

# Functions 11

## Teacher Notes

# Course Notes

# Unit 8 – Financial Math

***Compound interest is the greatest Mathematical discovery of all time***

Albert Einstein (maybe)

*We will learn*

- understand the connections between simple interest and Arithmetic Sequences and linear growth
- understand the connections between compound interest and Geometric Sequences and exponential growth
- solve problems using the formulae, or using spreadsheets, for simple and compound interest, and for annuities/mortgages



# Chapter 8 – Financial Math

*Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.*

**8.1:** Pg. 481 – 482 #5bce, 6, 7, 11, 8 (yes, try #8 last!)

**8.2:** Pg. 490 – 491 #1ab, 3cd (no timeline), 6 (guess and check!), 7 (option two is tricky), 9, 10

**8.3:** Pg. 498 – 499 #1ab, 3cd, 4, 6, 8 (ask for help if you need it), 10

**8.4:** Pg. 511 – 512 #3, 5bd, 6, 8 (guess and check!), 9, 10

**8.5:** Pg. 520 – 521 #1ac (*no timelines, no series*), 3bc, 4, 6 (*yes – CD/DVD players did cost that much in the past!*), 7, 10

## Chapter 8 – Financial Applications (Discrete Functions)

### 8.1 Simple Interest

**Learning Goal:** We are learning to calculate simple interest.

This *should* be review, but it never hurts to review previously learned material.

Most people are *interested* in their personal financial situations. Obviously that's why we call the money earned on investments **interest**.

Simple **I**nterest is calculated using an interest **r**ate, **r** (%/a), over a period of time, **t** (in years).

We call the amount invested (borrowed) the **P**rincipal.

↑ **r** is rate per year. BUT % are not numbers. convert to a decimal by  $\div$  by 100.

The Interest Formula

$$I = Prt$$

Both **r** & **t** must be based on the same time period. Here, in years.

The **A**mount (of money) Formula

$$A = P + I = P + Prt = P(1 + rt)$$

#### Example 8.1.1

From your text: Pg. 481 #5f

For each investment, calculate the interest earned and the total amount.

	Principal	Rate of Simple Interest per Year	Time
a)	\$500	4.8%	8 years
b)	\$3 200	9.8%	12 years
c)	\$5 000	3.9%	16 months
d)	\$128	18%	5 months
e)	\$50 000	24%	17 weeks
f)	\$4 500	12%	100 days

→  $I = Prt = (500)(0.048)(8) = \$192$

$A = P + I = 500 + 192 = \$692$

→  $I = (500)(0.039)\left(\frac{16}{12}\right)$  \*convert to years

$= \$260$

$\therefore A = P + I = \$5260$

$I = (4500)(0.12)\left(\frac{100}{365}\right)$   
 $= \$147.95$

means per annum (year)

### Example 8.1.2

Jasmine invests \$4850 at 7.6%/a simple interest. If she wants her money to increase to \$8000, for how long will she need to keep her money invested?

(A)

find t

Given	want
$P = 4850$	time
$r = 0.076$	
$A = 8000$	

$$A = P(1 + rt)$$

$$8000 = 4850(1 + 0.076t) \quad (\div 4850)$$

$$1.649 = 1 + 0.076t \quad (-1)$$

$$\frac{0.649}{0.076} = \frac{0.076t}{0.076}$$

$$8.5 = t$$

8.5 years

### Example 8.1.3

Philip (the unwise) borrows \$1540 for 90 days by taking a cash advance from the company YourCashIsOurCash. The interest rate Philip (unwisely) agrees to is 26%/a (simple interest). How much money will Philip have to pay back at the end of 90 days, and how much interest does he pay?

Given	want
$P = 1540$	A
$t = \frac{90}{365}$	then I
(need t in years)	
$r = 0.26$	

$$A = P(1 + rt)$$

$$= 1540 \left( 1 + 0.26 \left( \frac{90}{365} \right) \right)$$

$$= \$1638.73$$

$$I = A - P = 1638.73 - 1540$$

Philip Pays

$$= \$98.73$$

### Success Criteria:

- I can recognize that simple interest is calculated only on the principal
- Simple interest is an example of a linear function



## Chapter 8 – Financial Mathematics

### 8.2 – Compound Interest and Future Value

**Learning Goal:** We are learning to determine the future value of a principal amount, using compound interest. This applies to both savings and loans.

Last day we look at the idea of Simple Interest (with formula  $A = P(1 + rt)$ ). Now we consider the notion of Compound Interest. Compound Interest is such that your savings grows much more quickly than it could if you were just earning simple interest.

Compound Interest Formulae:

Future Value

$$A = P(1 + i)^n$$

where:  $A = \text{Amount (in future)}$

$P = \text{Principal (present value)}$

$i = \text{interest rate per compounding period}$

$n = \text{total number of compounding periods}$

Now, there is one major challenge to any calculations. Interest is normally stated on a yearly basis, but interest is actually charged more frequently (the bank earns more money when you pay interest on interest ☹). So if you pay 12% interest per year, but pay interest monthly, you actually pay 1% each month. You must incorporate this into your calculations.

Here is a helpful chart:

	How often you pay (Times per year)	$i$ value in your equation	$n$ value in your equation
Annually	1	$i$	$n$
Semi-annually	2	$\frac{i}{2}$	$2n$
Quarterly	4	$\frac{i}{4}$	$4n$
Monthly	12	$\frac{i}{12}$	$12n$

$n$  is normally given in years. It must match the interest period.

Ex (If is monthly, need  $n$  in months.)

### Example 8.2.1

- a) You deposit \$10 000 into an account which pays 2.4% per year, compounded annually. What is the amount of money in the account in 10 years?

Given	Want
$P = 10,000$	$A$
$i = \frac{0.024}{1}$	
$n = 10(1)$	

$$\begin{aligned} A &= P(1+i)^n \\ &= 10000(1.024)^{10} \\ &= \$12,676.51 \end{aligned}$$

- b) You deposit \$10 000 into an account which pays 2.4% per year, compounded monthly. What is the amount of money in the account in 10 years?

Given	Want
$P = 10,000$	$A$
$i = \frac{0.024}{12}$	
$= 0.002$	
$n = 10(12) = 120$	

$$\begin{aligned} A &= P(1+i)^n \\ &= 10,000(1.002)^{120} \\ &= \$12,709.44 \end{aligned}$$

### Example 8.2.2

Beth deposits \$500 into an account which pays 6% compounded monthly. Fred deposits \$500 into an account which pays 6% simple interest. What is the difference in the value of their accounts after 5 years?

Beth

$$\begin{aligned} P &= 500 \\ i &= \frac{0.06}{12} = 0.005 \\ n &= 5(12) = 60 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 500(1.005)^{60} \\ &= \$674.43 \end{aligned}$$

Fred

$$\begin{aligned} P &= 500 \\ i &= 0.06 \quad t = 5 \end{aligned}$$

$$\begin{aligned} A &= P(1+rt) \\ &= 500(1+(0.06)(5)) \\ &= \$650 \end{aligned}$$

∴ Beth made an additional \$24.43.

**Example 8.2.3**

On her 15<sup>th</sup> birthday, being very wise, Susan invests \$10,000 in an account which pays 2.4% compounded monthly. The not-so-wise John waits until his 45<sup>th</sup> birthday to invest \$10,000 in an account which pays 2.4% compounded monthly. How much is each account worth when they reach 65 years old?

Susan

$$P = 10,000$$

$$i = \frac{0.024}{12}$$
$$= 0.002$$

$$n = (50)(12)$$
$$= 600$$

$$A = ?$$

$$A = P(1+i)^n$$
$$= 10,000(1.002)^{600}$$
$$= \$33,161.40$$

John

$$P = 10,000$$

$$i = \frac{0.024}{12}$$

$$= 0.002$$

$$n = (20)(12)$$
$$= 240$$

$$A = ?$$

$$A = P(1+i)^n$$
$$A = 10,000(1.002)^{240}$$
$$= \$16,153$$

### Example 8.2.4

You find yourself in a furniture store. Looking around you become dazzled by the adverts promising a better and happier life if you only had one of their beautiful couches. The advertisement reads:

No Money Down!  
No Payments for 3 years!!  
Take this couch home today!!!!  
Only \$1599\*!!!!!!!!!!!!!!

You decide it's a good deal, but you neglect to read the asterisk until it's too late. You've already signed on the dotted line. After signing you decide to finally read the fine print which says:

*\*Financed at 18% compounded monthly*

How much do you have to pay after 3 years?

How much of that is **just** interest?

$$P = 1599$$

$$i = \frac{0.18}{12}$$

$$= 0.015$$

$$n = 3(12)$$
$$= 36$$

$$A = ?$$

$$A = P(1+i)^n$$
$$= 1599(1.015)^{36}$$
$$= \$2732.91$$

### Success Criteria:

- I can determine the related interest ( $i$ ) per compounding period ( $n$ )
- I can use the Future Value formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking  $A - P$ .



## Chapter 8 – Financial Mathematics

### 8.3 – Compound Interest and Present Value

How much money must I invest now to get X later?

**Learning Goal:** We are learning to determine the present value remaining on a loan or investment.

Last day we looked at the idea of Compound Interest (Future Value). Today we continue looking at Compound Interest with a focus on Present Value.

Compound Interest Formulae:

IF  $r = \text{rate / year}$   
 $i = \frac{r}{N}$  ← Annual interest rate  
↑ # of periods in 1 year

Future Value

$$A = P(1+i)^n$$

where:  $A = \text{Amount (Future Value)}$   $P = \text{Principal (Present Value)}$

$i = \text{interest rate per compounding period}$

Present Value

$$P = \frac{A}{(1+i)^n}$$

$n = \text{Total \# of Compounding Periods.}$   
 $i$  and  $n$  must be in the same units.

#### Example 8.3.1

You want \$10 000 in your bank account 20 years from now. Your account pays 1.8% per year, compounded annually. What is the amount of money you have to deposit today?

Given  
 $A = 10,000$   
 $t = 20$   $n =$   $P = ?$   
 $r = 0.018$   
 $i = \frac{r}{n} = \frac{0.018}{1} = 0.018$

$$P = \frac{A}{(1+i)^n} = \frac{10,000}{(1.018)^{20}} = \$6999.14$$

#### Example 8.3.2

You want to buy a house at 30 years old (14 years from now). You estimate that you will need a down payment of \$150 000. You find a bond which matures in 14 years paying 3.6% interest compounded monthly. How much do you need to invest in the bond today?

Given  
 $A = 150,000$   
 $i = \frac{0.036}{12} = 0.003$   
 $n = (14)(12) = 168$

want  
 $P$

$$P = \frac{A}{(1+i)^n} = \frac{150,000}{(1.003)^{168}} = \$90,684.80$$

### Example 8.3.3

Today Henry invests some money in an account which pays 2% compounded quarterly. 10 years from now he takes the money in his account and reinvests it in an account which pays 4.8% compounded monthly. After an additional 15 years Scott has \$125,345 in his account. How much did he originally invest?

Two Steps

Start w/ the second acct

Account 1	Account 2
2% (10 yrs)	4.8% (15 yrs)
$A_{\text{end}} = \$125,345$	$A_{\text{end}} = 125,345$
$i = \frac{0.02}{4}$	$i = \frac{0.048}{12} = 0.004$
$= 0.005$	
$n = (10)(4)$	$n = (15)(12)$
$= 40$	$= 180$
	need $P_1$

$$\textcircled{1} P_1 = \frac{A}{(1+i)^n}$$
$$= \frac{125,345}{(1.004)^{180}}$$

$$= 61,099.65$$

This is how much the 1st account ended with.

$$\textcircled{2} P_1 = \frac{A}{(1+i)^n}$$
$$= \frac{61,099.65}{(1.005)^{40}}$$

$$= \$50,049.09$$

He began with  
\$50,049.09

### Success Criteria:

- I can use the Present Value formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking  $A - P$ .

## Chapter 8 – Financial Mathematics

### 8.4 – Annuities: Future Value

**Learning Goal:** We are learning to determine the future value of an annuity earning compound interest.

The problem with the examples in 8.2 and 8.3 is that not many people have large amounts of money to be depositing into savings accounts. People usually make regular deposits of smaller amounts of money. An account into which (or out of!) regular payments are made is called AN ANNUITY. We will study two aspects of annuities: Future Value and Present Value.

Future Value of an Annuity

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

Be very careful with the brackets

where:  $FV$  = Future Value

$n$  = total # of compounding periods

$R$  = Regular Payment

$PV$  = Present Value

$i$  = interest per compounding period  
= ~~rate~~  
rate

#### Example 8.4.1

Dylan decides to deposit \$200 monthly into an account which pays 3.6% per year, compounded monthly. What is the value of his annuity after 25 years? How much interest is earned? If Dylan leaves the money in his account for another 20 years, but makes no more regular payments, how much money will be in the account at the end of 45 years?

1st 25 yrs = Annuity

Given

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

$R = 200$   
 $i = \frac{0.036}{12} = 0.003$   
 $n = (25)(12) = 300$

$$= \frac{200[(1.003)^{300} - 1]}{0.003}$$

$$= \$97,086.10$$

$i = (25)(12)$

$= 300$

FV?

$I = 97,086.10 - (200)(300)$  what Dylan deposited

$$= \$37,086.10$$

2nd 20 yrs = Compounding

Given

$$P = 97,086.10 \quad A = P(1+i)^n$$

$i = 0.003$   
 $n = (20)(12) = 240$

$$= 97,086.10(1.003)^{240}$$

$$= \$149,242.04$$



!! unrealistic

#### Example 8.4.2

Annuity

Tingyi invests \$300, every three months (i.e. quarterly) into an RRSP which pays (on average) 12% compounded quarterly. What is the value of her RRSP after 45 years?

Given

$$R = 300$$

$$i = \frac{0.12}{4}$$

$$= 0.03$$

FV?

$$n = (45)(4) \\ = 180$$

$$FV = \frac{R((1+i)^n - 1)}{i} \\ = \frac{300((1.03)^{180} - 1)}{0.03} \\ = \$2,035,033.59$$

#### Example 8.4.3

An absolutely wonderful student decides to give Mr. Templeton \$100,000 when he retires in 15 years. The student finds an investment which pays 9% compounded monthly. What is the regular payment s/he would need to make to have \$100,000 in 15 years?

Given

$$FV = 100,000$$

$$i = \frac{0.09}{12}$$

$$= 0.0075$$

$$n = (12)(15) \\ = 180$$

$$R = ?$$

$$FV = \frac{R((1+i)^n - 1)}{i}$$

$$R = \frac{(FV)(i)}{[(1+i)^n - 1]} \\ = \frac{(100000)(0.0075)}{(1.0075)^{180} - 1} \\ = \$388.03$$

#### Success Criteria:

- I can recognize that an annuity is a "regular payment" on a loan or investment
- I can use the Future Value Annuity formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking  $I = FV - (n \times R)$ .



## Chapter 8 – Financial Mathematics

### 8.5 – Annuities: Present Value

**Learning Goal:** We are learning to calculate the present value of an annuity earning compound interest.

Today we continue with the idea of an annuity, but we will be looking at **PRESENT VALUE** (TODAY'S VALUE OF THE INVESTMENT/LOAN)

#### Future Value of an Annuity

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

#### Present Value of an Annuity

$$PV = \frac{R[1 - (1+i)^{-n}]}{i}$$

→ What must I invest today?

→ Also for loans.

"What did I borrow?"

"How much can I afford based on a payment?"

where:  $FV$  = Future Value

$R$  = The Regular Payment into or out of the annuity

$i$  = The interest rate per compounding period

$n$  = The total number of compounding periods over the life of the account

$PV$  = Present Value

#### Example 8.5.1

While in college you want an annuity to pay you \$250 a month, every month over the four years you are studying. How much money would you need to invest today in an account which pays 2.4% compounded monthly to guarantee the monthly payment? What is the total amount of money you receive from the annuity? How much interest do you earn?

Annuity

PV!

$$PV = \frac{R(1 - (1+i)^{-n})}{i}$$

$$= \frac{250(1 - (1.002)^{-48})}{0.002}$$

$$= \$11,431.11 \text{ invested today.}$$

$$\text{Over 4 years, the annuity will pay } (250)(4)(12) = 12000$$

$$\therefore \text{Interest earned} = 12,000 - 11,431.11 = \$568.89$$

Given

$$R = 250$$

$$i = \frac{0.024}{12}$$

$$= 0.002$$

$$n = (4)(12) = 48$$

$$PV = ?$$

$$I = ?$$

we borrow money  
presently

every other week  
 $\frac{52}{2} = 26$

### Example 8.5.2

Henry borrows \$25,000 to buy a car. He pays \$3,000 down. He will take 6 years to pay off the loan. The bank charges 2.6% compounded biweekly. What are the regular payments Henry has to make to pay off the car? How much interest does he pay?

Givens

$$PV = 25,000$$

$$i = \frac{0.026}{26}$$

$$= 0.001$$

$$n = (6)(26)$$

$$= 156$$

$$R = ?$$

$$PV = \frac{R(1 - (1+i)^{-n})}{i}$$

Solve for R

$$(PV)(i) = R(1 - (1+i)^{-n})$$

$$R = \frac{(PV)(i)}{[1 - (1+i)^{-n}]}$$

$$= \frac{(25,000)(0.001)}{[1 - (1.001)^{-156}]}$$

$$= \$173.16$$

$$\text{Total Paid} = (173.16)(156)$$

$$= \$27,012.96$$

$$\therefore \text{Interest Pd} = \$27,012.96 - \$25,000$$

$$= \$2,012.96$$

### Success Criteria:

- I can recognize the difference between present value and future value
- I can use the Future Value Annuity formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking  $I = (n \times R) - PV$ .