Advanced Functions

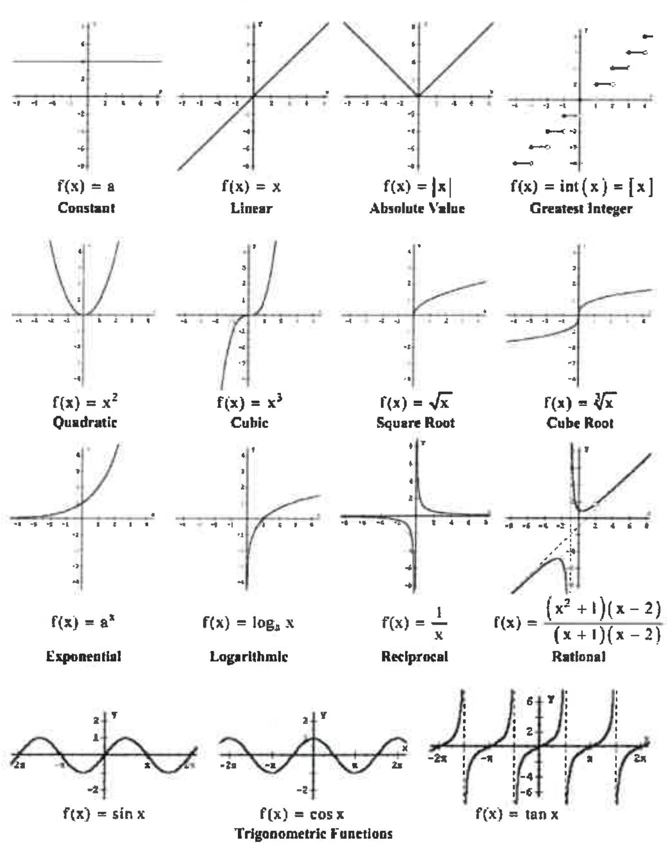
Course Notes

Chapter 1 – Functions

***Learning Goals**: We are learning to...*

- *identify the characteristics of functions and how to represent them*
- *apply transformations to parent functions and how to use transformations for sketching*
- *to determine the inverse of a function*

PARENT FUNCTIONS



A∞Ω
MATH@TD

Chapter 1 – Introduction to Functions

Contents with suggested problems from the Nelson Textbook. These problems are not going to be checked, but you can ask me any questions about them that you like.

1.1 Functions – ***Pg 1 - 3***
Read Example 3 on Page 9 - Pg. 11 – 13 #1 – 3, 5, 6, 7b-f, 9, 10, 12

1.2 Properties of Functions – ***Pg 4 – 12***
Pg. 23 – 24 #5, 7 – 11 (*1.3 in Nelson Text*)

1.3 Transformations of Functions Review – ***Pg 13 - 15***
Worksheet and graphs

1.4 Inverses of Functions – ***Pg 16 - 20***
Pg. 43 – 45 #2 – 4, 7, 9, 12, 13, 15 (*1.5 in Nelson*)

1.5 Piecewise Defined Functions – ***Pg 21 - 25***
(Abs. Value.) Pg. 16 #2, 4 – 8 (think about transformations!), 10 (*1.2*)
(Piecewise) Pg. 51 – 53 #1 – 5, 7 – 9 (*1.6*)

1.1 Functions

Learning Goal: We are learning to represent and describe functions and their characteristics.

There are some people who argue that mathematics has just two basic building blocks: Numbers and Operations. This course is concerned with functions which can be considered number generators. A function takes a given number, and using mathematical operations generates another number. We will be examining the **relationship** between the given numbers, and the generated numbers for various functions.

Definition 1.1.1

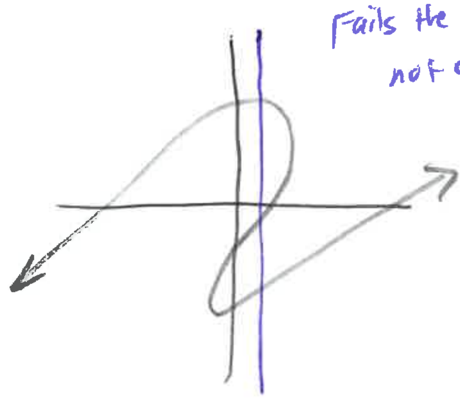
A **Function** is an algebraic rule which assigns exactly one element in a set called the range to each element in a set called the domain

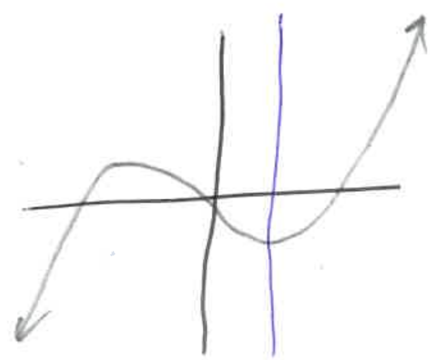
→ Each "x" value produces one "y" value

Pictures

Vertical Line Test

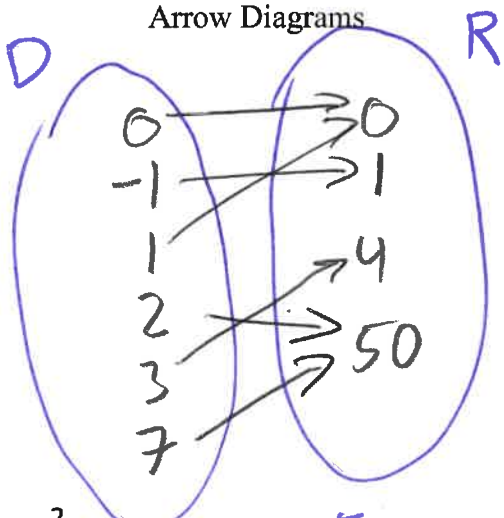
Fails the VLT
not a function

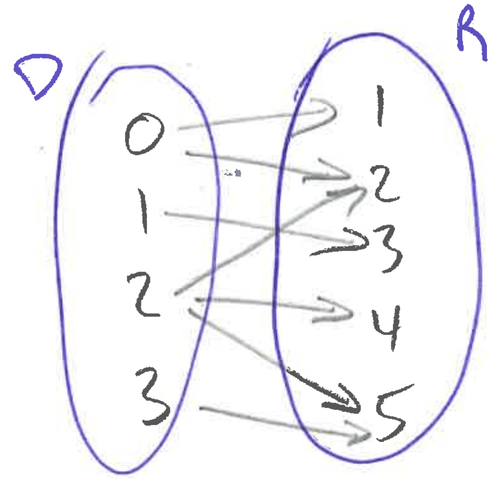




passes
Is a
function

Arrow Diagrams





Is a Fn

Not a Fn

$f(x) = \sqrt{x - 1}$

Definition 1.1.2

Domain of a Function: A set of "x" values which "make sense" when plugged into $f(x)$

Range of a Function: The set of functional values which are calculated or derived from the domain.

Function Notation

We use the notation $f(x)$ to "name" a function. This notation is powerful because it contains both the domain and the range. For example we might write $f(2)$, which shows that the domain value is $x = 2$, and that the range value (which we must calculate) is denoted $f(2)$.

Definition 1.1.3

The **Graph** of a function is a set of points $(x, f(x))$ and denoted $f(x) = \{ (x, f(x)) \mid x \in D_f \}$

Example 1.1.1

Given the graph of the function $f(x) = \{(3,4),(2,-1),(7,8),(4,2),(5,4)\}$ determine:

a) $D_f = \{3, 2, 7, 4, 5\}$

b) $R_f = \{4, -1, 8, 2\}$ ← skip repeats

c) Is $f(x)$ a function?

Yes

Example 1.1.2

Consider the sketch of the graph of $g(x)$, and determine:

a) $D_g = \{ x \in \mathbb{R} \mid -4 \le x \le 5 \}$

b) $R_g = \{ g(x) \in \mathbb{R} \mid -2 \le g(x) \le 3 \}$

c) Is $g(x)$ a function?

No. Fails V.L.T

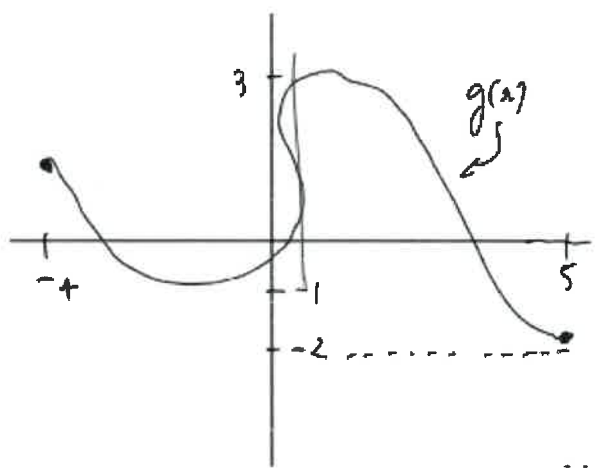


Figure 1.1.2

Note: In the above examples we have seen functions (and non-functions which we call relations) depicted graphically and numerically. We now turn to algebraic representations of functions. It is much more difficult to determine domain and range for functions given in an algebraic form, but the algebraic form is incredibly useful!

Example 1.1.3

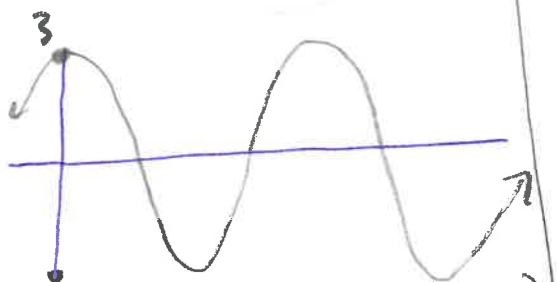
State the domain and range of the functions given in algebraic form.

a) $f(x) = 3\cos(2x)$

b) $g(t) = (t-2)^2 + 1$

c) $h(x) = \dfrac{2}{x-1}$

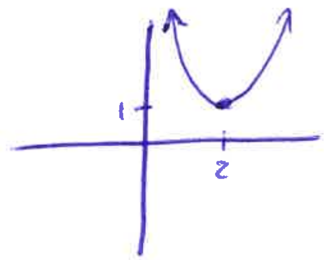
a)



D: $\{x \in \mathbb{R}\}$ or $x \in (-\infty, \infty)$

R: $\{f(x) \in \mathbb{R} \mid -3 \leq f(x) \leq 3\}$ or $f(x) \in [-3, 3]$

b.



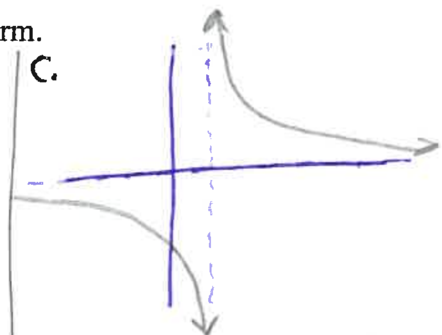
D: $\{t \in \mathbb{R}\}$

R: $\{g(t) \in \mathbb{R} \mid g(t) \geq 1\}$

D: $t \in (-\infty, \infty)$

R: $g(t) \in [1, \infty)$

c.



D: $\{x \in \mathbb{R} \mid x \neq 1\}$

R: $\{h(x) \in \mathbb{R} \mid h(x) \neq 0\}$

D: $x \in (-\infty, 1) \cup (1, \infty)$

Notations for Domain and Range

Use example 1.1.2

Interval Notation	Set Notation	Pseudo-set Notation
D_g: $x \in [-4, 5]$	D: $\{x \in \mathbb{R} \mid -4 \leq x \leq 5\}$	$D_y = -4 \leq x \leq 5$
R_g: $g(x) \in [-2, 3]$	R: $\{g(x) \in \mathbb{R} \mid -2 \leq g(x) \leq 3\}$	$R_y = -2 \leq g(x) \leq 3$

Success Criteria

- I can use function notation to represent the values of a function
- I can apply the vertical line test
- I can identify the domain and range for different types of functions
- I can recognize and apply restrictions on the domains of functions

Real #'s are assumed

$[$ = $\geq$ or $\leq$

$($ = $>$ or $<$

1.2 Properties of Functions

Learning Goal: We are learning to compare and contrast the properties and characteristics of various types of functions.

Recall that we define the graph of a function to be the SET of Ordered Pairs:

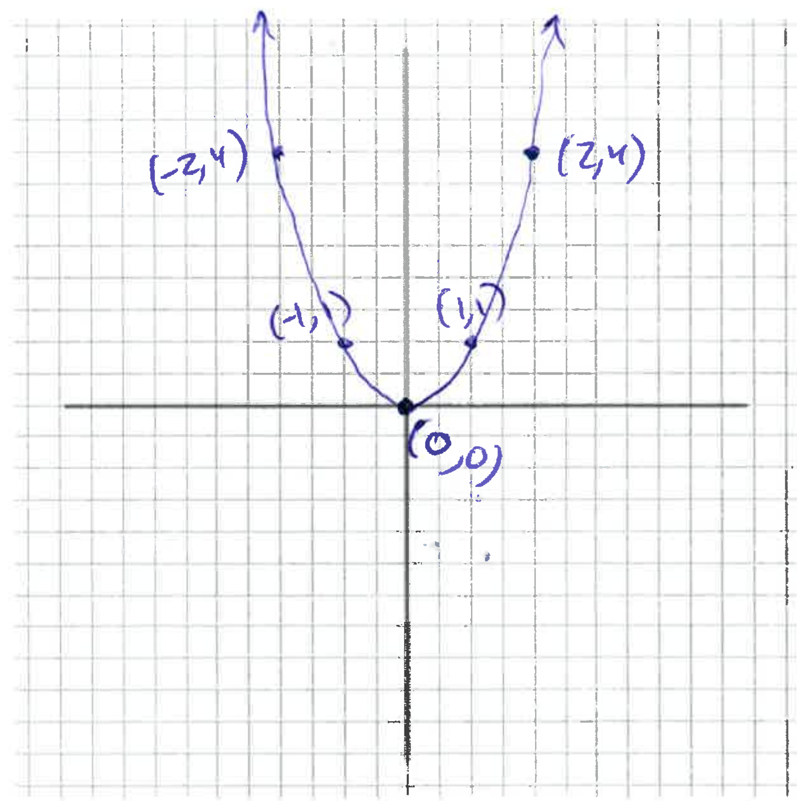
graph: $\{(x, f(x)) \mid x \in D_f\}$

We can visualize the graph of a function by plotting its ordered pairs on the Cartesian axes.

Example 1.2.1

e.g. $f(x) = x^2$ has the graph $\{(x, x^2) \mid x \in \mathbb{R}\}$

and looks like



Characteristics of a Function's Graph

Over the course we will be studying Polynomial, Rational, Trigonometric, Exponential and Logarithmic Functions. For now we are focussed on Polynomial and Rational Functions, but for each type of function we will try and understand various functional (fnal) behaviours (or characteristics).

The characteristics (behaviours) we are primarily interested in studying are:

1. Domain + Range
2. Axis Intercepts
3. Intervals of Increase / Decrease → "Chunks" of the domain
4. Odd + Even functions (symmetry)
5. Continuity + Discontinuity
6. Function End Behaviours

> Note: *Generally a geometric point of view will just mean that we'll look at pictures, but Geometry is actually **much** deeper than that!*

Intervals of Increase and Decrease

↳ only concern the domain

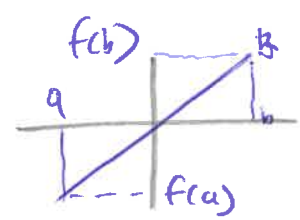
We will examine (when possible) functional behaviour from both algebraic and geometric points of view.

Definition 1.2.1

Interval notation - Both are x-values

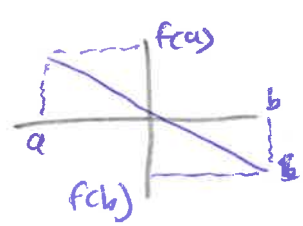
A function $f(x)$ is said to be increasing on the open interval (a,b) when

$f(a) < f(b)$



A function $f(x)$ is said to be decreasing on the open interval (a,b) when

$f(a) > f(b)$



Note the difference between open and closed intevals:

An open interval does NOT contain endpoints

(a, b)

A closed interval DOES contain endpoints

$[a, b]$

Example 1.2.2

Consider the function $f(x)$, represented graphically:

Determine where $f(x)$ is increasing and decreasing.

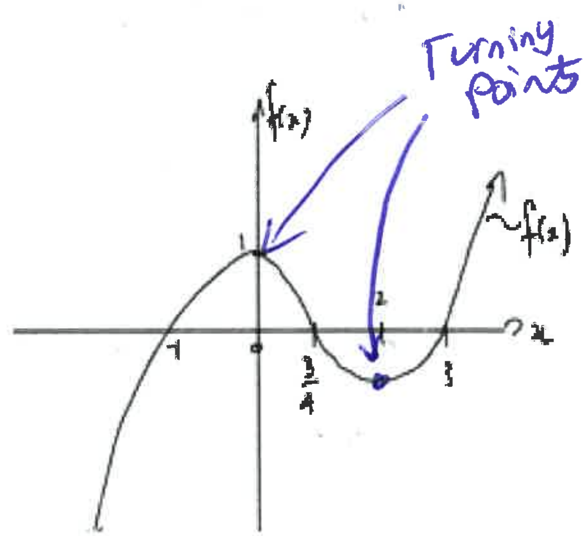


Figure 1.2.2

Increasing: $(-\infty, 0)$ and $(2, \infty)$

so

$(-\infty, 0) \cup (2, \infty)$

Decreasing: $(0, 2)$

At turning points, function is neither increasing nor decreasing!

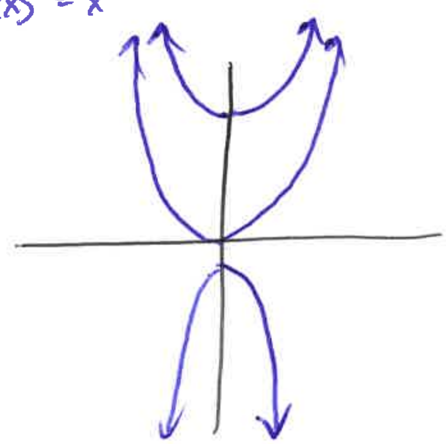
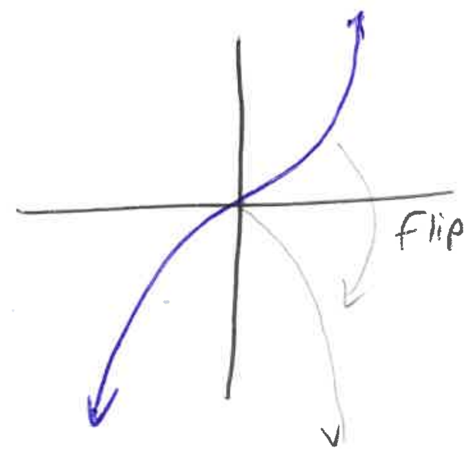
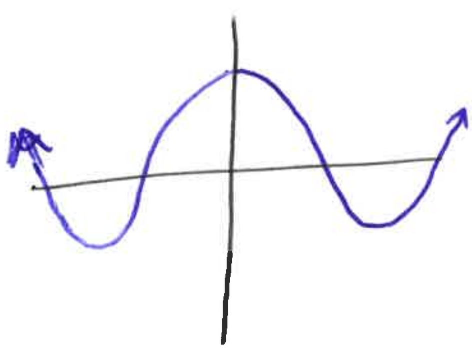
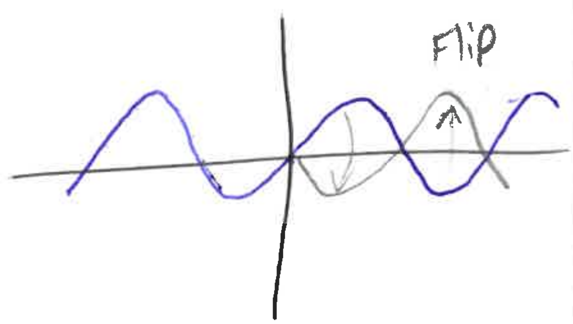
Odd vs. Even Functions

Note: This functional behaviour deals with SYMMETRY rather than the "power(s)" that you might see in various terms of the function.

Basic Definitions:

- Even Functions are symmetric around the "y-axis"
- Odd Functions are symmetric around the origin (0, 0)

Graphical point of view:

Even Functions	*Odd Functions*
$f(x) = x^2$ 	$g(x) = x^3$ 
$y = \cos(x)$ 	$y = \sin(x)$  Now Symmetric on y-axis

Notice the Rotation

Algebraically we will consider definitions for Even and Odd Functions:

Definition 1.2.2

A function $f(x)$ is **even** if $f(x) = f(-x)$ for every $x \in D_f$

A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for every $x \in D_f$

If a vertical flip makes the 2 sides Symmetric

Example 1.2.3

a) Show $f(x) = 3x^4 + 2x^2 + 5$ is even.

Consider $f(-x) = 3(-x)^4 + 2(-x)^2 + 5$

$= 3x^4 + 2x^2 + 5$

$= f(x)$

$\therefore$ $f(x)$ is even!

b) Show $g(x) = 5x^3 - 2x$ is odd. $\rightarrow$ we need to see $-(5x^3 - 2x)$

Consider $g(-x) = 5(-x)^3 - 2(-x)$

$= -5x^3 + 2x$

Factor out the negative

$= -(5x^3 - 2x)$

$= -g(x)$

$\therefore$ $g(x)$ is odd

Remember: $\frac{-2}{3} = \frac{2}{-3} = -\frac{2}{3}$

c) Are i) $f(t) = 5t^3 - 2t + 1$ and

ii) $h(x) = \dfrac{3x^3 - 2x}{x^2 - 1}$ odd or even?

i Consider $f(-t) = 5(-t)^3 - 2(-t) + 1$

$= -5t^3 + 2t + 1$ $\quad f(-t) \neq f(t)$, ∴ not even

OR

$= -(5t^3 - 2t - 1)$ $\quad f(-t) \neq -f(t)$, ∴ not odd

Consider $h(-t) = \dfrac{3(-x)^3 - 2(-x)}{(-x)^2 - 1}$

$= \dfrac{-3x^3 + 2x}{x^2 - 1}$ ← NOT EVEN

$= -\dfrac{(3x^3 - 2x)}{x^2 - 1}$

$= -\dfrac{3x^3 - 2x}{x^2 - 1} = -h(x)$ ∴ ODD

Continuity

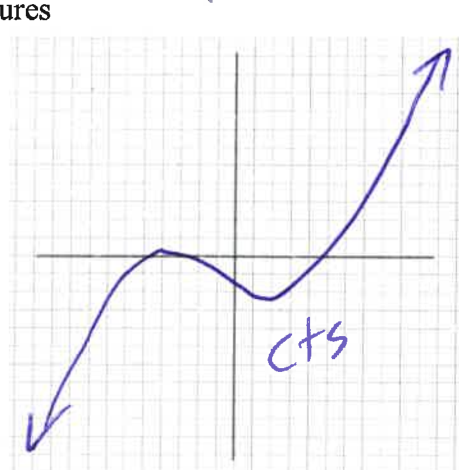
Pause?

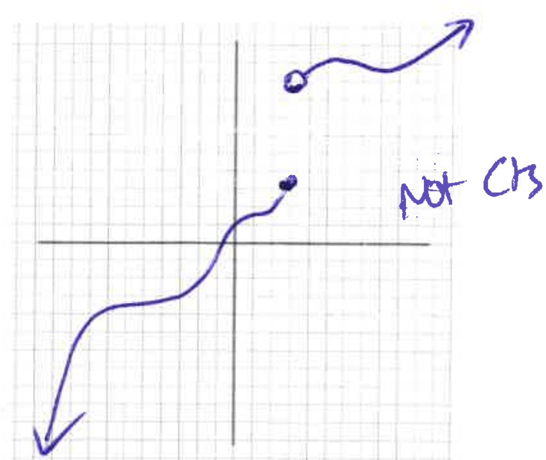
For the time being we will consider a (quite) rough definition of what it means for a function to be continuous. In fact, we will see that understanding what it means for a function to be discontinuous may be more helpful for now. In the course ***Calculus and Vectors***, a formal, algebraic definition of continuity will be considered.

Rough Definition

A function $f(x)$ is **continuous** (cts) on its domain D_f if when sketching the graph, you do not need to lift your pen/pencil from the paper.

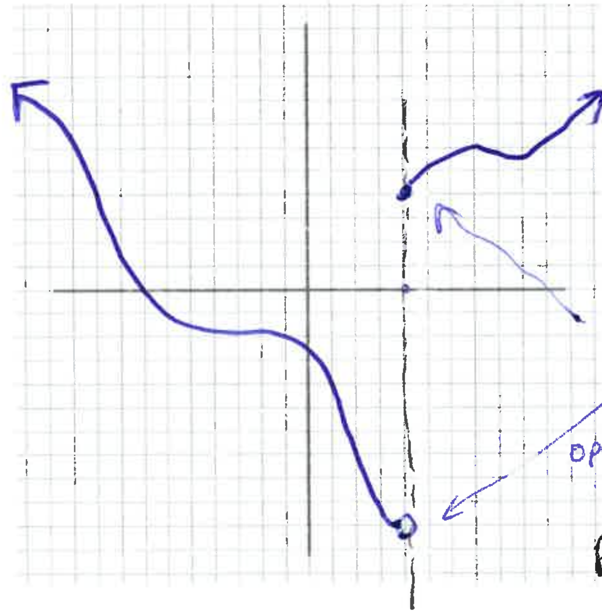
Pictures





There are 3 types of **dis**continuities:

1) Jump



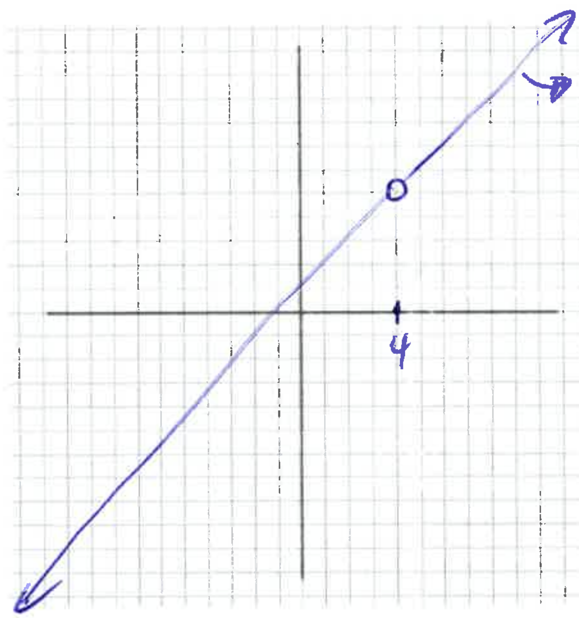
a Jump at 4.

$f(4) = 4$

open dot not include

passes V.L.T

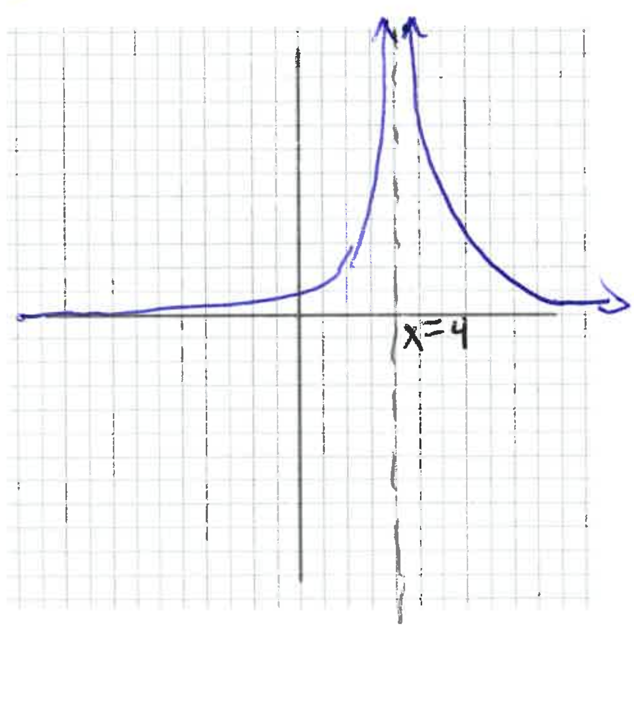
2) Hole



$g(x) = \frac{x^2 - 16}{x - 4}$

- cancel $x-4$ factors
- But originally you can't have $x=4$

3) Infinite (Asymptote)



End Behaviour of Functions

Here we are concerned with how the function is behaving as x gets HUGE.

As x gets Huge (which we write $x \to \infty$, or $x \to -\infty$)

approaches

the functional values (for whatever function we are studying) can do one of three things:

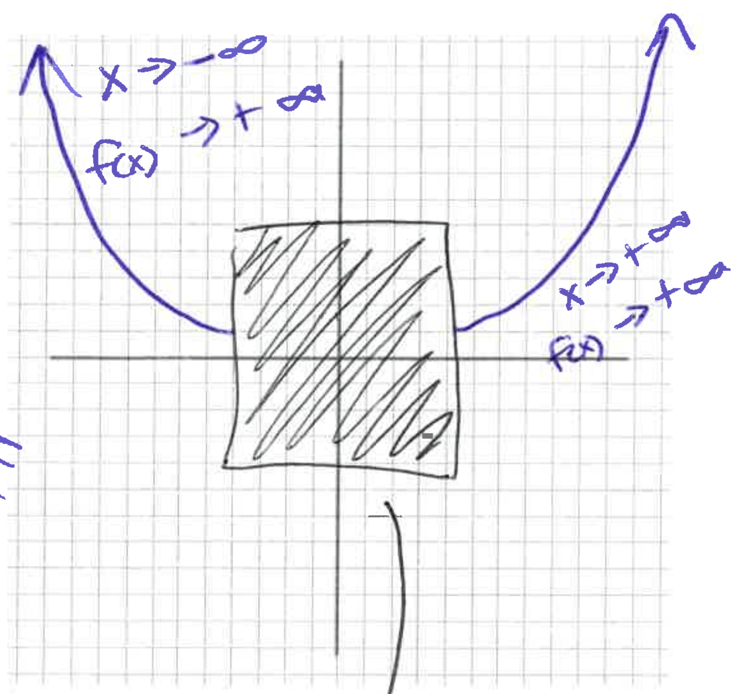
1) Go to Infinity e.g. as $x \to \pm\infty$, $f(x) \to \pm\infty$

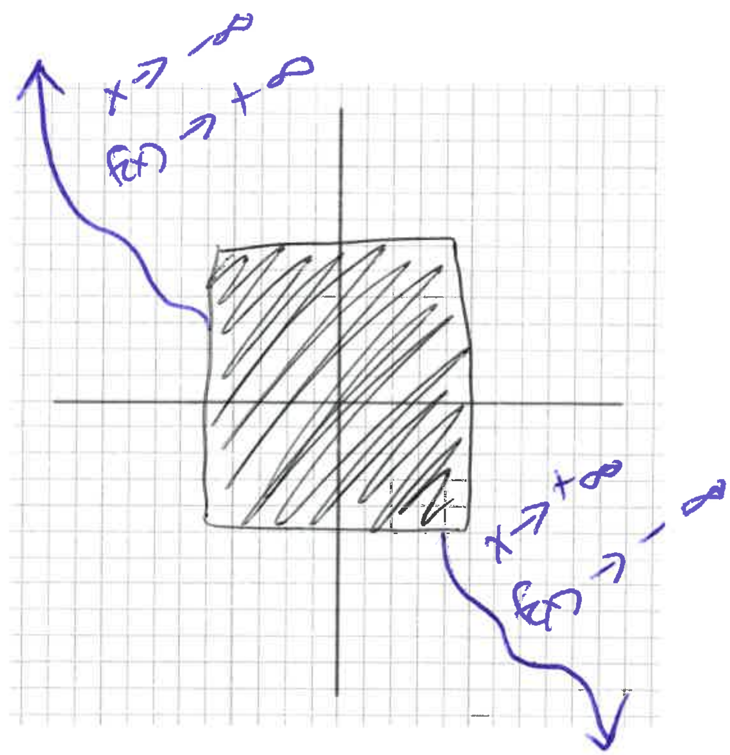
2) Settle down to one Value (Hor. Asymptote)

3) Oscillates (bouncing between #'s)

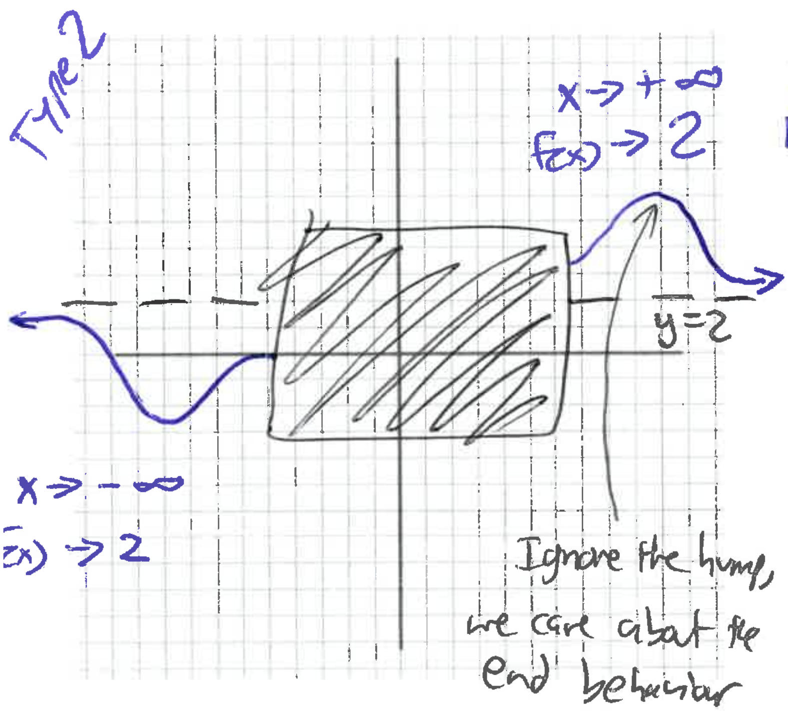
Pictures:

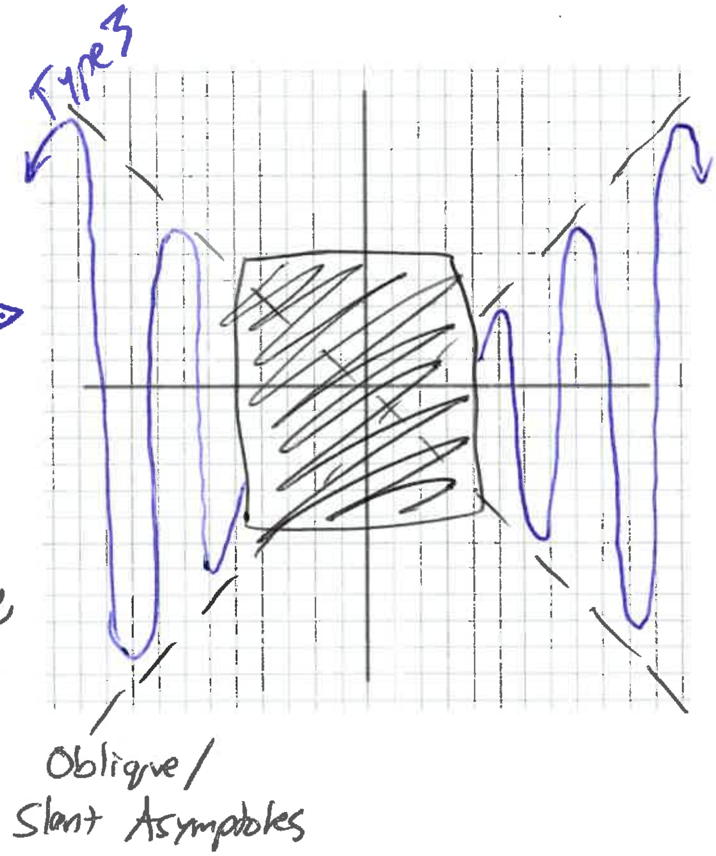
Type I





Black box of Mystery





Oscillation. No true end behaviour bc changes

Success Criteria

- I can identify types of functions based on their graphical characteristics
- I can use different characteristics (intervals of increase/decrease, odd/even, end behaviour, continuity) to help me identify types of functions

Review Gr. 11

1.3 Transformations of Functions

Learning Goal: We are learning to apply transformations to parent functions and how to use transformations for sketching

This section is pure review of material from Grade 11. If you've forgotten certain aspects of the concepts, ask for help. Recall that there are three basic transformations of functions. You've probably heard of Flips, Stretches and Shifts. More formal mathematical terms would be Reflections, Dilations and Translations, respectively. Recall also that transformations can occur both vertically and horizontally.

y's ↲ ↳ x's

3 Basic Transformations: Flips, Stretches, Shifts (Translations)
(Reflections) ↓ (Dilations)

Definition 1.3.1

Given a function $f(x)$, then we denote transformations to $f(x)$ as

$$F(x) = \underline{\underline{a}}\, f(\underline{k}(x-\underline{d})) + \underline{\underline{c}}$$

a + c control verticale   k + d control horizontal

a = vertical stretch/dilation w/ stretch factor of a
If a < 0, we have a flip. → multiplying w/ y's

c = vertical shift/translation → up if c > 0
→ down if c < 0

k = horizontal Stretch w/ stretch factor of $\frac{1}{k}$
If k < 0, we have a flip (horizontally)

d = horizontal shift ... but equation must be in the form x − d
warning: to know d, we must factor out k!

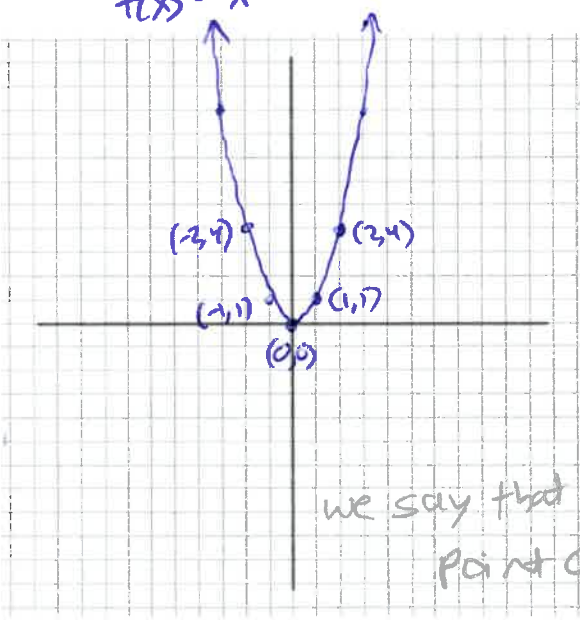
So (x + 4) becomes (x − (−4))

Success Criteria:

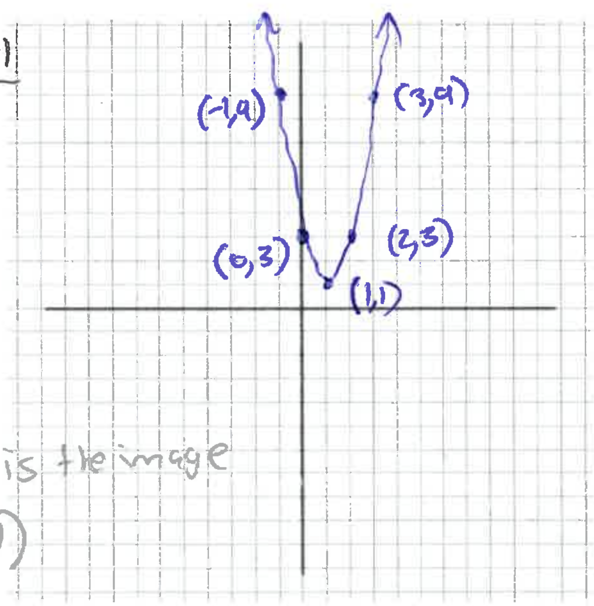
- I can use the value of a to determine if there is a vertical stretch/reflection in the x-axis
- I can use the value of k to determine if there is a horizontal stretch/reflection in the y-axis
- I can use the value of d to determine if there is a horizontal translation
- I can use the value of c to determine if there is a vertical translation

Parent $f(x) = x^2$ $\Longrightarrow$ $f(x) = \overset{a}{2}(\overset{d}{x-1})^2 \overset{c}{+1}$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



$x+1$	$2f+1$
-1	9
0	3
1	1
2	3
3	9



we say that $(-1,9)$ is the image point of $(-2,4)$

warning: Factor the coefficient

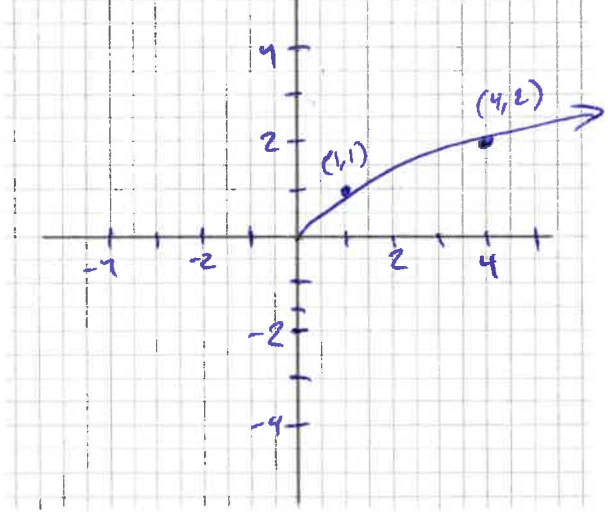
$f(x) = -\sqrt{3(x+1)} + 2$

Parent $f(x) = \sqrt{x}$ $\Longrightarrow$ $f(x) = -\sqrt{3x+3} + 2$

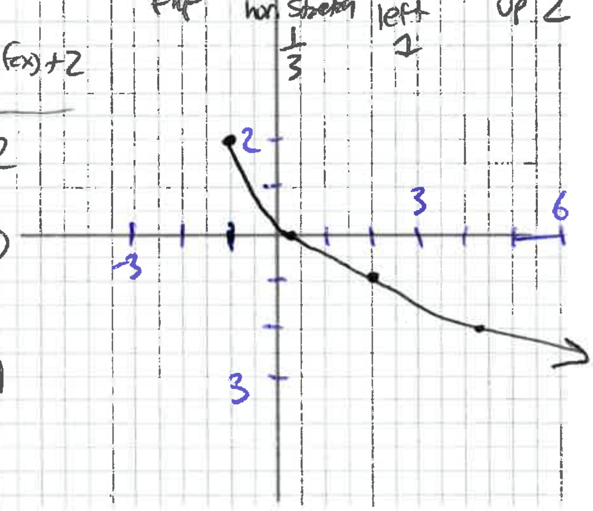
$3(x+1)$

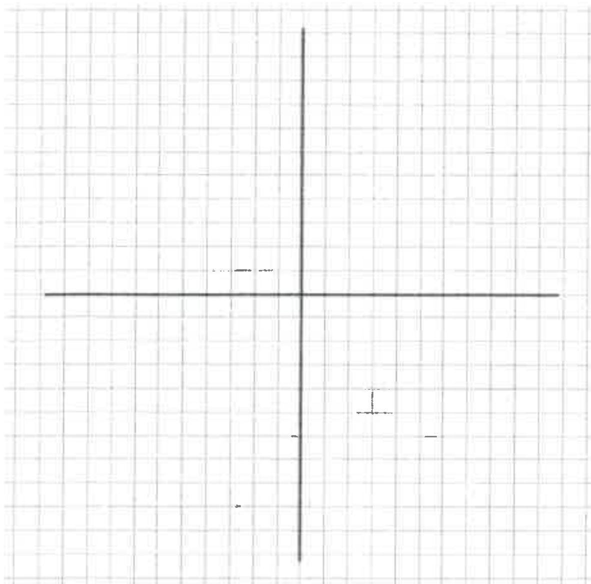
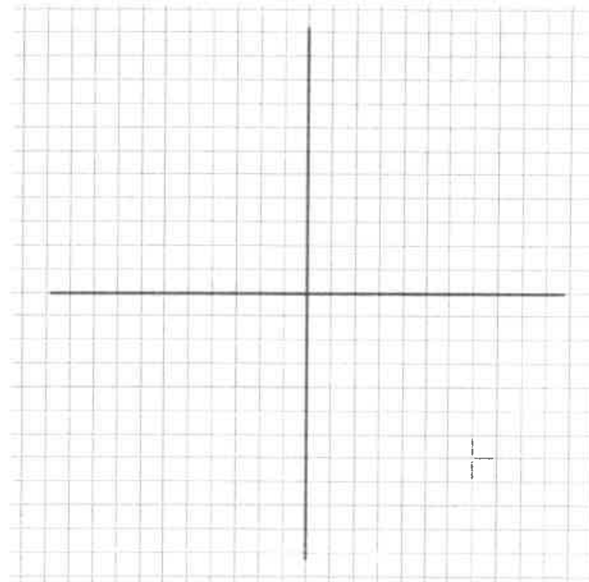
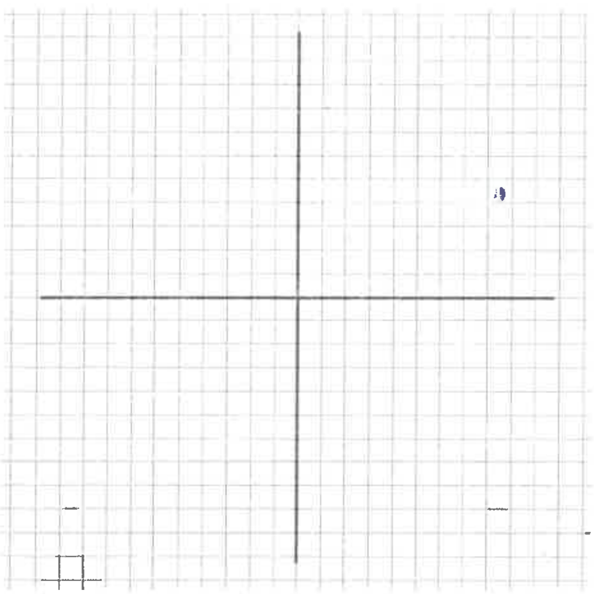
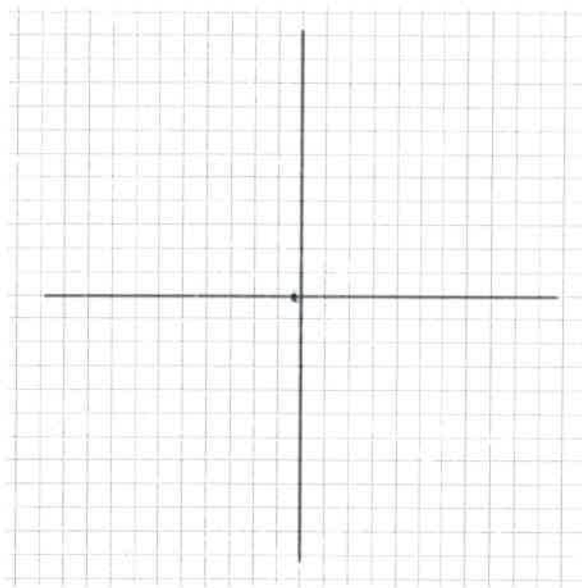
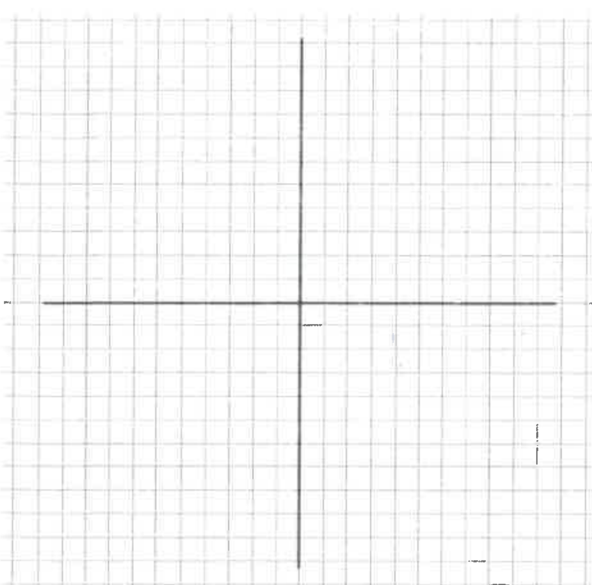
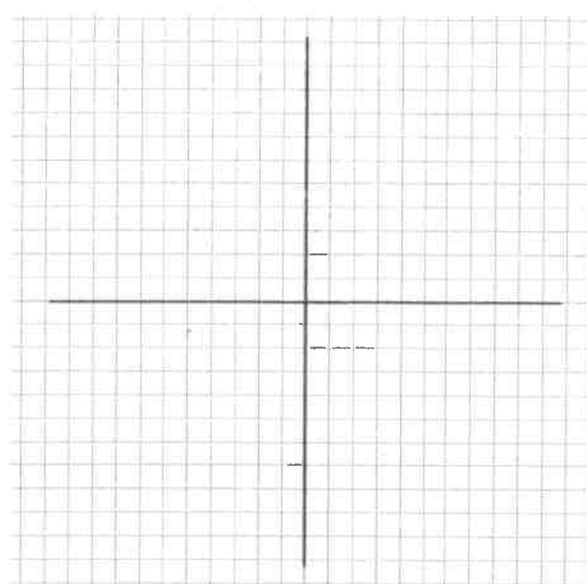
vert flip; hor stretch $\frac{1}{3}$; left 1; up 2

x	$f(x)$
0	0
1	1
4	2
9	3
16	4



$\frac{x}{3}-1$	$-f(x)+2$
-1	2
$-\frac{1}{3}$	0
2	-1
$4\frac{1}{3}$	-2



1.4 Inverses of Functions

Learning Goal: We are learning to determine the equation of an inverse relation and the conditions for an inverse relation to be a function.

The inestimable William Groot has a saying:

An Inverse *Relation* is an UNDO

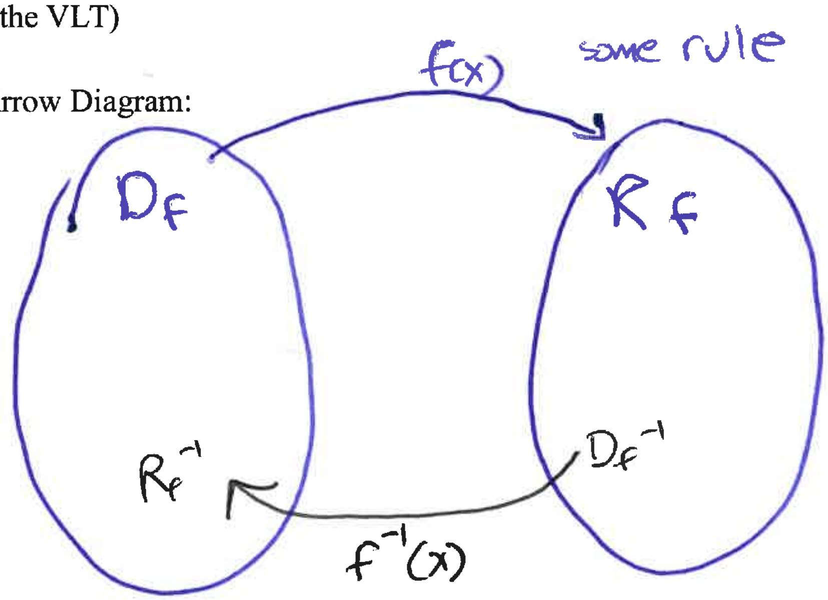
Definition 1.4.1

A **relation** is simply an algebraic relationship between domain values and range values.

Note: All functions are relations, but not all relations are functions

e.g. $x^2 + y^2 = 25$ is a relation, but it is not a function (it's a circle and so doesn't pass the VLT)

Consider the Arrow Diagram:



Big Concept

To determine the inverse of a function, switch x and y.

Example 1.4.1

Given the graph of $f(x)$ determine: $D_f, R_f, f^{-1}(x), D_{f^{-1}}, R_{f^{-1}}$

$$f(x) = \{(2,3),(4,2),(5,6),(6,2)\}$$

$D_f : \{2, 4, 5, 6\}$

$R_f : \{3, 2, 6\}$

$f^{-1}(x) = \{(3,2), (2,4), (6,5), (2,6)\}$

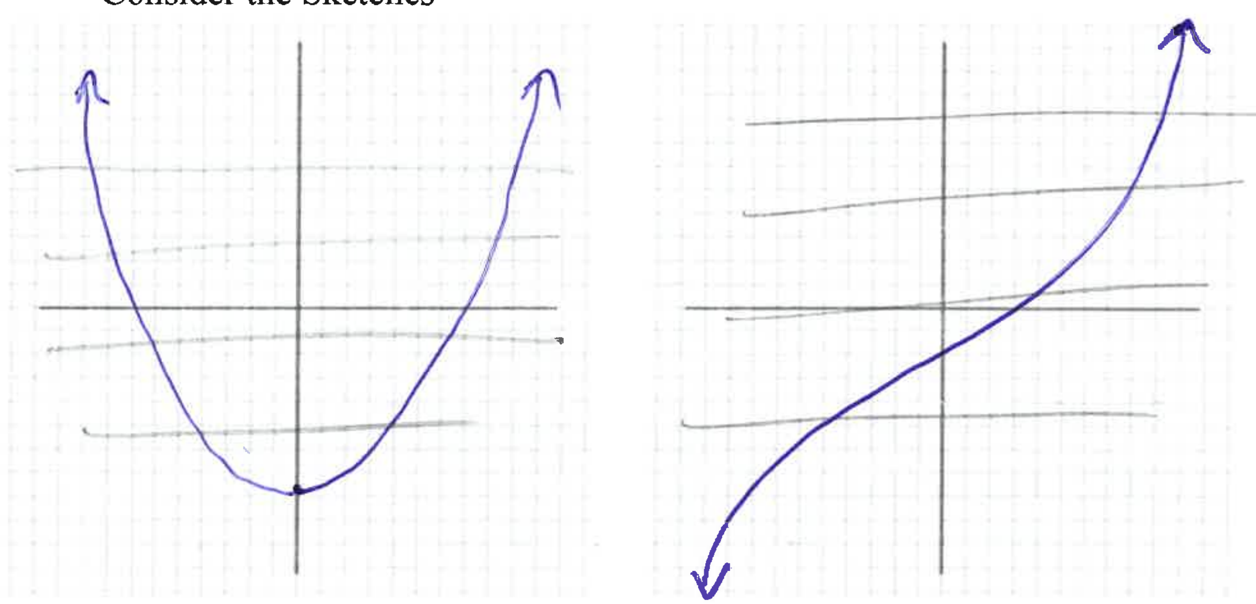
$D_{f^{-1}} = \{2, 3, 6\}$

$R_{f^{-1}} = \{2, 4, 5, 6\}$

Note: the inverse is NOT a function

Horizontal Line Test

Consider the Sketches



VLT of inverse would be a horizontal line

Inverse is not a function

original is.

Both inverse + original are functions.

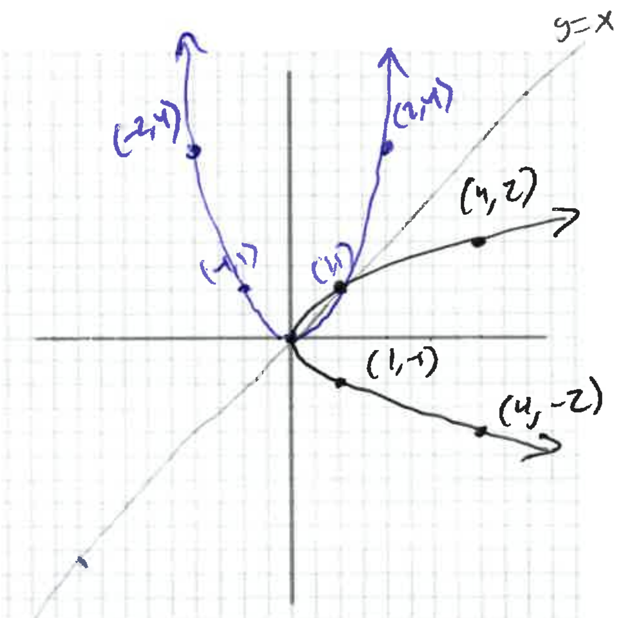
Determining the Inverse of a Function

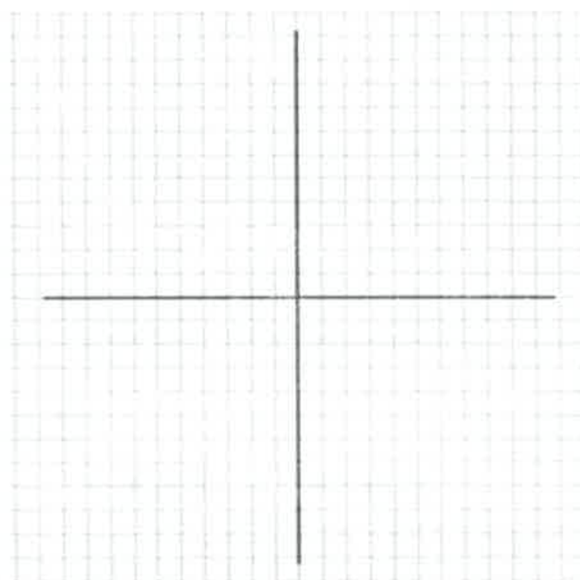
We can determine the inverse of some given function in either of two ways: Graphically and Algebraically.

> *Note: Finding a function inverse graphically is not a very useful method, but it can be instructive.*

Function Inverses Graphically

Let's use $f(x) = x^2$





To find inverse, flip the coordinates

Notice the symmetry about the line $y=x$.

Is there a way to force $f^{-1}(x)$ to be a function?

Yes! →

Restricting the Domain

- Do the inverse on only a portion of the domain
- Restrict up to and including the turning point

For $f(x) = x^2$, use $(-\infty, 0]$ or $[0, \infty)$

Function Inverses Algebraically

Determining algebraic representations of inverse relations for given functions can be done in (at least) two ways:

1) Use algebra in a "brute force" manner (keeping in mind the Big Concept)
2) Use Transformations (keeping in mind "inverse operations")

Example 1.4.2

Determine the inverse of $f(x)=2\sqrt{\frac{1}{3}(x-1)}+2$.

State the domain and range of both the function and its inverse.

Here we will use "brute force". Method:

1) Switch x and $f(x)$, and call "$f(x)$", $f^{-1}(x)$.
2) Solve for $f^{-1}(x)$

$$x=2\sqrt{\tfrac{1}{3}(y-1)}+2$$

$$x-2=2\sqrt{\tfrac{1}{3}(y-1)}$$

$$\frac{x-2}{2}=\sqrt{\tfrac{1}{3}(y-1)}$$

$$\left(\frac{x-2}{2}\right)^2=\tfrac{1}{3}(y-1)$$

$$3\left(\frac{x-2}{2}\right)^2=y-1$$

$$y=3\left(\frac{x-2}{2}\right)^2+1$$

$$\therefore f^{-1}(x)=3\left(\frac{x-2}{2}\right)^2+1$$

$D_f: x\in[1,\infty)$	$D_{f^{-1}}: x\in(-\infty,\infty)$ study eqn'
$R_f: f(x)\in[2,\infty)$ Use domain to determine Range	$R_{f^{-1}}: f^{-1}(x)\in[1,\infty)$ Parabola opening up

Example 1.4.3

Using transformations determine the inverse of $f(x)=2\sqrt{\frac{1}{3}(x-1)}+2$.

→ vertical becomes horizontal

→ Flip everything / do inverse operation (+ become −, etc....)

V Stretch → H Stretch

H Stretch → V Stretch

H Shift → V Shift

V Shift → H Shift

Square root becomes squared

$$f(x)=x^2 \longrightarrow f(x)=3\left(\tfrac{1}{2}(x-2)\right)^2+1$$

Example 1.4.4

Determine the inverse of $g(x) = -2(x-1)^2 + 3$.

Note that the natural domain of $g(x)$ is $(-\infty, \infty)$. However, $g(x)$ does not pass the HLT so its inverse is not a function. Determine a restricted domain for $g(x)$ so that $g^{-1}(x)$ *is* a function.

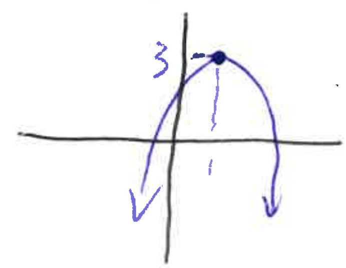
Determine $g^{-1}(x)$

	$g^{-1}(x)$
-1	$+1$
$(\ )^2$	$\pm\sqrt{\ }$
$\cdot -2$	$\div -2$
$+3$	-3

$$g^{-1}(x) = \pm\sqrt{\frac{x-3}{-2}} + 1$$

$$= \pm\sqrt{-\tfrac{1}{2}(x-3)} + 1$$

Domain Restriction? Study original



Use Domain of $(-\infty, 1]$ or $[1, \infty)$

Example 1.4.5

Given $f(x) = kx^2 - 3$ and given $f^{-1}(5) = 2$, find k.

~~Two Methods~~: First, $f^{-1}(x) \Rightarrow (5, 2)$

So $f(x) \Rightarrow (2, 5)$

$$f(2) = k(2)^2 - 3 = 5$$

$$4k - 3 = 5$$

$$4k = 8$$

$$\boxed{k = 2}$$

Success Criteria:

- I can determine the equation of an inverse function using various methods
- I can determine whether an inverse relation is a function, and whether or not the domain needs to be restricted

1.5 Piecewise Defined Functions

Learning Goals: We are learning to understand, interpret, and graph situations that are described by piecewise functions; and learning the properties of the absolute value function.

Some aspects of "reality" exhibit different (as opposed to changing) $\boxed{\text{behaviours}}$

To capture those different behaviours mathematically may require using different $\boxed{\text{functions}}$ over different $\boxed{\text{pieces/intervals}}$ of the domain.

Absolute Value

Before discussing piecewise defined functions in general, we will first review the concept of *absolute value*.

Definition 1.5.1

The absolute ***value*** of a number, *x*, is given by

$$|x| = \begin{cases} x, & x \geq 0 \quad [0, \infty) \\ -x, & x < 0 \quad (-\infty, 0) \end{cases}$$

e.g.'s

$|22| = 22$

$|-8| = 8$

$|8-13| = |-5| = 5$

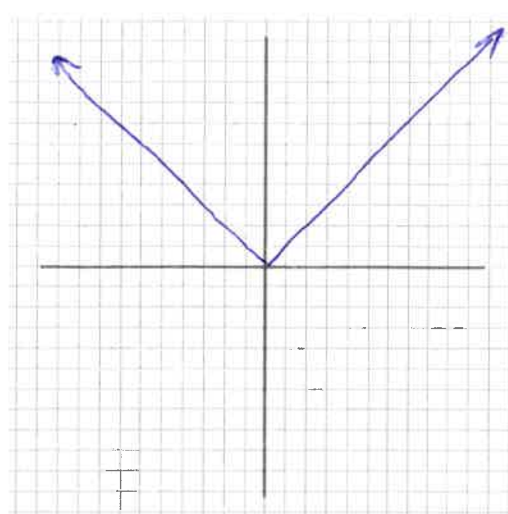
Absolute Value Functions

We can define the function which returns the absolute value for any given number as

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

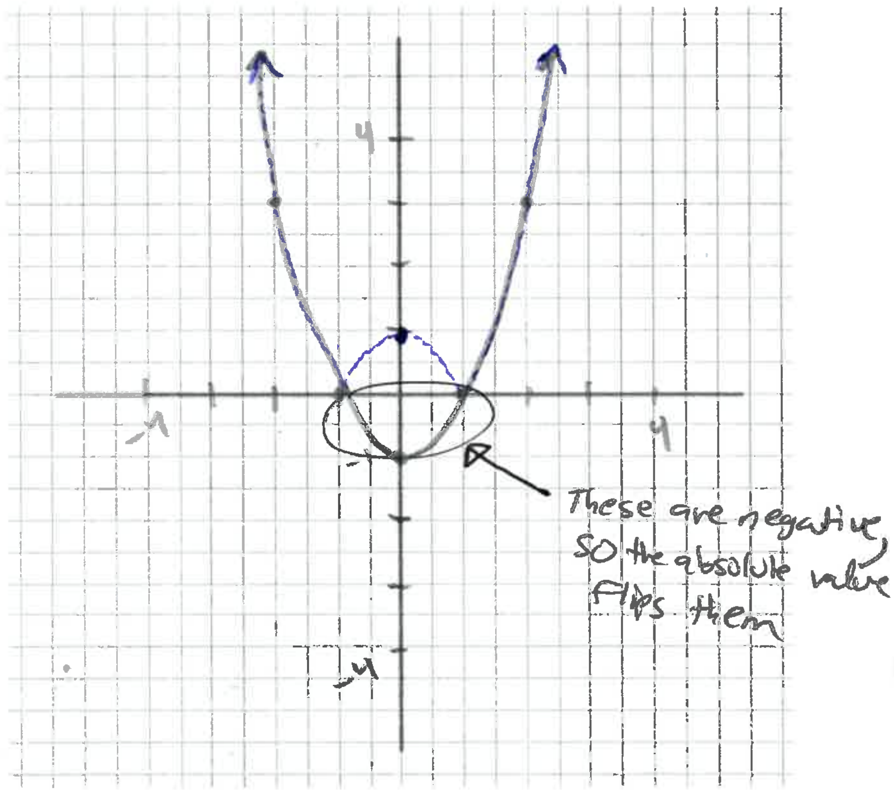
(Two behaviours!)

Picture



We can go further and define functions which return the absolute value for more complicated expressions.

e.g. Sketch $g(x) = |x^2 - 1|$ (note: g(x) takes the absolute value of the ***functional values*** for the "basic" function $f(x) = x^2 - 1$)



$$g(x) = \begin{cases} x^2-1, & (-\infty, -1] \\ -(x^2-1), & (-1, 1) \\ x^2-1, & [1, \infty) \end{cases}$$

(Three functional behaviours)

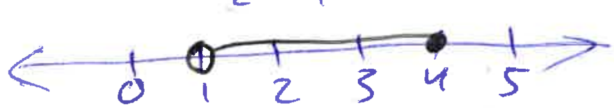
Absolute Value and Domain Intervals (and Quadratic Inequalities)

e.g.'s Sketch the solution sets of the following inequalities:

a) $x > -1$

b) $x \leq 2$

c) $1 < x \leq 4$



d) $-2 < x < 2$

Note the symmetry in part d)! Sometimes it's useful to think of absolute value as

Using the above notion we can thus use absolute value to denote the interval $-2 < x < 2$ as $|x| < 2$

$\boxed{\text{distance from the origin}}$

e.g. Solve the quadratic equation

$x^2 = 4$ $\quad x = \pm 2 \quad$ OR $\quad |x| = 2$

e.g. Solve the quadratic inequalities, and sketch the solution sets:

a) $x^2 < 4$ $\quad |x| < 2 \quad$ or $\quad -2 < x < 2$

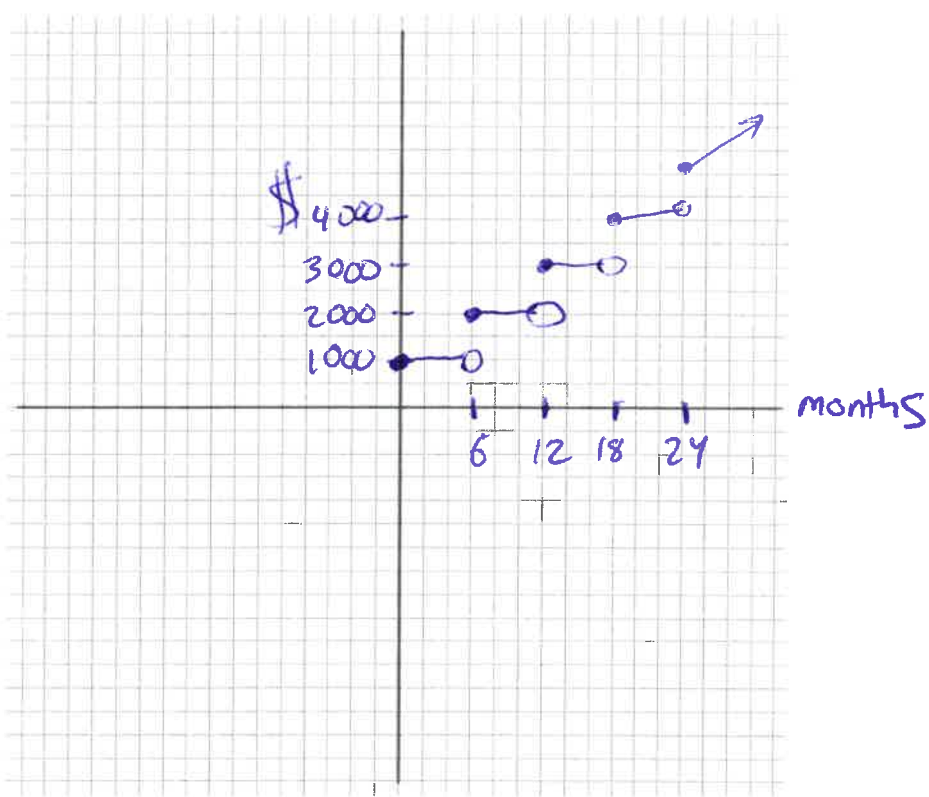
b) $x^2 \geq 3$ $\quad |x| \geq \sqrt{3} \quad$ OR $\quad -\sqrt{3} \leq x \leq \sqrt{3}$

And now we return our attention to general **Piecewise Defined Functions**

Example 1.5.1

You are saving for university, and place \$1000 into a sock every six months. After 18 months you wake up and put the money in your sock into an interest bearing bank account. You continue making deposits. Give a graphical representation of this situation.

What is the behaviour of the amount of money you have saved? How is the behaviour changing?

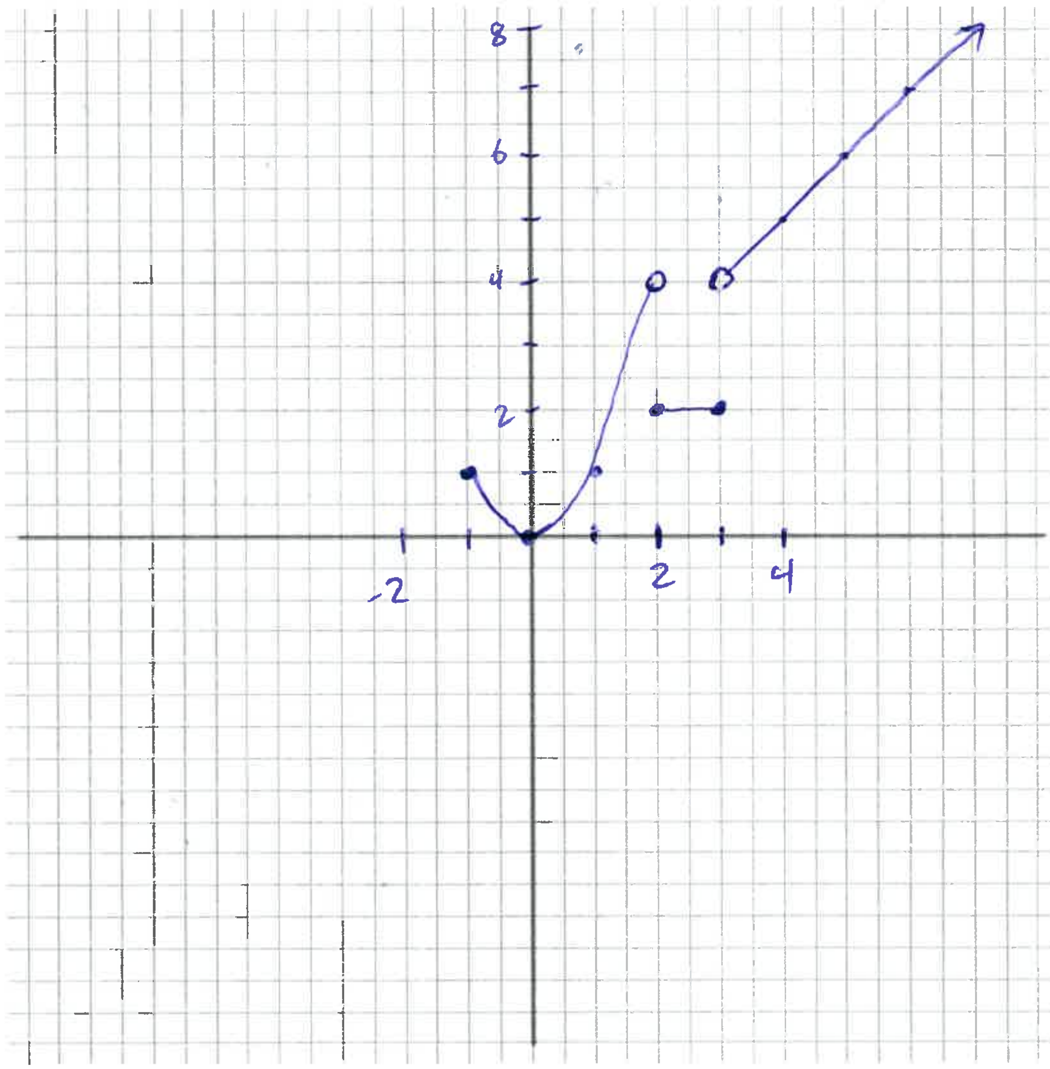


Example 1.5.2

Determine the graphical representation for:

$$f(x)=\begin{cases} x^2, & x\in[-1,2) \\ 2, & x\in[2,3] \\ x+1, & x\in(3,\infty) \end{cases}$$

Note the notation we use for piecewise defined functions. Each functional behaviour has a mathematical representation, defined over its own piece of the domain (just like the Absolute Value function we considered earlier.



Example 1.5.3

Determine a possible algebraic representation which describes the given functional behaviour.

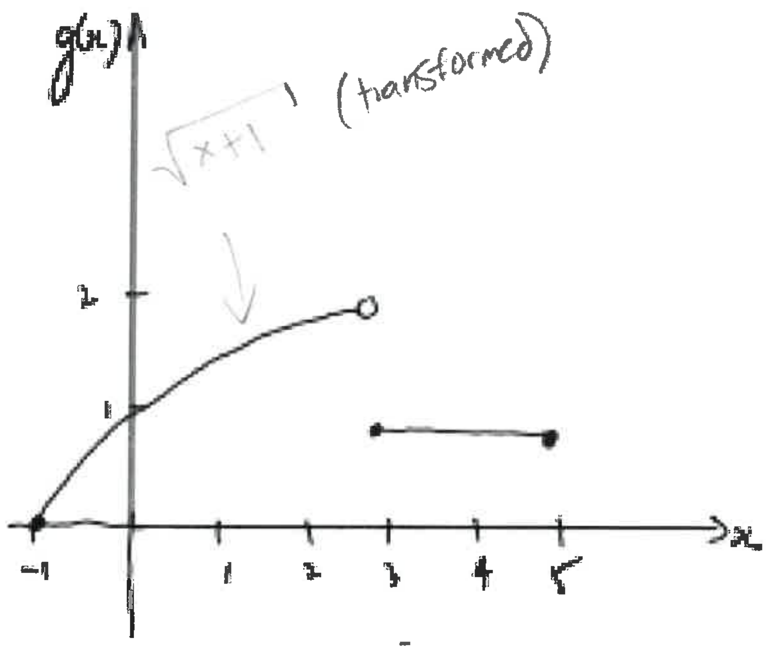


Figure 1.5.3

$$g(x) = \begin{cases} \sqrt{x+1}, & [-1, 3) \\ 1, & [3, 5] \end{cases}$$

Success Criteria:

- I can absolutely understand the absolute value function
- I can graph the piecewise function by graphing each piece over the given interval
- I can determine if a piecewise function is continuous or not