) = \frac{13}{5}$

$\cos(A) = \frac{12}{13}$ $\quad$ $\sec(A) = \frac{13}{12}$

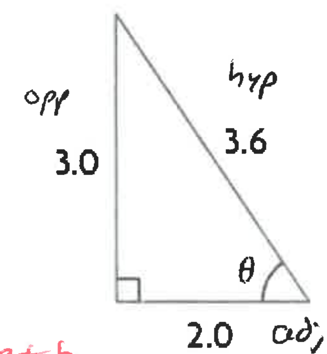
$\tan(A) = \frac{5}{12}$ $\quad$ $\cot(A) = \frac{12}{5}$

Example 5.1.2

For the given right triangle determine:

a) $\csc(\theta)$, $\sec(\theta)$, and $\cot(\theta)$.

b) the angle θ to the nearest degree.



$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp} = \frac{3.6}{3.0} = 1.2$

$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj} = \frac{3.6}{2.0} = 1.8$

$\cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp} = \frac{2}{3} = 0.\overline{6}$

can use any for θ, convert to primary ratio

$\sec\theta = 1.8$

$\frac{1}{\cos\theta} = 1.8$

$\frac{1}{1.8} = \cos\theta$

$\cos^{-1}\left(\frac{1}{1.8}\right) = \theta \doteq 56°$

Example 5.1.3

a) Determine the corresponding reciprocal ratio:

i) $\sin(\theta) = \frac{2}{5}$ $\quad$ ii) $\tan(\theta) = -3$

$\csc\theta = \frac{5}{2}$ $\quad$ $\cot\theta = -\frac{1}{3}$

b) Calculate to the nearest hundredth: $\sec(34°)$

$= \frac{1}{\cos(34°)} = \frac{1}{0.829} \doteq 1.21$

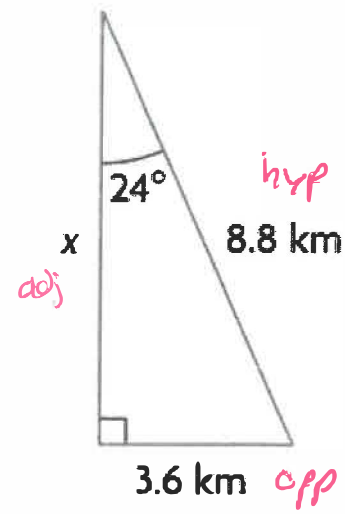
c) Determine the value of θ to the nearest degree: $\csc(\theta) = 2.46$

$\frac{1}{\sin\theta} = 2.46 \Rightarrow \frac{1}{2.46} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2.46}\right)$

$\doteq 24°$

Example 5.1.4

Given the right triangle, determine the unknown side using two different trig ratios:



① $\tan 24° = \frac{3.6}{x}$

$x = \frac{3.6}{\tan(24°)}$

$= 8.1\text{ km}$

② $\cos 24° = \frac{x}{8.8}$

$8.8\cos(24°) = x$

$8.0\text{ km} = x$

Discrepancy due to rounding.

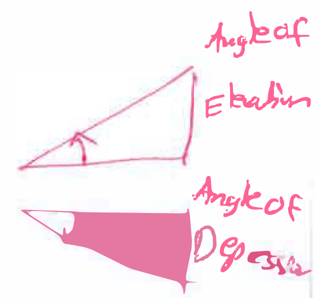
Example 5.1.5

From your text, Pg. 282 #11

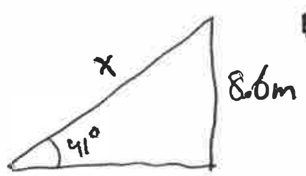
A kite is flying 8.6 m above the ground at an angle of elevation of 41°. Calculate the length of string, to the nearest tenth of a metre, needed to fly the kite using

a) a primary trigonometric ratio

~~b) a reciprocal trigonometric ratio~~



Let:



a) $\sin(41) = \frac{8.6}{x}$

$x = \frac{8.6}{\sin(41)}$

$x = 13.1\text{ m}$

b) $\csc(41) = \frac{x}{8.6}$

Success Criteria:

- I can use SohCahToa to determine the primary and reciprocal trigonometric ratios
- I can evaluate problems using the reciprocal trigonometric ratios
- I cannot use my calculator to directly solve a reciprocal trigonometric ratio

opener: Put up a completed Δ and have students find the 3 ratios.

Chapter 5 – Trigonometric Ratios

5.6: The Sine Law

Learning Goal: We are learning to use the Sine law to solve non-right angle triangles.

Last year you learned the Sine Law. It is a "formula" we can use to **solve triangles which are not right angle triangles**. There is one requirement to be able to use the Sine Law.

You Must Have an Angle With Its Corresponding Side!

So far we have been using Right Angle Triangles along with SOH CAH TOA to "solve" triangles. BUT right angle triangles aren't always the best triangle to use;

Sometime using a right angle triangle just can't be done. We then need to use so-called "**OBLIQUE TRIANGLES**". Oblique triangles come in two forms:

1) Acute (all angles are less than 90°
2) Obtuse (one angle is more than 90°

The Sine Law (*for oblique triangles*)

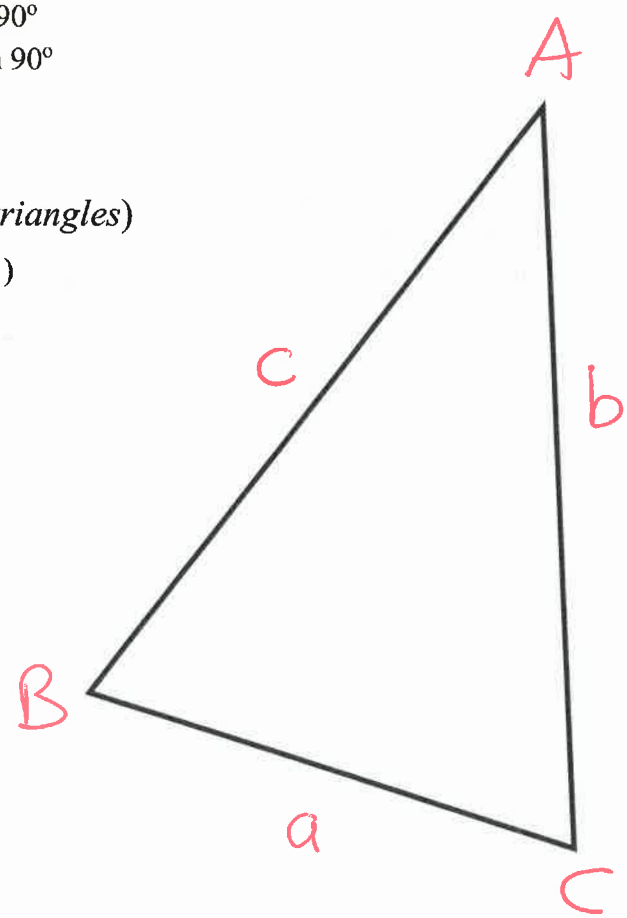
(There are **TWO FORMS** you should know!!)

Given the non-right triangle, ΔABC, then:

$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$

or

$$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$$



Notes:

1) Memorize the SINE LAW!
2) If we are trying to **find an angle**, use the first form of the Sine Law **(angles on top)**
3) If we are trying to **find the length** of a side, use the second for of the law **(with sides on top)**
4) In order to use the Sine Law, you must have the correct information in the triangle. You must have:
   a) 3 pieces of information
   b) One "**CORRESPONDING PAIR**" – an angle with its opposite side (for example you might have side *a* and angle *A*)

Note: **There is a problem with the Sine Law**
(Have not learned yet)

Recall that for trig ratios, "sine" is positive in quadrants 1 ***and*** 2.

e.g. $\sin(51^\circ) =$ 0.777 $\sin(129^\circ) =$ 0.777

But,
$\sin\theta = 0.777$
$\theta = \sin^{-1}(.777)$
$= 51^\circ$
(only 1 angle)

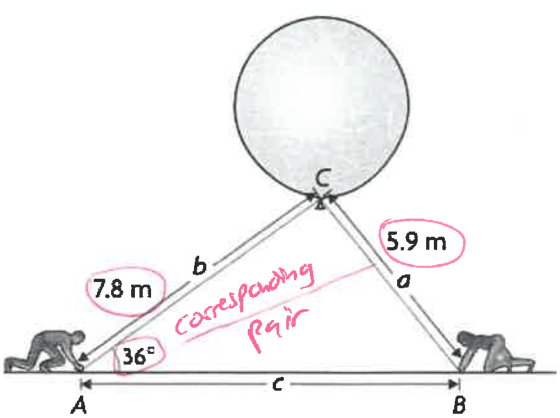
Consider Example 1 in your text: Pg. 312 – 314 .

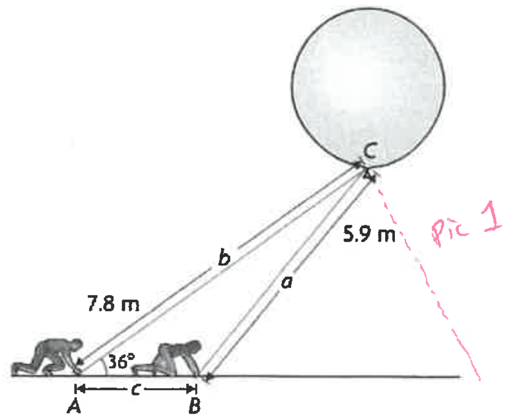
Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long.

The question we are asked is: **How far is Albert from Belle, to the nearest meter?**

Possible Pictures:

If info Given is
S S A
(Side Side Angle)





Using your calculator means that ONLY the acute angle of B is given. Use supplementary angle theorem to find the other possible angle of B.

Both pictures describe the problem completely. So which is correct? Well…BOTH ARE POSSIBLE solutions. This is known as the "AMBIGUOUS CASE". Because both are possible solutions, you must find both.

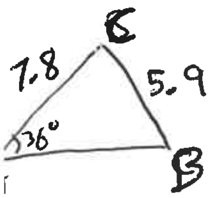
Note: If the GIVEN ANGLE is ACUTE, then this so called Ambiguous Case MAY APPLY. But, if the Given Angle is Obtuse, then the Ambiguous Case CANNOT APPLY. (*And **Sometimes**, there is no triangle which solves the problem.*)

Why? If we are given an obtuse angle, the other two MUST be acute. (So no second sol'n)

Example 5.6.1

Solve the triangles above.

Means find all 3 sides + angles



Answers: $\angle A = 36°$ (given) $\quad a = 5.9$ (g)

$\angle B = 51°$ (S.L) $\quad b = 7.8$ (g)

$\angle C = 93°$ (ASΔ) $\quad c = 10$ (SL)

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{7.8} = \frac{\sin(36)}{5.9}$$

$$B = \sin^{-1}\left(\frac{7.8\sin(36)}{5.9}\right)$$

$$\boxed{\angle B = 51°}$$

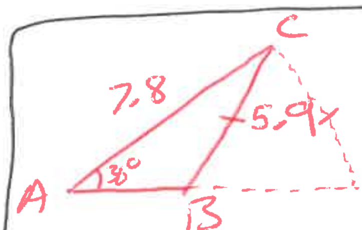
$$\angle C = 180 - 36 - 51$$

$$\boxed{\angle C = 93°}$$

$$\frac{c}{\sin(93)} = \frac{5.9}{\sin(36)}$$

$$c\sin(36) = 5.9\sin(93)$$

$$c = \frac{5.9\sin(93)}{\sin 36} = 10.02\,m$$



Isosceles Δ. Two bottom angles equal

$$\angle B = 180 - 51°$$

$$\boxed{\angle B = 129°}$$

$$\angle C = 180 - 129 - 36$$

$$\boxed{\angle C = 15°}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin(15)} = \frac{5.9}{\sin(36)}$$

$$c\sin(36) = 5.9\sin(15)$$

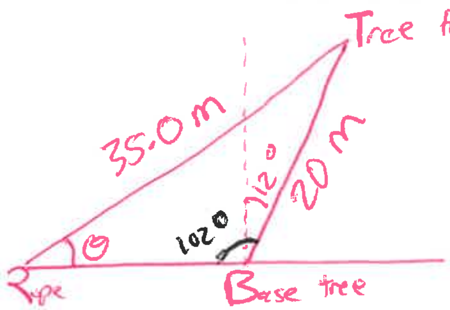
$$c = \frac{5.9\sin(15)}{\sin 36}$$

$$\boxed{c = 2.6\,m}$$

Example 5.6.2

From your text: Pg. 319 #6

The trunk of a leaning tree makes an angle of 12° with the vertical. To prevent the tree from falling over, a 35.0 m rope is attached to the top of the tree and is pegged into level ground some distance away. If the tree is 20.0 m from its base to its top, calculate the angle the rope makes with the ground to the nearest degree.



Not ambiguous case. For SSA, the angle (B) is obtuse.

$90 + 12° = 102°$

$$\frac{\sin R}{r} = \frac{\sin B}{b}$$

$$b \sin R = r \sin B$$

$$\sin R = \frac{r \sin B}{b}$$

$$\sin R = \frac{20 \sin(102)}{35}$$

$$R = \sin^{-1}(0.55894...)$$

$$= 33.98°$$

$$\boxed{R = 34°}$$

Success Criteria

- I can recognize when the sine law applies and use it to solve for an unknown value
- I can identify, given S-S-A, that there will be two solutions (the ambiguous case)

Chapter 5 – Trigonometric Ratios

5.7: The Cosine Law

Learning Goal: We are learning to use the cosine law to solve non-right angle triangles.

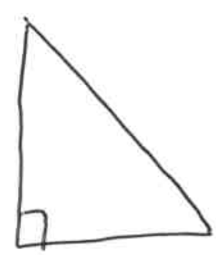
The Cosine Law is another "formula" for solving Oblique Triangles. Remember, to "solve" a triangle you MUST be given **3 PIECES OF INFORMATION** about the triangle (and I should note that one of those given pieces **MUST BE A SIDE LENGTH**).

The main question you will have to be able to answer is this:

When do you use

1) SOH CAH TOA

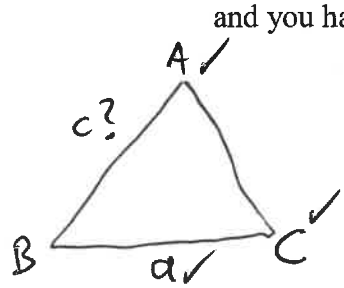
   When you have a Right Triangle



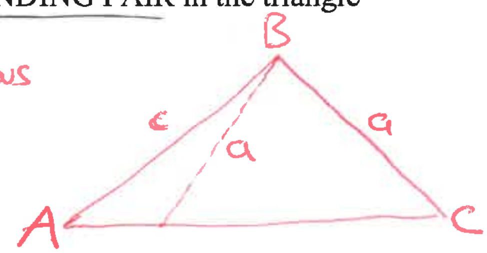
2) The SINE LAW

   When you have an Oblique △

   and you have a CORRESPONDING PAIR in the triangle



Ambiguous Case:



3) The COSINE LAW

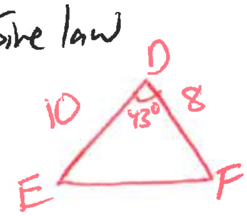
   ① Oblique △

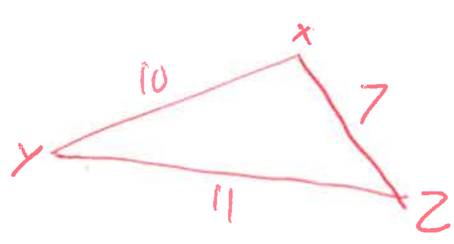
   ② Can't use the Sine law

Info Given is SAS

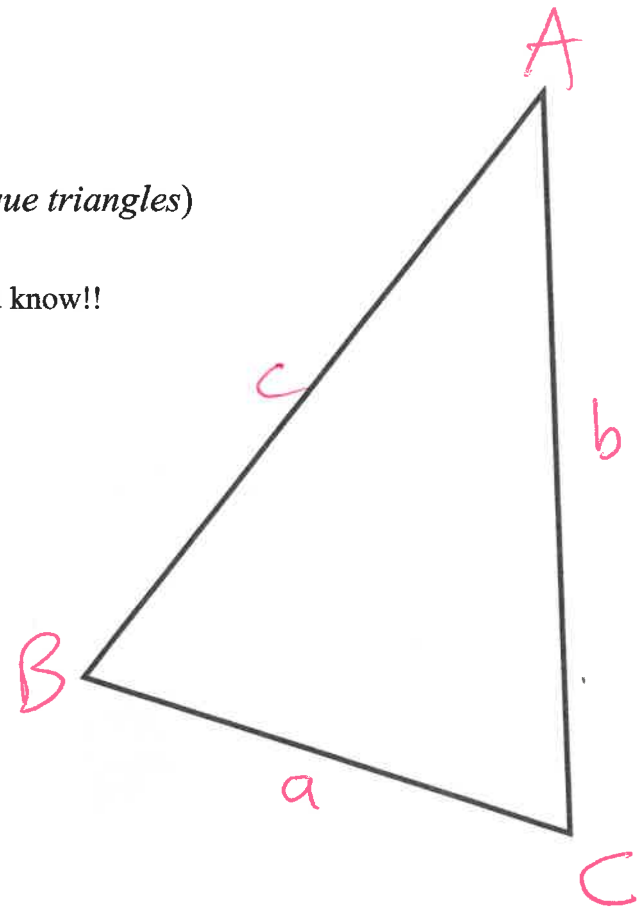
or

SSS





The Cosine Law *(for oblique triangles)*



There are **THREE SIDE FORMS** you should know!!

Given the non-right triangle, ΔABC, then:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

or

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

or

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Also, there are **THREE ANGLE FORMS** you should know!! (rearranged equations)

$$\cos(A) = \frac{a^2 - b^2 - c^2}{-2bc}$$

or

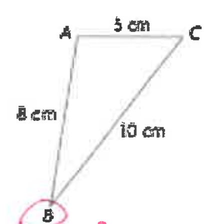
$$\cos(B) = \frac{b^2 - a^2 - c^2}{-2ac}$$

or

$$\cos(C) = \frac{c^2 - a^2 - b^2}{-2ab}$$

The formula you use depends on which side or angle you are looking for!!!

e.g. Determine angle B



$$\cos B = \frac{(5)^2 - (10)^2 - (8)^2}{-2(10)(8)}$$

$$= \frac{25 - 100 - 64}{-160}$$

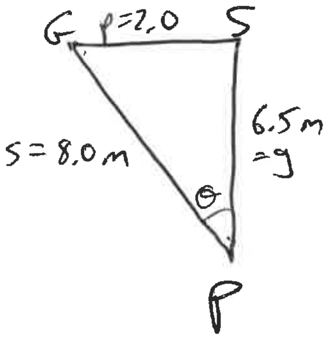
$$B = \cos^{-1}\left(\frac{-139}{-160}\right)$$

$$B = 29.68^\circ \quad = 30^\circ$$

Example 5.7.1

From your text: Pg. 326 #5

The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.



$$\cos(P) = \frac{p^2 - g^2 - s^2}{-2gs}$$

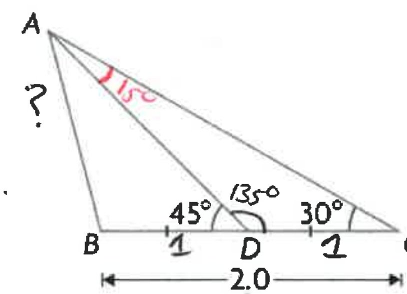
$$= \frac{(2)^2 - (6.5)^2 - (8)^2}{-2(6.5)(8)}$$

$$P = \cos^{-1}\left(\frac{-102.25}{-104}\right) = 10.525^\circ = 11^\circ$$

Example 5.7.2

From your text: Pg. 327 #7

Given $\triangle ABC$ at the right, $BC = 2.0$ and D is the midpoint of BC. Determine AB, to the nearest tenth, if $\angle ADB = 45^\circ$ and $\angle ACB = 30^\circ$.



① Find angles in $\triangle ADC$

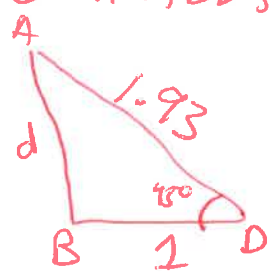
② Find AD

$$\frac{AD}{\sin 30^\circ} = \frac{1}{\sin(15)}$$

$$\therefore AD = \frac{\sin 30}{\sin 15}$$

$$\boxed{AD = 1.93}$$

③ In ABD, use Cosine law. (SAS)



$$d^2 = b^2 + a^2 - 2ab\cos D$$

$$= 1.93^2 + 1^2 - 2(1)(1.93)\cos 45$$

$$d = \sqrt{4.7249 - 3.86\cos 45}$$

$$d = 1.4126$$

AB is 1.4 units

Success Criteria:

- I can use the cosine law, given S-A-S or S-S-S
- I can rearrange the cosine law to solve for an unknown angle

Chapter 5 – Trigonometric Ratios

5.8: 3D Problems

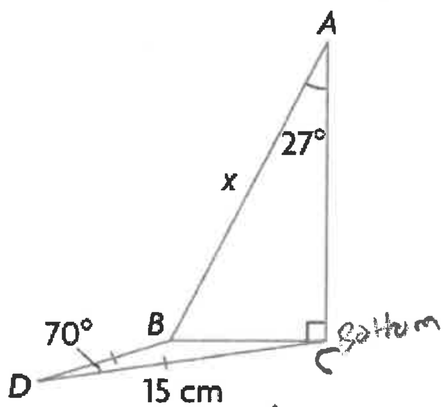
Learning Goal: We are learning to use trigonometry to solve 3-dimensional problems.

We will be using SOH CAH TOA, the Sine Law, and the Cosine Law for these problems. We'll jump right in by solving some problems since we already know how to use the various techniques! **One thing to keep in mind, though, is that these sorts of problems can be difficult to draw, or even simply visualize because we are working in 3D! Art specialists – rejoice!**

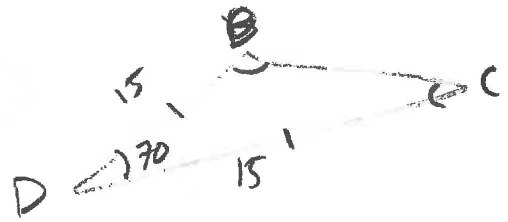
Example 5.8.1

From your text: Pg. 332 #4b

Solve for x



- Need another side in △ ABC.
- Find BC



Can use cosine law

$d^2 = b^2 + c^2 - 2bc\cos D$

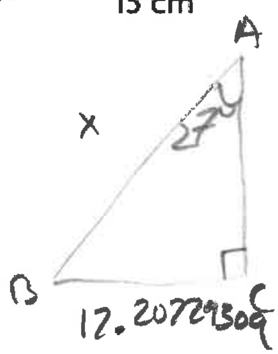
$d^2 = (15)^2 + (15)^2 - 2(15)(15)\cos(70)$

$= 450 - 450\cos 70$

$d^2 = 296.0909355$

$d = 17.20729309$ ✓

$\angle B = \angle C = \frac{180 - 70}{2} = 55°$



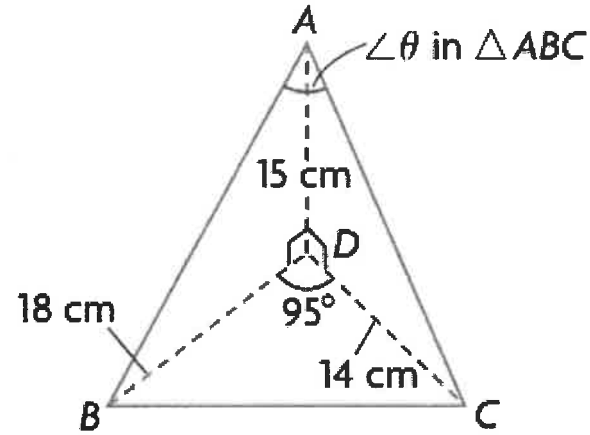
use Sine

$\sin A = \frac{BC}{BA}$

$\sin(27) = \frac{17.20729309}{x}$

$x = 37.9\text{ cm} \doteq 38\text{ cm}$

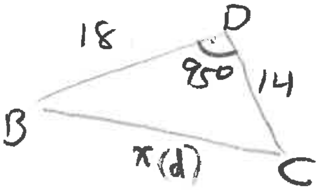
d) Solve for θ



θ is on the sloping $\triangle$ ABC.

Need sine or cosine law

we can find BC



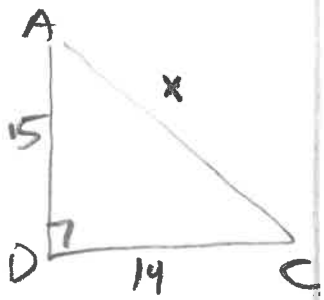
$x^2 = 18^2 + 14^2 - 2(18)(14)\cos 95$

$x^2 = 563.9264443$

$x = 23.74713655\text{ cm} = BC$

Use cosine for AC

Turns into pythag theorem !! :)

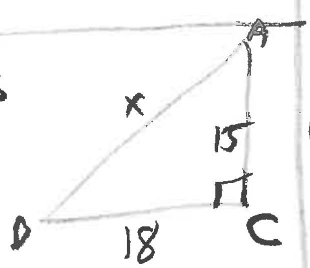


$x^2 = 15^2 + 14^2 - \cancel{2(15)(14)\cos 90}$

$x = \sqrt{15^2 + 14^2}$

$x = 20.51828453\text{ cm} = AC$

Use pythag for AB

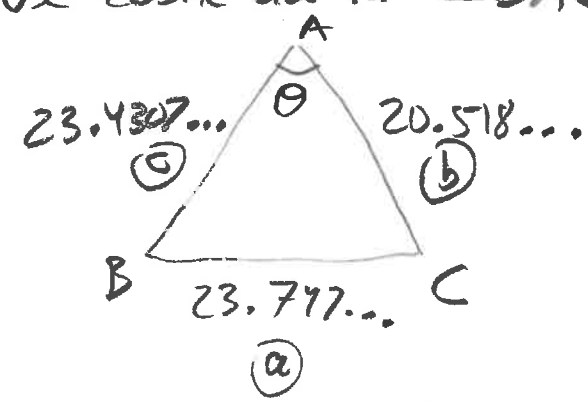


$x^2 = 18^2 + 15^2$

$x = \sqrt{18^2 + 15^2}$

$= 23.43074903 = AB$

Use cosine law for $\angle BAC$



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 - b^2 - c^2 = -2bc\cos A$

$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A$

$\cos A = \frac{(23.747)^2 - (20.518)^2 - (23.4307)^2}{-2(20.518)(23.4307)}$

$\cos A = \frac{-406.0660125}{-961.5022052}$

$A = \cos^{-1}(.4223245826)$

$A = 65^0 = \theta$

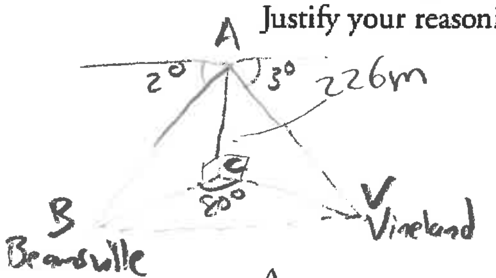
Example 5.8.2

From your text: Pg. 333 #5

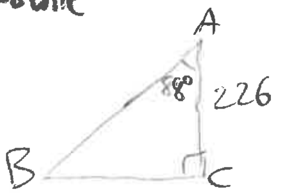
While Travis and Bob were flying a hot-air balloon from Beamsville to Vineland in southwestern Ontario, they decided to calculate the straight-line distance, to the nearest metre, between the two towns.

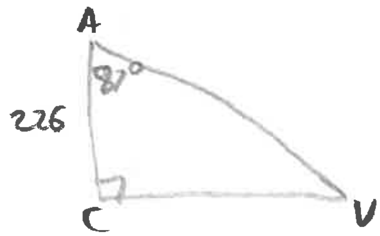
- From an altitude of 226 m, they simultaneously measured the angle of depression to Beamsville as 2° and to Vineland as 3°.
- They measured the angle between the lines of sight to the two towns as 80°.

Is there enough information to calculate the distance between the two towns? Justify your reasoning with calculations.



- Find BC + CV.
- use cosine for BV





$\tan(88) = \frac{BC}{226m}$

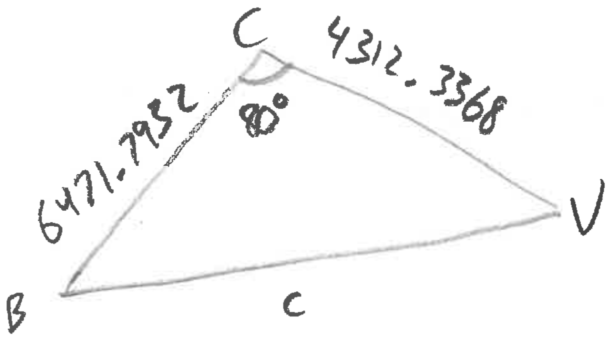
$226 \tan(88) = BC$

$6471.7932 = BC$

$\tan 87 = \frac{CV}{226}$

$226 \tan 87 = CV$

$4312.3368 = CV$



$c^2 = v^2 + b^2 - 2vb \cos C$

$= (6471.7932)^2 + (4312.3368)^2 - 2(6471.7932)(4312.33)\cos 80$

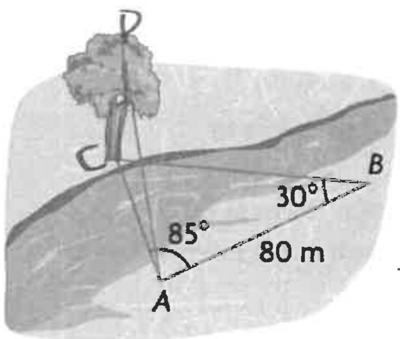
$\sqrt{c^2} = \sqrt{50787817.52}$

$c = 7126.55$

Yes there is enough info. The towns are 7127 m apart

Example 5.8.3

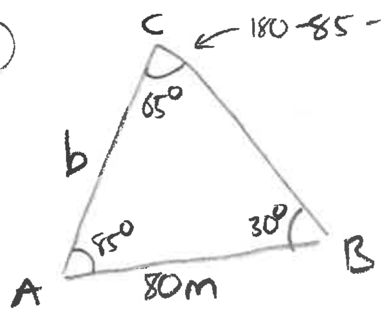
From your text: Pg. 334 #11



Bert wants to calculate the height of a tree on the opposite bank of a river. To do this, he lays out a baseline 80 m long and measures the angles as shown at the left. The angle of elevation from A to the top of the tree is 28°. Explain if this information helps Bert to calculate the height of the tree to the nearest metre. Justify your reasoning with calculations.

- Need side AC
- we can use sine law.

(1)

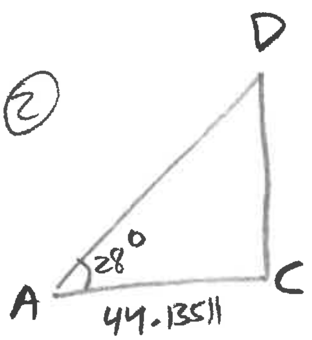


$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{\sin(30)(80)}{\sin(65)}$$

$$b = 44.1351\text{ m}$$

(2)



$$\tan(28) = \frac{DC}{AC}$$

$$(44.13511)\tan(28) = DC$$

$$23.467\text{m} = DC$$

yes, we can find it. The tree is about 23 m tall.

Success Criteria:

- I can sketch, to the best of my ability, a representation of the question
- I can identify the correct method to solve the unknown(s) in a given problem

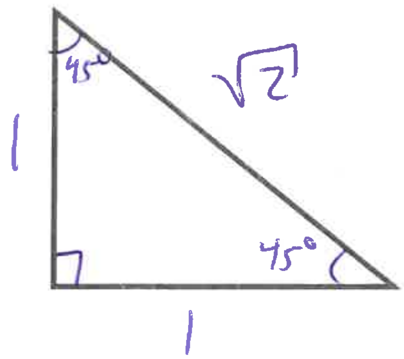
Chapter 5 – Trigonometric Ratios

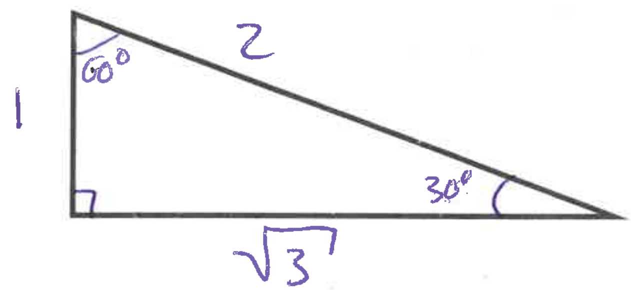
5.2 – Trigonometric Ratios and Special Triangles

Learning Goal: We are learning to find the EXACT values of sin, cos, and tan for specific angles.

There are two "**Special Triangles**"

1 – 60 – 2





MEMORIZE THESE!

The Primary Trigonometric Ratios of the Special Angles

$\sin(30°) = \frac{1}{2}$

$\sin(60°) = \frac{\sqrt{3}}{2}$

$\sin(45°) = \frac{1}{\sqrt{2}} \left(\times \frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$

$\cos(30°) = \frac{\sqrt{3}}{2}$

$\cos(60°) = \frac{1}{2}$

$\cos(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan(30°) = \frac{1}{\sqrt{3}}$ Irrational

$\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

"Rationalizing the denominator"

$\tan(60°) = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\tan(45°) = \frac{1}{1} = 1$

NO calculator use special $\triangle$!

what is being squared

$\cos^2(30)$
$= (\cos(30))(\cos(30))$

Example 5.2.1

Evaluate exactly

a) $\sin(45)\cdot\cos(60)$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}}{2(\sqrt{2})(\sqrt{2})} = \frac{\sqrt{2}}{2\cdot 2} = \frac{\sqrt{2}}{4}$

b) $\cos^2(30)+\sin^2(30)$

$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} + \frac{1}{4}$

$= \frac{4}{4} = 1$

c) $\tan(60)\cdot\cos(60)-\sin(60)$

$= \left(\frac{\sqrt{3}}{1}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= 0$

d) $\tan(30)\cdot\frac{\sin(60)}{\cos(45)}$

$= \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}}\right)$

$= \left(\frac{1}{\cancel{\sqrt{3}}}\right)\left(\frac{\cancel{\sqrt{3}}}{2} \cdot \frac{\sqrt{2}}{1}\right)$

$= \frac{\sqrt{2}}{2}$

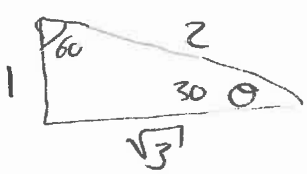
Example 5.2.2

Determine the angle θ (where $0 \le \theta \le 90^\circ$) given:

a) $\sec(\theta) = \frac{2}{\sqrt{3}}$ $\quad \frac{h}{a}$

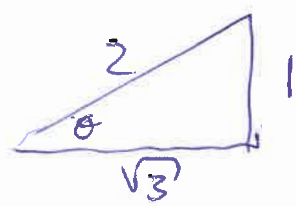
How? Convert to Primary trig ratio

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \frac{a}{h}$



$\therefore \theta = 30^\circ$

b) $\tan(\theta) = \frac{\sqrt{3}}{3}$ $= \frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$



$\theta = 30^\circ$

Unrationalize the denominator! Need a special $\triangle$!

Success Criteria:

- I can draw the two special triangles
- I can identify the EXACT values for 30º, 45º, 60º, using the special triangles
- I can evaluate EXACTLY (no calculators…OR capes!!!) problems involving the special triangles

Chapter 5 – Trigonometric Ratios

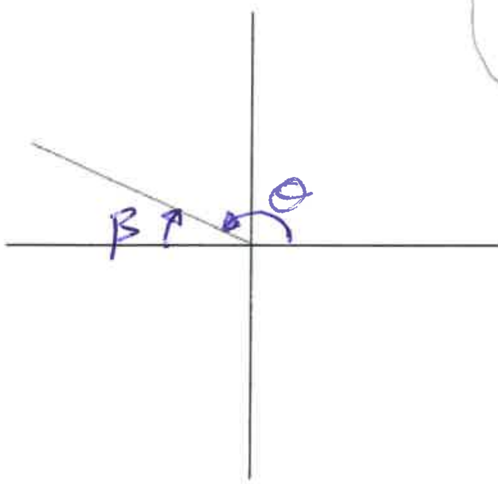
5.3 – 5.4 – Trigonometric Ratios for Angles Larger than 90°

Learning Goal: We are learning to use a Cartesian plane to evaluate trig ratios for angles between 0° and 360°.

Angles Larger than 90°

Consider the following sketch of the angle $\theta = 150°$

- θ is called an angle of rotation in standard position.
- Measured counterclockwise from +x-axis



e.g. Calculate

$\sin(150) = \frac{1}{2}$

$\sin(30) = \frac{1}{2}$

$\cos(150) = -0.866$

$\cos(30) = +0.866$

$\tan(150) = -0.577$

$\tan(30) = +0.577$

Clearly some connection between 30° + 150°

In this example, we call $\theta = 150°$ the **PRINCIPAL ANGLE**, or the angle in standard position.

Note: The angle $\beta = 30°$ is called the

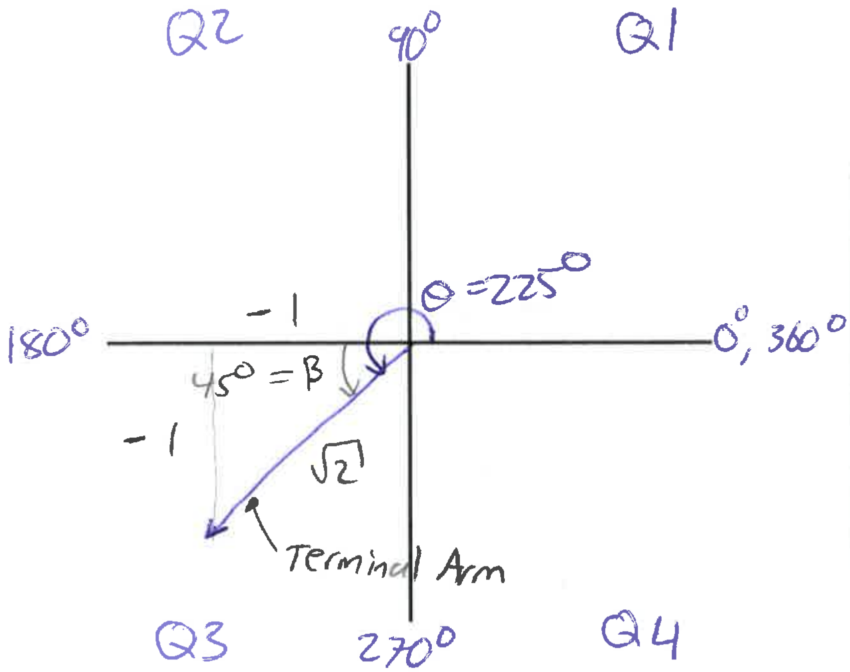
RELATED ACUTE ANGLE

Always measured from x-axis, less than 90°

Example 5.3.1

Sketch the angle of rotation $\theta = 225°$ and determine the related acute angle. β

↳ Always between the pole (x-axis) + line.



e.g. Calculate

sin(225) = −0.707

sin(45) = +0.707

cos(225) = −.707

cos(45) = +.707

tan(225) = 1

tan(45) = 1

What is up with these signs??? (**BE CAREFUL WITH YOUR SIGNS!!!!!!!!!**)

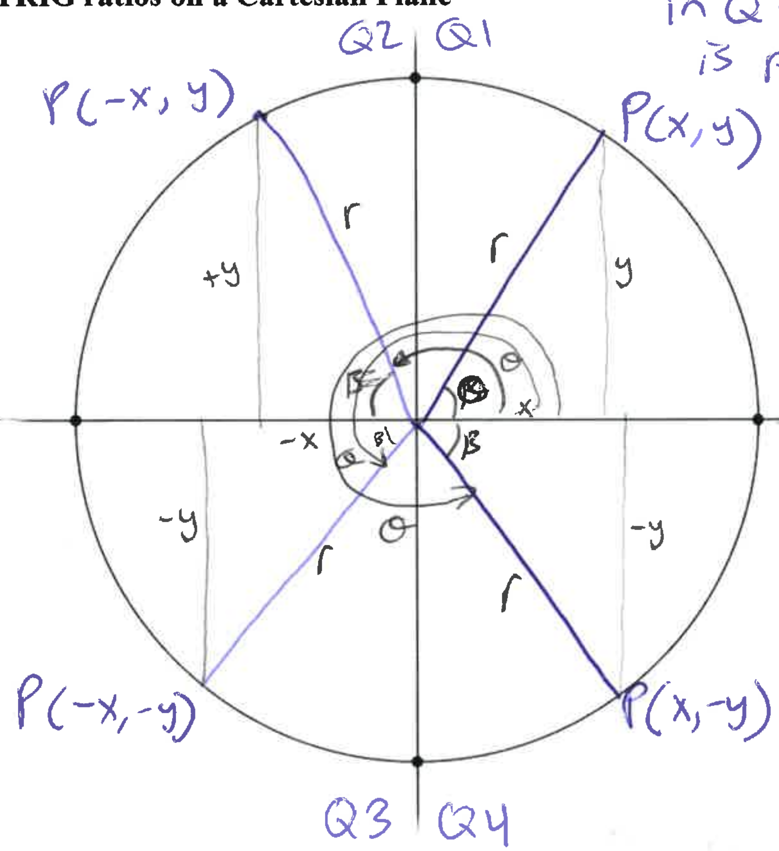
↳ When we use β, it is as though we are in Q1, where everything is positive.

Looking at the TRIG ratios on a Cartesian Plane

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{-x}{r}$

$\tan\theta = \frac{y}{-x}$



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

$\sin\theta = \frac{-y}{r}$

$\cos\theta = \frac{-x}{r}$

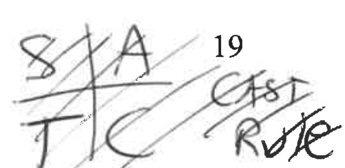
$\tan\theta = \frac{-y}{-x} = \frac{y}{x}$

$\sin\theta = \frac{-y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{-y}{x}$

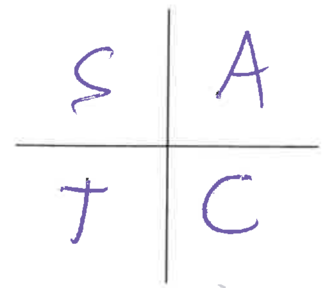
The terminal arm will always have a + length.



The **CAST RULE** determines the sign (+ or -) of the trig ratio

↳ of the Standard angle,

working with β is as though you are in Q1, where $\theta = \beta$



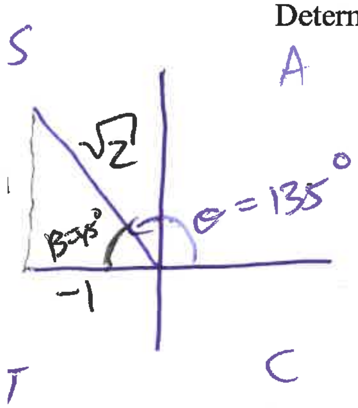
We now have enough tools to calculate the trigonometric ratios of any angle!

For any given angle θ we will:

1) Draw θ in **STANDARD POSITION** (i.e. draw the principal angle for θ)
2) Determine the **RELATED ACUTE ANGLE** (β) (between the terminal arm and the x-axis (also called the polar axis)
3) Use the related acute angle and the **CAST RULE** (and SOH CAH TOA) to determine the trig ratio (along with its sign…BE CAREFUL WITH YOUR SIGNS) in question

Example 5.3.2

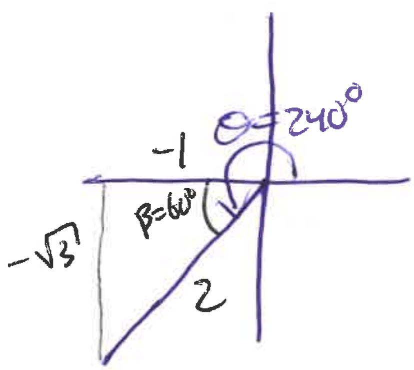
Determine the trig ratio $\sin(135)$



$\sin(135) = \frac{1}{\sqrt{2}}$

$\cos(135) = \frac{-1}{\sqrt{2}}$

Determine the trig ratio tan (240)



$\tan(240) = \frac{-\sqrt{3}}{-1}$

$= \sqrt{3}$

This is tan's ⊕ Quadrant

Example 5.3.3

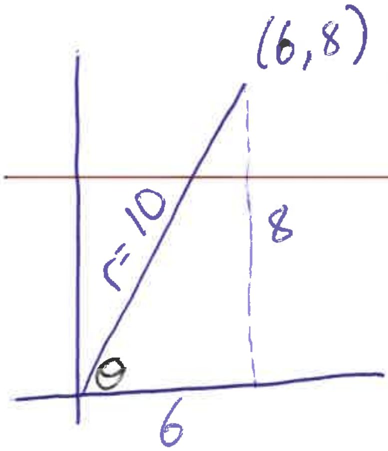
The point $P(x,y)=(6,8)$ lies on the terminal arm (of length *r*) of an angle of rotation.

Sketch the angle of rotation.

Determine: a) the value of *r*

b) the primary trig ratios for the angle

c) the value of the angle of rotation in degrees, to two decimal places



a) $r^2 = 6^2 + 8^2$

$r = \sqrt{36+64}$

$r = 10$

b) $\sin(\theta) = \frac{8}{10} = \frac{4}{5}$

$\cos(\theta) = \frac{6}{10} = \frac{3}{5}$

$\tan(\theta) = \frac{8}{6} = \frac{4}{3}$

c) $\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

Example 5.3.4

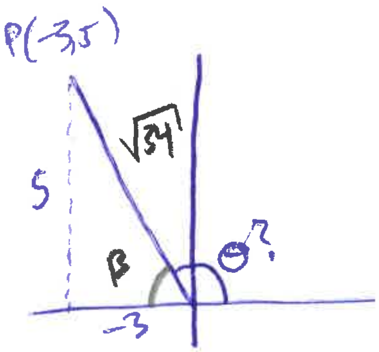
The point $(-3,5)$ lies on the terminal arm (of length *r*) of an angle of rotation. Sketch the angle of rotation.

Determine: a) the value of *r*

b) the primary trig ratios for the angle

c) the value of the angle of rotation in degrees, to two decimal places

Give Exact values. Use $\sqrt{34}$



a) $r^2 = (-3)^2 + (5)^2$

$r^2 = 34$

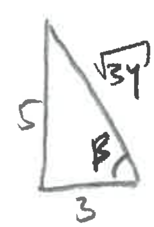
$r \doteq 5.83$ or $\sqrt{34}$

b) $\sin\theta = \frac{5}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$

$\cos\theta = \frac{-3}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = \frac{-3\sqrt{34}}{34}$

$\tan\theta = \frac{5}{-3}$

we cannot directly calculate the angle, only works in Q1. So find β + use it for θ



$\tan\beta = \frac{5}{3}$

$\beta = \tan^{-1}\left(\frac{5}{3}\right)$

$\beta = 59°$

$\theta = 180° - 59°$

$\boxed{= 121°}$

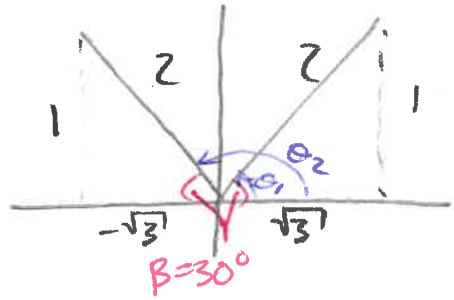
Note $\tan^{-1}\left(-\frac{5}{3}\right)$ gives $-59°$, bc we are in Quad 2 sin cos tan are all positive in two quadrant, thus there is normally two solutions.

Example 5.3.5 (going backwards!)

a) Given $\sin(\theta) = +\frac{1}{2}$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

Q1 + Q2 !

1) Sketch in the appropriate quadrants
2) label your Δ's
3) Find β
4) use β to get θ



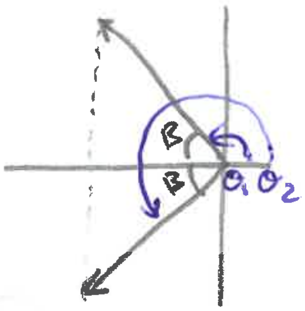
$\beta = 30^\circ$ (special Δ)

$\therefore \theta_1 = 30^\circ$

$\theta_2 = 180 - 30 = 150^\circ$

b) Given $\cos(\theta) = -0.5372$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

Q2 + Q3



Get β by ignoring the negative

$\cos(\beta) = 0.5372$

$\beta = \cos^{-1}(0.5372)$

$\beta = 57.51^\circ$

$\therefore \theta_1 = 180 - 57.51$

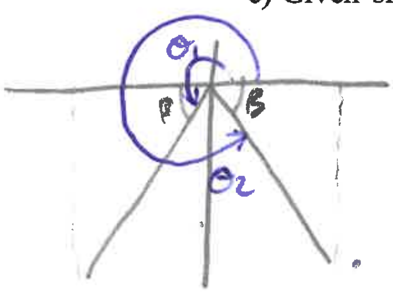
$= 122.49^\circ$

$\therefore \theta_2 = 180 + 57.51$

$= 237.51^\circ$

c) Given $\sin(\theta) = -0.4567$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

Q3 + 4



Get β

$\sin\beta = 0.4567$

$\beta = \sin^{-1}(0.4567)$

$= 27.2^\circ$

$\therefore \theta_1 = 180 + 27.2 = 207.2^\circ$

$\therefore \theta_2 = 360 - 27.2 = 332.8^\circ$

Success Criteria:

- I can identify a positive or negative angle based on the direction of rotation
- If the principal angle (Θ) lies in quadrants 2, 3, or 4 there is a related acute angle, β
- I can identify where a trigonometric ratio is + or – using the CAST Rule
- Every trigonometric ratio has two principal angles between 0° and 360°

Chapter 5 – Trigonometric Ratios

5.5 – Trigonometric Identities

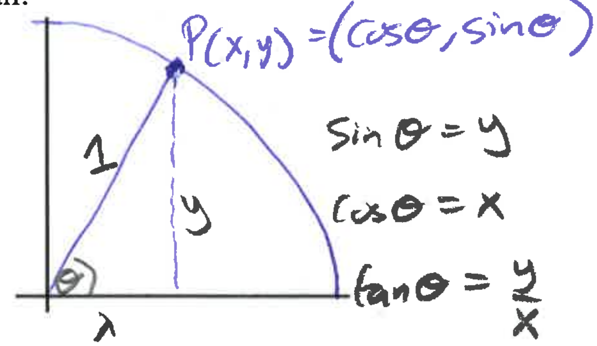
Learning Goal: We are learning to prove trigonometric identities

Proving Trigonometric Identities is so much fun it's **ridiculous**!

Let's start with a simple identity:

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

Recall:



$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \frac{y}{x}$

Our second identity: Pythagorean Identity

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

OR

$\cos^2\theta = 1 - \sin^2\theta$

OR

$\sin^2\theta = 1 - \cos^2\theta$

More commonly written as

$x^2 + y^2 = 1^2$

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\cos^2\theta + \sin^2\theta = 1$

This notation says sin is squared, not the angle

When proving trig identities, it's helpful to keep a few things in your mind. Things such as:

- The Reciprocal Trig Identities
- Converting everything to sin and cos can be helpful
- Start with the side which has the most "stuff" to work with, and work toward the other side
- A few special formulas, which we need to find…

*NEVER cross the = sign

Example 5.5.1 LS RS

Prove $\cos(x)\tan(x) = \sin(x)$

$$\cancel{\cos(x)}\left(\frac{\sin(x)}{\cancel{\cos(x)}}\right) = RS$$

$$\sin(x) = \sin(x)$$

$$LS = RS$$

Example 5.5.2 LS RS

Prove $1+\cot^2(x) = \csc^2(x)$

Start: Remember

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

All exponent rules still apply!

$$LS: = 1 + \frac{1}{\tan^2(x)}$$

$$= \frac{1}{1} + \frac{\cos^2(x)}{\sin^2(x)}$$

$$= \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}$$

$$= \frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)}$$

$$= \frac{1}{\sin^2(x)} = \csc^2(x) = RS$$

$$LS = RS$$

Example 5.5.3

From your text: Pg. 310 #8b

Prove $\dfrac{\tan^2\alpha}{1+\tan^2\alpha}=\sin^2\alpha$

LS: change tan to sin + cos

$$= \frac{\frac{\sin^2(a)}{\cos^2(a)}}{1+\frac{\sin^2(a)}{\cos^2(a)}}$$

$$= \frac{\frac{\sin^2(a)}{\cos^2(a)}}{\frac{\cos^2(a)}{\cos^2(a)}+\frac{\sin^2(a)}{\cos^2(a)}}$$

$$= \frac{\frac{\sin^2(a)}{\cos^2(a)}}{\frac{\cos^2(a)+\sin^2(a)}{\cos^2(a)}}$$

$$= \frac{\frac{\sin^2(a)}{\cos^2(a)}}{\frac{1}{\cos^2(a)}}$$

$$= \frac{\sin^2(a)}{\cancel{\cos^2(a)}} \cdot \frac{\cancel{\cos^2(a)}}{1}$$

$$= \frac{\sin^2 a}{1}$$

$= RS$

Example 5.5.4

Prove $1-2\cos^2\phi=\sin^4\phi-\cos^4\phi$

Ø is phi

LS ①

$= \boxed{1} - 2\cos^2 Ø$

Pythag identity

$= \sin^2 Ø + \cos^2 Ø - 2\cos^2 Ø$

$= \sin^2 Ø - \cos^2 Ø$

stop

RS ②

$= \sin^4 Ø - \cos^4 Ø$

Difference of squares

x^2-9

$=(x-3)(x+3)$

$= \underbrace{(\sin^2 Ø + \cos^2 Ø)}_{1}(\sin^2 Ø - \cos^2 Ø)$

$= 1(\sin^2 Ø - \cos^2 Ø)$

$= \sin^2 Ø - \cos^2 Ø$

LS = RS !! ✓

$\tan = \frac{\sin}{\cos}$ $\cot = \frac{\cos}{\sin}$

Example 5.5.5

Prove $\sin\theta + \sin\theta\cot^2\theta = \csc\theta$

Change to sin / cos

LS:

$$= \sin\theta + \cancel{\sin\theta}\left(\frac{\cos^2\theta}{\sin^{\cancel{2}}\theta}\right)$$

$$= \sin\theta\left(\frac{\sin\theta}{\sin\theta}\right) + \frac{\cos^2\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta}$$

$$= \csc\theta$$

$$= RS \checkmark$$

Example 5.5.6

Prove $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$

Get common Den. Need a single term.

LS: $= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

$$= \frac{\sin^2\theta}{\cos^2\theta\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}$$

$$= \csc^2\theta\cdot\sec^2\theta$$

$$= \sec^2\theta\cdot\csc^2\theta$$

$$= RS \checkmark$$

Success Criteria:

- I can prove trig identities using a variety of strategies:
  - Using the reciprocal, quotient, and Pythagorean identities
  - Factoring
  - Converting to sin and cos
  - Common denominators
- I can recognize the proper form to proving trigonometric identities