Functions 11

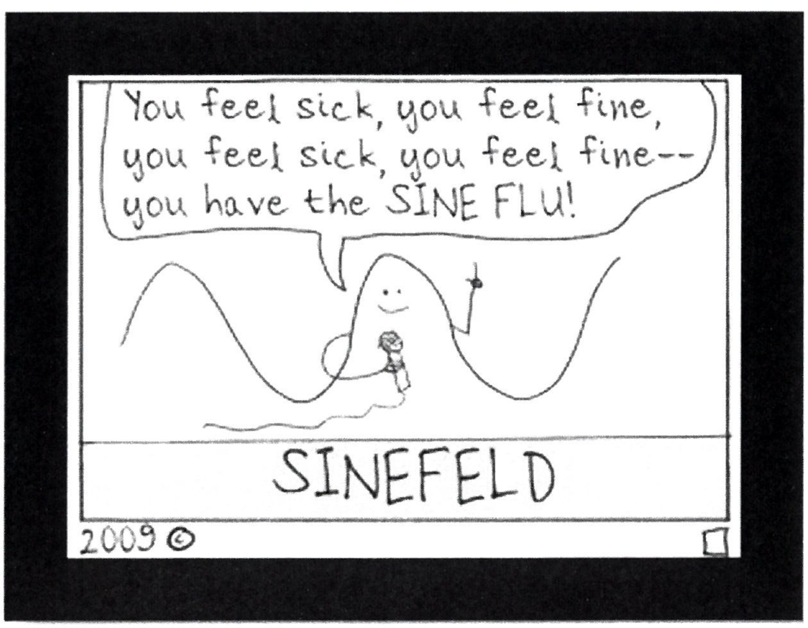
Teacher Notes

Course Notes

Unit 6 – Sinusoidal Functions

We will learn how to

- *identify situations that can be modelled by sinusoidal or periodic functions*
- *interpret the graphs of sinusoidal or periodic functions*
- *graph sinusoidal functions with transformations*
- *determine the equations of sinusoidal functions from real-world situations*



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Chapter 6 – Sinusoidal Functions

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

Section 6.1

Pg. 352 – 355 #4, 5, 7 – 10

Section 6.5

Pg. 383 – 3385 #1, 2, 4 – 7, 9

Section 6.6

Pg. 391 – 393 #4b, 5bcd, 6acd, 7, 11

Section 6.7

Pg. 398 – 401 #4 – 6, 8, 10 (*a question of* ***beauty***)

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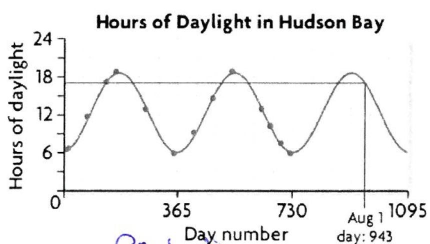
Chapter 6 – Sinusoidal Functions
6.1 – Properties of Periodic Functions

Learning Goal: We are learning to interpret and describe graphs that repeat at regular intervals.

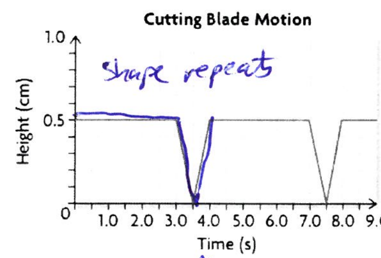
Definition 6.1.1

A PERIODIC FUNCTION is one in which *the functional values repeat.* in a "regular" way

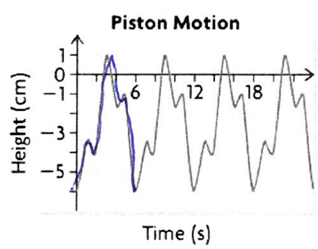
e.g. Consider the following pictures: Determine which are periodic.

a) 

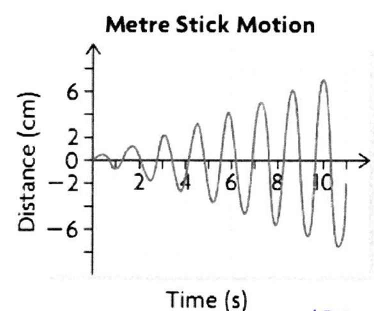
Periodic

b) 

periodic

c) 

periodic

d) 

Not periodic

Definition 6.1.2

The **Period** of a periodic function is the amount of the **domain values** where **one cycle** takes place.

the repeating shape

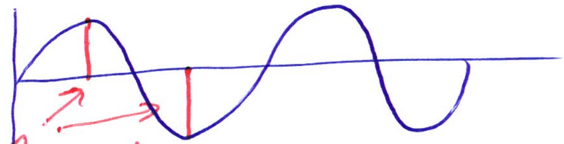
Example 6.1.1

Determine the periods of the above periodic functions:

a) 365 days

b) 4 seconds

c) 6 seconds



Amplitude

Definition 6.1.3

a) The **Amplitude** of a periodic function is half of the distance between a maximum value and a minimum value.

$$\text{Amplitude} = \frac{\max - \min}{2}$$

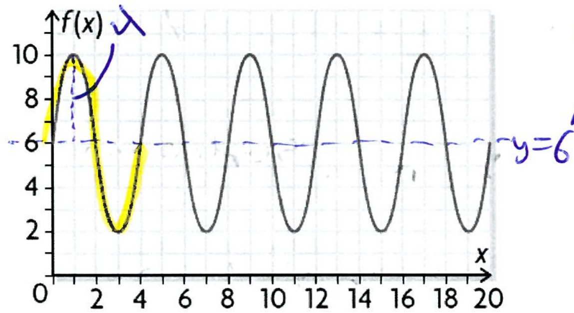
from central axis to max/min

b) The **Central Axis** is half way between the maximum value and the minimum value.

The equation of The Central Axis is given by $y = \dfrac{\max + \min}{2}$. (Average)

Example 6.1.1

Determine the range, period, equation of the central axis, and amplitude of the function shown.



Max: 10

Min: 2

$$\text{Amplitude} = \frac{\max - \min}{2} = \frac{10-2}{2} = 4$$

one period = 4 seconds / cycle

$$C.A. = \frac{\max + \min}{2} = \frac{10+2}{2}$$

$$\boxed{y=6}$$

Range: $\{f(x) \in \mathbb{R} \mid +2 \le f(x) \le 10\}$

Example 6.1.2

3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.

a) What is the period of one complete cycle?

b) What is the maximum distance between the device and the appliance?

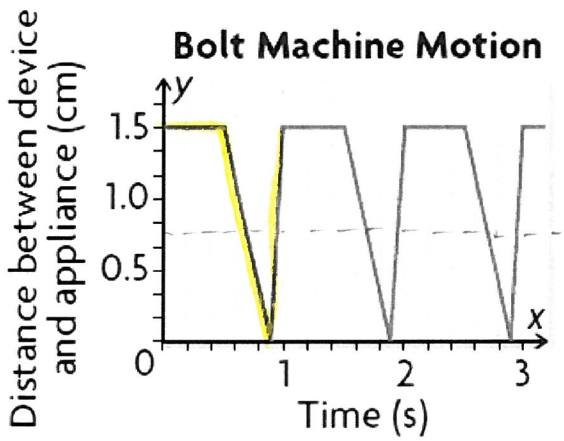
c) What is the range of this function?

d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.

e) Determine the equation of the axis.

f) Determine the amplitude.

g) There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of "attaching the bolt."



a) Period = 1 second / cycle

b) 1.5 cm (furthest away)

c) Range: $\{f(x) \in \mathbb{R} \mid 0 \le f(x) \le 1.5\}$

d) Domain is 5 periods (5 sec)

$\{x \in \mathbb{R} \mid 0 \le x \le 5\}$

e) $C.A. = \frac{max + min}{2} = \frac{1.5 + 0}{2} \overset{y=}{=} 0.75$

f) $Amp = \frac{max - min}{2} = \frac{1.5 - 0}{2} = 0.75$ cm

Success Criteria:

- I can find the range, period, central axis, and amplitude of a periodic function
- I can determine IF a function is periodic

Chapter 6 – Sinusoidal Functions

6.5 – Sketching Sinusoidal Functions

Learning Goal: We are learning to sketch the graphs of sinusoidal functions using transformations.

By this point in your illustrious High School careers, you have a solid understanding of Transformations of Functions in general. In terms of the sinusoidal functions Sine and Cosine in particular, the concepts are as you expect, but the transformations have specific meanings relating to nature of the sinusoidal "wave".

General Form of the Sine and Cosine Functions

$$f(\theta)=a\sin\left(k(\theta-d)\right)+c \qquad g(\theta)=a\cos\left(k(\theta-d)\right)+c$$

Transformation	Properties
$a=$ amplitude $=\lvert a\rvert$ (vertical stretch) If $a<0$, we also have a vertical flip!	$a=\dfrac{\max-\min}{2}$
$k=$ period factor (horizontal stretch)	Period $=\dfrac{360}{k}$ $k=\dfrac{360}{P}$ k converts a real life period (seconds) into degrees.
$d=$ phase shift (horizontal shift)	Note: To determine d you MUST factor "k" away from angle θ
$c=$ Central Axis $y=c$ (vertical shift)	$c=\dfrac{\max+\min}{2}$

Patterns

a	Sine	Cosine
+	CA - max - CA - min - CA	max - CA - min - CA - max
−	CA - min - CA - max - CA	min - CA - max - CA - min

Example 6.5.1

Determine the amplitude, period, phase shift and the equation of the central axis for:

a) $f(\theta) = 2\sin(\theta + 60^{\circ}) + 1$ $k=1$

Amplitude, $a = 2$

Period $= \frac{360}{1} = 360^{\circ}$

Central Axis $y = 1$

Phase Shift 60° Left

b) $g(\theta) = 3\cos(2\theta - 90^{\circ})$ Factor k!

$= 3\cos(2(\theta - 45))$

Amplitude $= 3$

Period $P = \frac{360^{\circ}}{2} = 180^{\circ}$

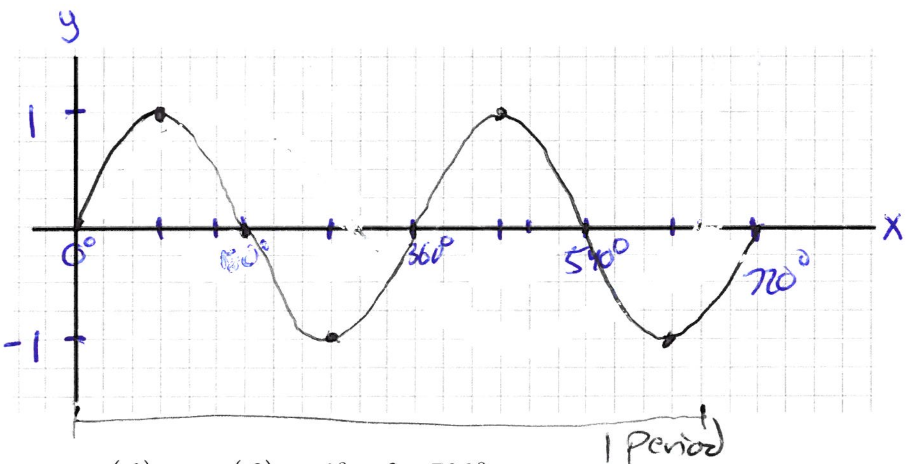
Central Axis $y = 0$

phase shift 45° Right

The Graphs of Sin and Cos

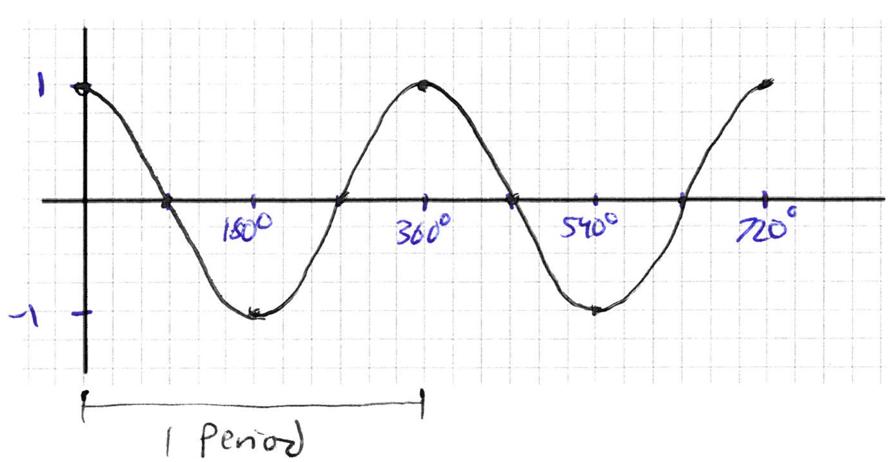
Using the special angles of 0º, 90º, 180º, 270º, and 360º, graph the following functions:

$f(\theta) = \sin(\theta) \quad 0^{\circ} \le \theta \le 720^{\circ}$



x	y
0°	0
90°	1
180°	0
270°	-1
360°	0

$g(\theta) = \cos(\theta) \quad 0^{\circ} \le \theta \le 720^{\circ}$



x	y
0°	1
90°	0
180°	-1
270°	0
360°	1

Apply hor + vert transformations to parent function.

Example 6.5.2

Sketch $f(x)=\sin(x)$ and $g(x)=\sin(2x)$ for $0^{\circ}\leq x\leq 360^{\circ}$ on the same set of axes.

Same as our earlier unit, apply transformations to Parent Function !!

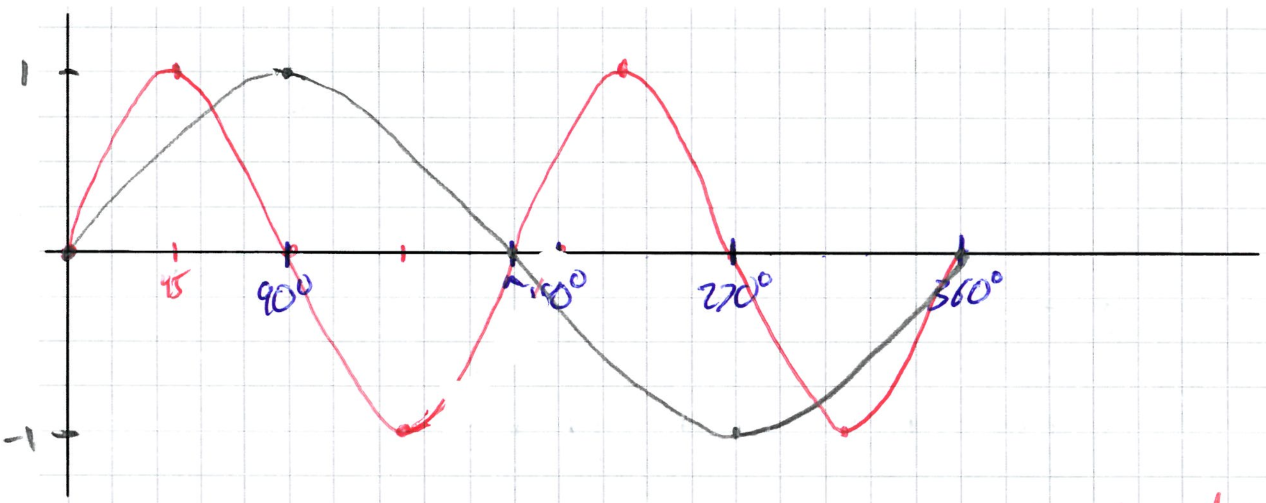
$a=1$ $k=2$

$c=0$ phase shift $=0$

x	y
0	0
90	1
180	0
270	-1
360	0

$\frac{x}{k} \rightarrow$

$\frac{x}{2}$	y
0	0
45°	1
90°	0
135°	-1
180°	0



The k value of 2 compresses the graph in half. We can fit two in there!

Now…what would $\sin\left(\frac{1}{2}x\right)$ look like?

would stretch it out (double the length)

Notes about Domain and Range: Consider the function $f(x)=-2\cos(3x+90^{\circ})+3$.

Determine all the transformations for this function. Without graphing, determine the range of the function. Determine the domain of the function for: 1 cycle; 2 cycles; 3 cycles.

Since sinusoidal functions repeat, the domain, in general, is $\{\theta = \mathbb{R}\}$

But we can restrict the domain, to number of periods.

$P=\frac{360}{3}=120^{\circ}$

One cycle: $\{x\in\mathbb{R} \mid 0^{\circ}\leq x\leq 120^{\circ}\}$

Two cycles: $\{x\in\mathbb{R} \mid 0^{\circ}\leq x\leq 240^{\circ}\}$

Three : $\{x\in\mathbb{R} \mid 0^{\circ}\leq x\leq 360^{\circ}\}$

For range, study the max & min amplitudes + center line

$a=-2$ so flip + stretch

5

$y=3$

1

Max @ $5=3+2$ Min @ $1=3-2$

So Range $=\{f(x)\in\mathbb{R} \mid 1\leq f(x)\leq 5\}$

$a=2$ (flipped) — Right 60° — $y=1$ center — $k=1$, Period $=360^\circ$

Example 6.5.3

Sketch $f(\theta) = -2\cos(\theta - 60^\circ) + 1$ on $0^\circ \le \theta \le 360^\circ$. State transformations, create tables, and state domain and range of the function.

Max $= 1+2 = 3$

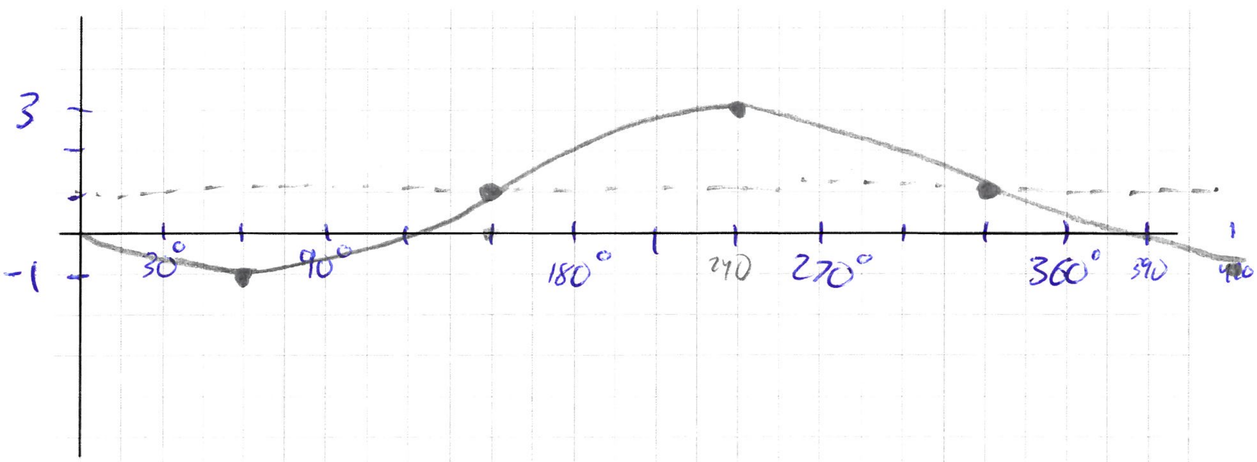
Min $= 1-2 = -1$

Cos θ

x	y
0	1
90	0
180	-1
270	0
360	1

Transformed

x + 60	-2y + 1
0+60 = 60	-2(1)+1 = -1
90+60 = 150	-2(0)+1 = 1
= 240	-2(-1)+1 = 3
= 330	1
= 420	-1



k Right 45°

center line

a $(2(\theta-45))-1$

Example 6.5.4

Sketch $f(\theta)=3\sin(2\theta-90)-1$. State transformations, create tables, and state domain and range of the function. (Give 2 cycles)

Period $= \frac{360}{k} = \frac{360}{2} = 180^\circ$

(compressed)

max $= -1+3 = 2$

min $= -1-3 = -4$

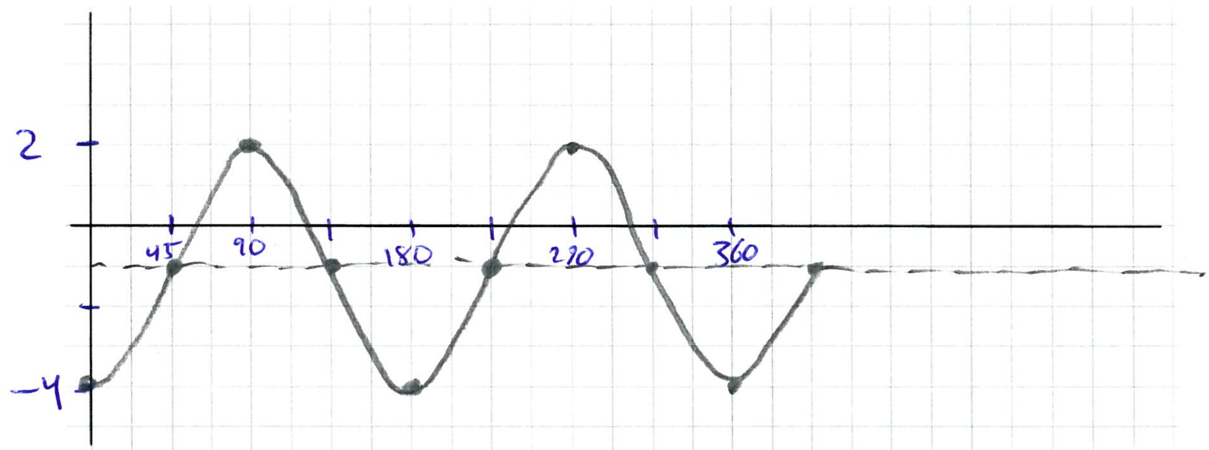
Parent

x	y
0	0
90	1
180	0
270	-1
360	0

Transformed

$\frac{x}{2}+45^\circ$	$3y-1$
$\frac{0}{2}+45=45$	$3(0)-1=-1$
$\frac{90}{2}+45=90$	$3(1)-1=2$
$\frac{180}{2}+45=135$	-1
180	-4
225	-1

Then, the cycle repeats. Follow the pattern in your sketch.



Success Criteria

- I can sketch the graph of a sinusoidal function by applying the transformations to the parent function.

Chapter 6 – Sinusoidal Functions

6.6 – Models of Sinusoidal Functions

Learning Goal: We are learning to create a sinusoidal function from a graph or table of values.

In this section we will look at how to develop a sinusoidal function which can explain given information. In essence we will be writing sine or cosine functions based on given transformations.

Just as a reminder:

General Form of the Sine and Cosine Functions

$$f(\theta) = a\sin(k(\theta - d)) + c \qquad g(\theta) = a\cos(k(\theta - d)) + c$$

Another reminder (about the pattern of sinusoidal functions):

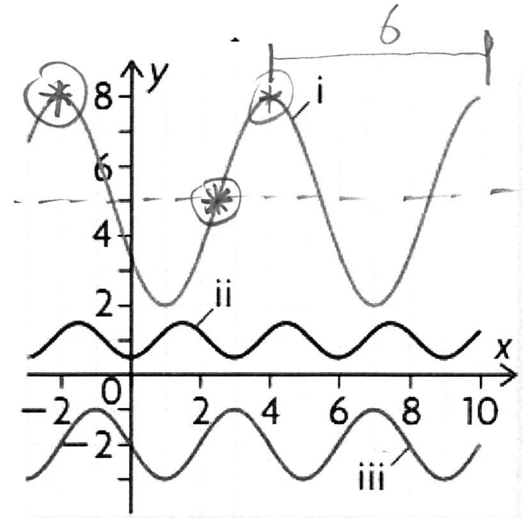
Sine functions "start" at the central axis and go up to a **max** if "*a*" is **positive**, or down to a **min** if "*a*" is **negative**.

Cosine functions "start" at a **max** if "*a*" is **positive**, or at a **min** if "*a*" is **negative**.

Example 6.6.1

From your text: Pg. 391 #4a

Determine a sinusoidal equation for each function:



could even do a negative cos!

Need 4 Things

- amplitude $\left(\frac{max - min}{2}\right)$
- central axis $\left(\frac{max + min}{2}\right)$
- period (how long it takes to get 1 cycle). Use this to find k. $k = \frac{360}{period}$
- Phase Shift

i) $a = \frac{max - min}{2} = \frac{8-2}{2} = 3$ $\quad$ C.A $= \frac{max + min}{2} = \frac{8+2}{2} = 5$ $\quad$ Period $= 6$ $\therefore k = \frac{360}{6} = 60$

Phase Shift as sine: R 2.5 $\quad$ as Cosine: R 4 (or L2)

Sine: $f(x) = 3\sin(60(x-2.5)) + 5$

Cosine: $f(x) = 3\cos(60(x-4)) + 5$ OR $f(x) = 3\cos(60(x+2)) + 5$

Example 6.6.2

From your text: Pg. 392 #5a)

hope for a 0 phase shift

5. For each table of data, determine the equation of the function that is the simplest model.

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1
	Max	CA	min	CA	Max	CA	min

max = 3

min = 1

amplitude = 1

$= \frac{3-1}{2} = \frac{2}{2} = 1$

Period? Study a max to max

$= 120^\circ$

$\therefore k = \frac{360}{120}$

$k = 3$

Phase shift

↳ None if you use a + cosine function.

$\therefore y = \cos(3x) + 2$

Example 6.6.3

From your text: Pg. 392 #6b)

6. Determine the equation of the cosine function whose graph has each of the following features.

	Amplitude	Period	Equation of the Axis	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	+ 30° (right)

a) $k = \frac{360}{360} = 1$ $\therefore f(x) = 3\cos(x) + 11$

b) $k = \frac{360}{180} = 2$ $\therefore g(x) = 4\cos(2(x - 30)) + 15$

Example 6.6.4
A sinusoidal function has an amplitude of 4 units, a period of 120º, and a maximum at (0,9). Determine the equation of the function.

max $= 9$

-4

$\boxed{\text{center} = 5}$

-4

min $= 1$

$k = \frac{360^\circ}{120^\circ}$

$\boxed{k = 3}$

No phase shift

$\therefore f(x) = 4\cos(3x) + 5$

Success Criteria:

- I can create an sinusoidal function based on information from a graph or table
- I can recognize when it is best to use a sine or cosine function

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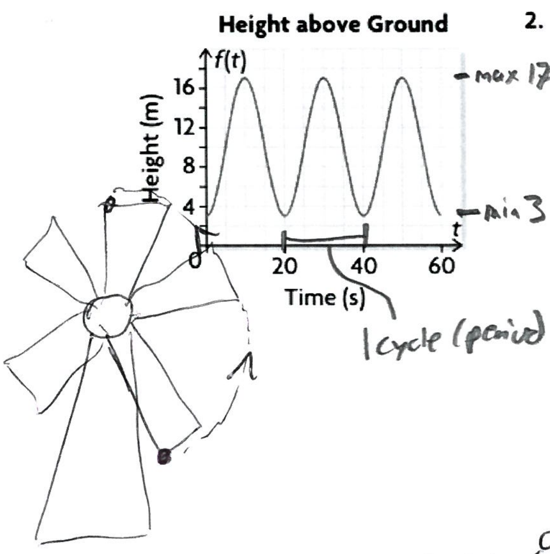
Chapter 6 – Sinusoidal Functions
6.7 – Problem Solving with Sinusoidal Functions

Learning Goal: We are learning to solve problems related to real-world applications of sinusoidal functions.

We can use the sinusoidal properties of **Period**, **Central Axis**, **Amplitude** and **Phase Shift** to describe and solve "real world" problems.

Example 6.7.1 (*From the text: Pg. 398 #2*)



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?

a) the center of the windmill (axle)

$c = \frac{max + min}{2}$

$= \frac{17+3}{2}$

$= 10$ $\boxed{y=10}$

b) Amplitude is the length of the blades

$a = \frac{max - min}{2}$

$= \frac{17-3}{2}$

$= 7$

c) Period is 20 seconds. This is one complete rotation of a blade.

d) Need $7 \times 20 = 140$

$\therefore D: \{t \in \mathbb{R} \mid 0 \le t \le 140\}$

$R: \{f(t) \in \mathbb{R} \mid 3 \le f(t) \le 17\}$

min max

e) Need $k = \frac{360}{period} = \frac{360}{20} = 18$

Phase shift? None if we use $-\cos$.

$f(t) = a \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (k(t-d)) + c$

$= -10\cos(18(t)) + 10$

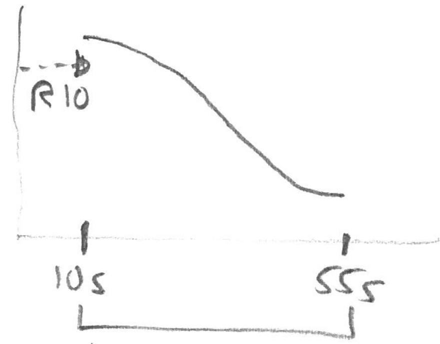
f) The period would increase, and the graph would stretch out. k decreases in size.

Example 6.7.2

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches a maximum height of 11m at 10s and then reaches a minimum height of 1m at 55s. How high is John after 2 minutes?

Part 1: Generate an Equation

k?



max = 11

min = 1

$a = \frac{11-1}{2}$ $\boxed{a = 5}$

$CA = \frac{11+1}{2}$ $\boxed{y = 6}$

½ a period = 45 sec

1 period = 90 sec

$$\boxed{\therefore k = \frac{360}{90} = 4}$$

Phase shift? Let's use +cos to start at maximum.

The first max is @ 10 seconds. ∴ phase shift is 10 right

Equation: $h(x) = a\cos(k(x-d)) + c$

$$= 5\cos(4(x-10)) + 6$$

↑ For right

Part 2: Use the Equation to solve.

Need t in seconds. 2 min = 120 sec

$$\therefore h(120) = 5\cos(4(120-10)) + 6$$

$$= 5\cos(4(110)) + 6$$

$$= 5\cos(440) + 6$$

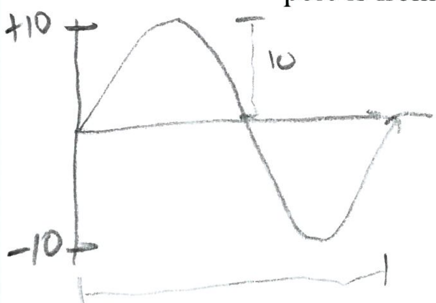
$$= 6.868$$

He is 6.87 m high after 2 minutes.

Example 6.7.3 (Text pg. 396)

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right (+10 cm) and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At t = 0, the pole was momentarily at its resting position. Then it started moving to the right.

a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.



• $a = 10$

• CA ⇒ $y = 0$

• Phase shift?
None needed if you use + sine

• Period?

Let's use time in seconds ☺

Need $\frac{\text{time}}{\text{cycle}} = \frac{60\text{ sec}}{240\text{ cycles}} = \frac{0.25\text{ sec}}{1\text{ cycle}}$

$\therefore k = \frac{360}{\text{period}} = \frac{360}{0.25}$

$k = 1440$

$\therefore y = 10\sin(1440x)$

b) How does the situation affect the domain and range?

Domain is restricted to positive values

Range is based on the amplitudes.

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Only the amplitude changes. 80% remains.

$10 \times 0.8 = 8$

$\therefore y = 8\sin(1440x)$

Success Criteria:

- I can create a sinusoidal function that represents information from a real-life scenario
- I can use the function to solve further problems