Advanced Functions

Teacher

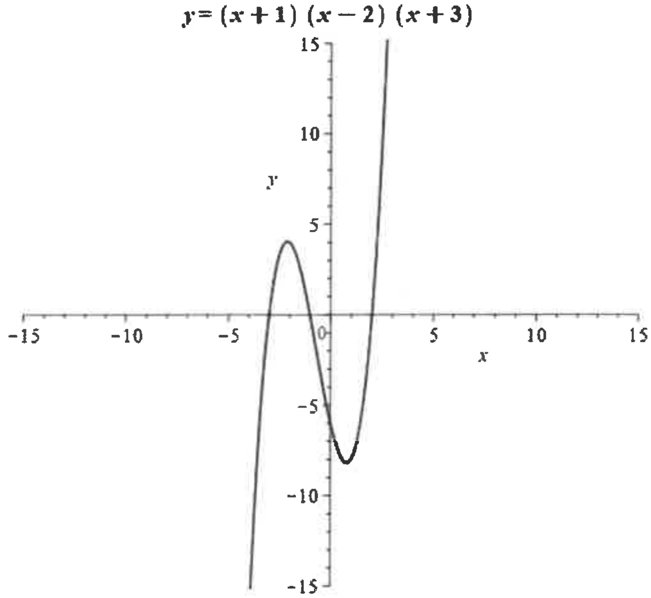
Course Notes

Chapter 2 – Polynomial Functions

***Learning Goals**: We are learning*

- *The algebraic and geometric structure of polynomial functions of degree three and higher*
- *Algebraic techniques for dividing one polynomial by another*
- *Techniques for using division to FACTOR polynomials*
- *To solve problems involving polynomial equations and inequalities*

$y = (x+1)(x-2)(x+3)$



A cubic polynomial

A∞Ω
MATH@TD

Chapter 2 – Polynomial Functions

Contents with suggested problems from the Nelson Textbook (Chapter 3)

Ex hi 3

2.1 Polynomial Functions: An Introduction

Learning Goal: We are learning to identify polynomial functions.

Definition 2.1.1

A **Polynomial Function** is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots a_2 x^2 + a_1 x^1 + a_0 \boxed{x^0} = 1$$

where a_i, $i = 0, 1, 2, \dots n$, are coefficients

Examples of Polynomial Functions

a) $f(x) = 8x^4 - 5x^3 + 2x^2 + 3x - 5$

$a_4 = 8$ $\quad a_3 = -5$ $\quad a_0 = -5$

b) $g(x) = 7x^6 - 4x^3 + 3x^2 + 2x$

$a_6 = 7$, $a_5 = 0$, $a_4 = 0$, $a_3 = -4$, $a_2 = 3$, $a_1 = 2$, $a_0 = 0$

Notes: The **TERM** $a_n x^n$ in any polynomial function (where n is the **highest power** we see) is

called the leading term, and then we write all the following terms

in descending order.

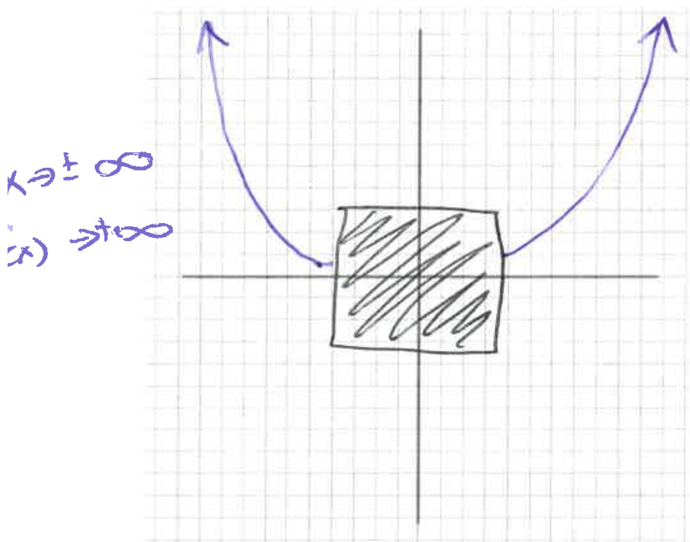
The leading term has two components:

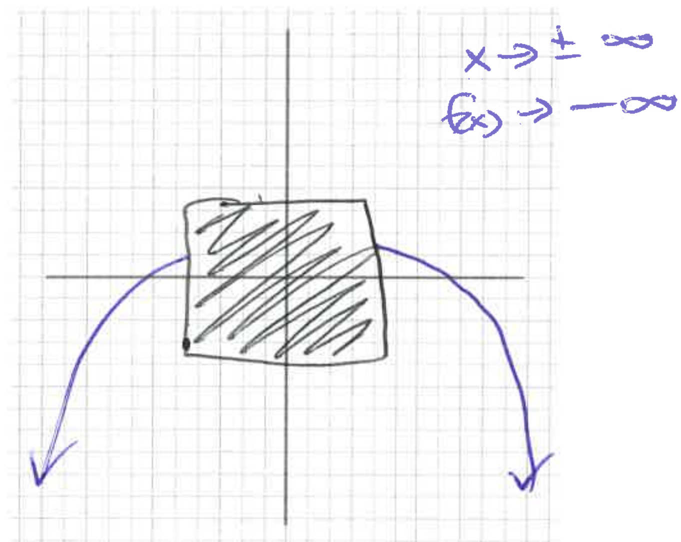
1) Leading coefficient, a_n, is positive or negative

2) $n \Rightarrow$ the highest power, it can be odd or even

The *leading term* tells us the **end behaviour** of the polynomial function.

Pictures

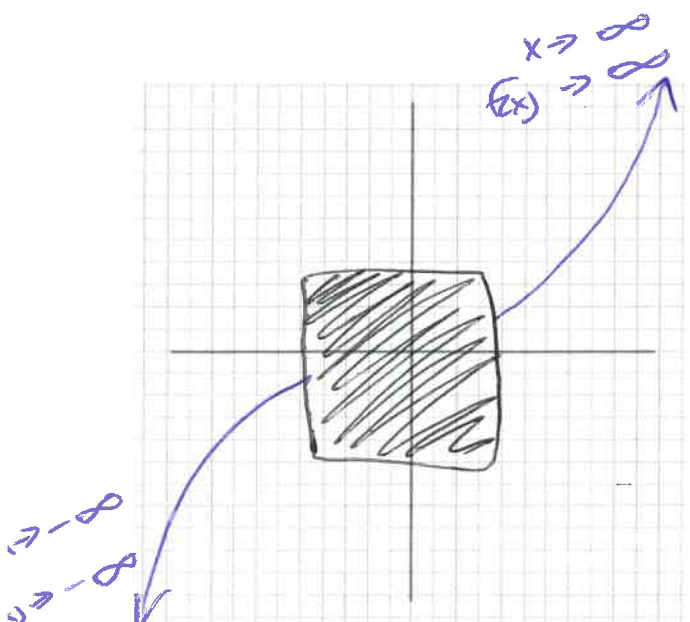


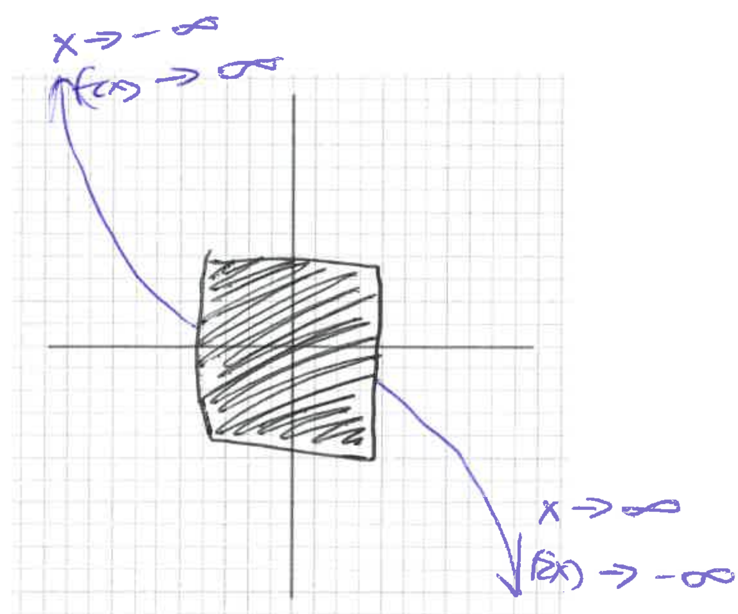


n is even
$a_n > 0$

Think of a parabola

n is even
$a_n < 0$





n is odd
$a_n > 0$

Think of a line

n is odd
$a_n < 0$

Definition 2.1.2

The **order** of a polynomial is f_n is the value of the highest power, or just the degree of the leading term.

ex: $g(x) = 2x^3 + 3x^2 - 8x^5 - 1$

The order of $g(x)$ is 5

Success Criteria:

- I can justify whether a function is polynomial or not
- I can identify the degree of a polynomial function
- I can recognize that the domain of a polynomial is the set of all real numbers
- I can recognize that the range of a polynomial function may be the set of all real numbers, or it may have an upper/lower bound
- I can identify the shape of a polynomial function given its degree

2.2 Characteristics (Behaviours) of Polynomial Functions

Today we open, and look inside the black box of mystery

Learning Goal: We are learning to determine the turning points and end behaviours of polynomial functions.

Consider the sketch of the graph of some function, $f(x)$:

Add the max terms when on the next page?

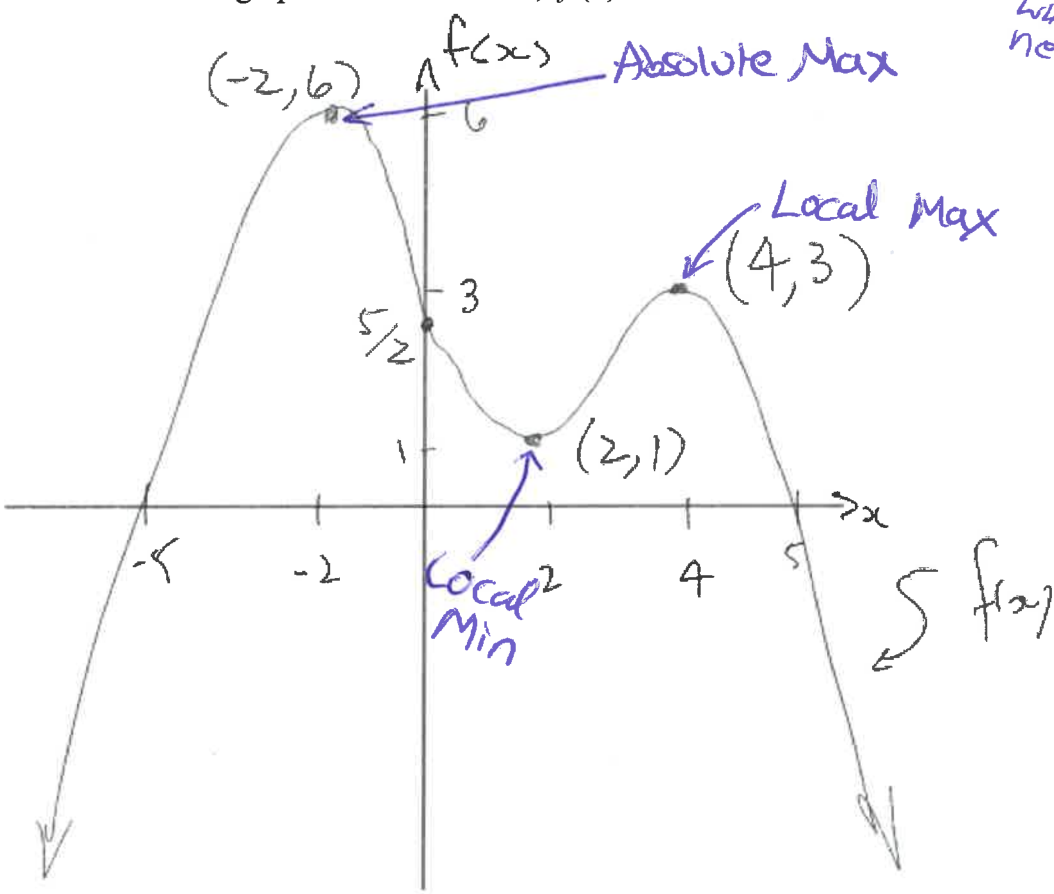


Figure 2.2.1

Observations about $f(x)$:

1) $f(x)$ is a polynomial of even order (degree). The end behaviours are the same.

2) The leading coefficient is negative

3) $f(x)$ has 3 turning points (where the functional behaviour of INCREASING/DECREASING switches from one to the other.)

4) $f(x)$ has 2 zeros $f(-5)=0$ & $f(5)=0$

zeros @ $x=-5$ & $x=5$

5) $f(x)$ is increasing on $x \in (-\infty,-2) \cup (2,4)$

$f(x)$ is decreasing on $x \in (-2,2) \cup (4,\infty)$

6) $f(x)$ has a maximum functional value. of 6.

The max occurs at $x=-2$
This max is also called the global maximum, meaning it is the absolute highest value.

In the neighbourhood of $x=4$, there is a local max of 3. It is a turning point, but not the highest value.

7) $f(x)$ has a local minimum at $(2,1)$ but

In the neighbourhood of $x=2$, there is a local min of 1.

Consider the sketch of the graph of some function $g(x)$:

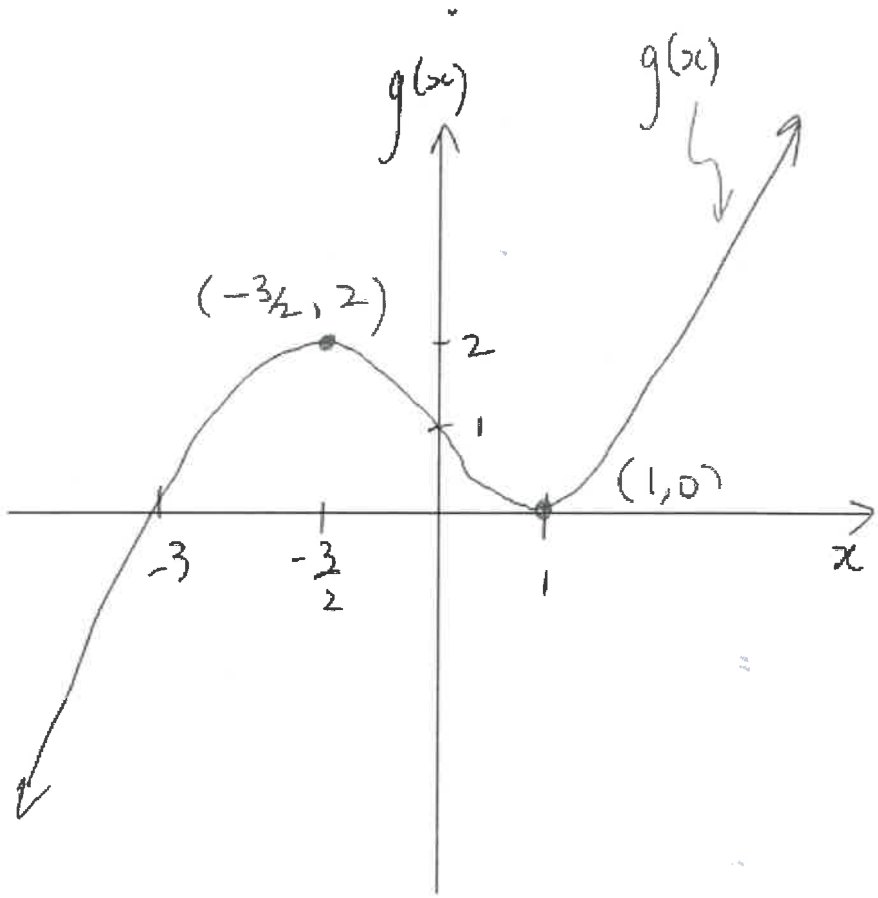


Figure 2.2.2

Observations about $g(x)$:

1. g(x) is odd (degree, not symmetry). End behaviours are opposite.

2. g(x) has two zeros at $x=3$ & $x=1$

3. g(x) has a local max of 2 around $x=-\frac{3}{2}$ $(-\frac{3}{2}, 2)$

g(x) " " " min " 0 around $x=1$ $(1,0)$

4. g(x) has two turning points

5. g(x) has a positive leading coefficient (Going up)

6. g(x) increasing on $(-\infty, -\frac{3}{2}) \cup (1, \infty)$

g(x) decreasing on $(-\frac{3}{2}, 1)$

General Observations about the Behaviour of Polynomial Functions

1) The Domain of all Polynomial Functions is

$\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$

2) The Range of ODD ORDERED Polynomial Functions is

$\{f(x) \in \mathbb{R}\}$ or $(-\infty, \infty)$

3) The Range of EVEN ORDERED Polynomial Functions

depends on 1. the sign of the L.C. → if positive, $\geq$

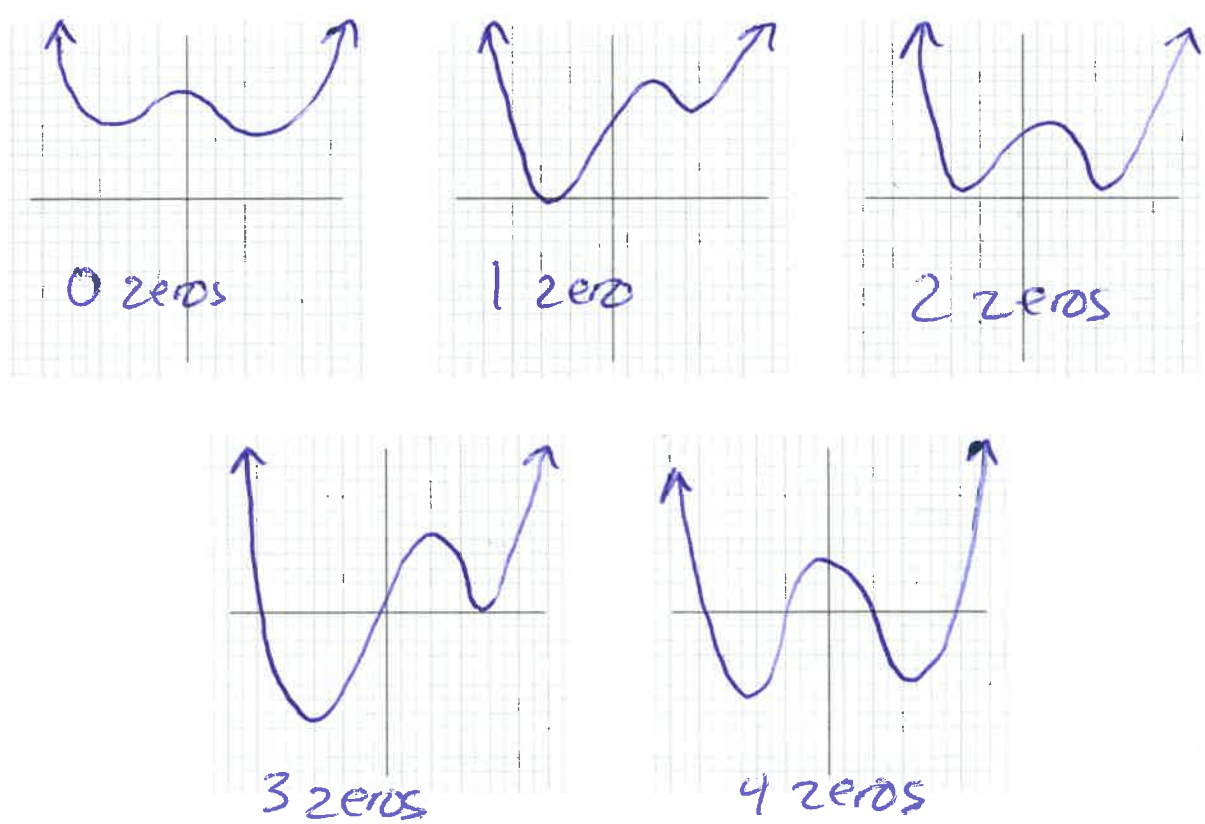
→ if negative, $\leq$

2. the value of the global max/min

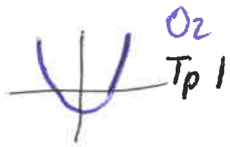
Even Ordered Polynomials

Zeros: A Polynomial Function, $f(x)$, with an even degree of "n" (i.e. $n = 2, 4, 6...$) can have $0, 1, 2, \ldots, n$ zeroes

e.g. A degree 4 Polynomial Function (with a positive leading coefficient) can look like:



Turning Points:



The minimum number of turning points for an Even Ordered Polynomial Function is ONE, because you must turn

The maximum number of turning points for a Polynomial Function of (even) order n is $n-1$ Even poly fn's turn an odd # of times

ex: If order is 6, max T.P. is 5

Odd Ordered Polynomials

Zeros: The min # of zeros is ONE. It must go through the x-axis at least once.

The max # of zeros is $\boxed{n}$

Turning Points:

min # of T.P. is zero (line)

max # of T.P. is $n-1$

Say $f(x) = 2(x+3)(2x-1)(3x+5)$

Leading term is $12x^3$. There are 3 zeros

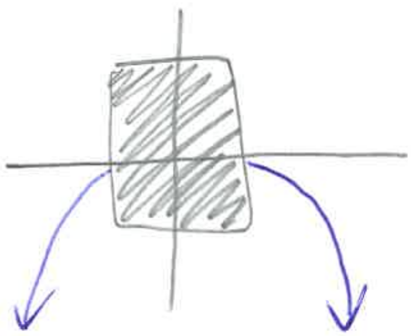
Example 2.2.1 (*#2, for #1b, from Pg. 136*)

Determine the minimum and maximum number of zeros and turning points the given function may have: $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

Order (n) is 5, $\therefore$ odd

Example 2.2.2 *(#4d from Pg. 136)*

Describe the end behaviour of the polynomial function using the order and the sign on the leading coefficient for the given function: $f(x) = \boxed{-2x^4} + 5x^3 - 2x^2 + 3x - 1$



Even + Negative

$\therefore\ x \to -\infty,\ f(x) \to -\infty$

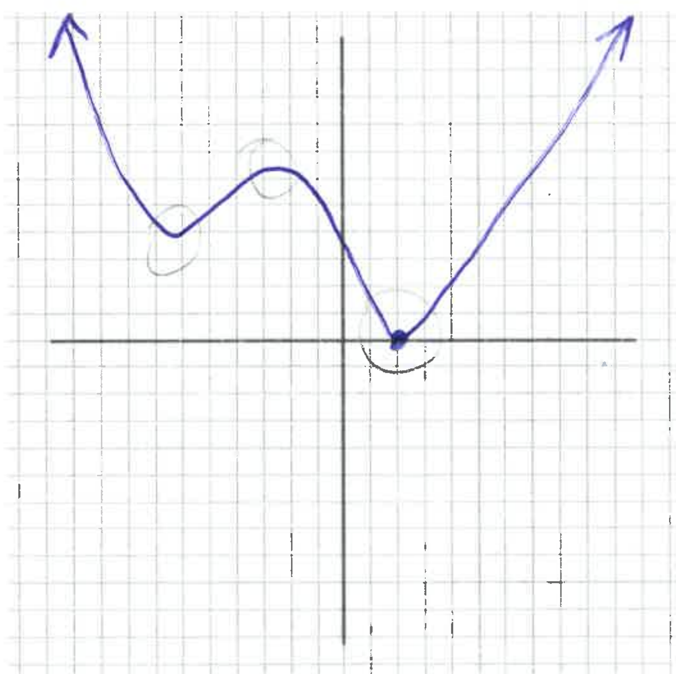
$x \to +\infty,\ f(x) \to -\infty$

Example 2.2.3 (*#7c from Pg. 137*)

Sketch a graph of a polynomial function that satisfies the given set of conditions:

Degree 4 - positive leading coefficient - 1 zero - 3 turning points.

-opens up



Success Criteria:

- I can differentiate between an even and odd degree polynomial
- I can identify the number of turning points given the degree of a polynomial function
- I can identify the number of zeros given the degree of a polynomial function
- I can determine the symmetry (if present) in polynomial functions

2.3 Zeros of Polynomial Functions

(Polynomial Functions in Factored Form)

Today we take a deeper look inside the Box of Mystery, ***carefully examining Zeros of Polynomial Functions***

Learning Goal: We are learning to determine the equation of a polynomial function that describes a particular situation or graph and vice-versa.

We'll begin with an **Algebraic Perspective**:

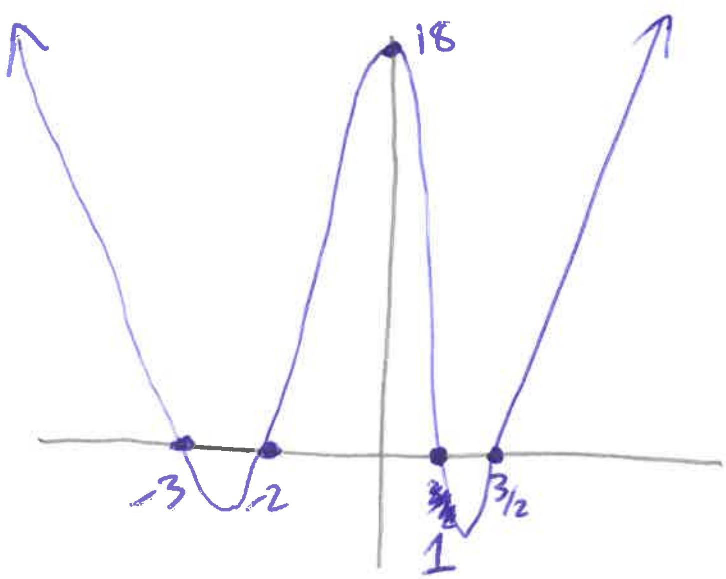
Consider the polynomial function in factored form: "zeros form"

$$f(x) = (2x-3)(x-1)(x+2)(x+3)$$

$(2x)(x)(x)(x) = 2x^4$ = Leading Term

Observations:

1. f(x) has order 4 (even)
2. Leading coefficient is 2 (positive)
3. End behaviours (opens up) As $x \to +\infty$, $f(x) \to \infty$; $x \to -\infty$, $f(x) \to \infty$
4. f(x) has 4 zeros (the factors) at $x = \frac{3}{2}$, $x = 1$, $x = -2$, $x = -3$
5. y-int is $f(0) = (-3)(-1)(2)(3) = +18$



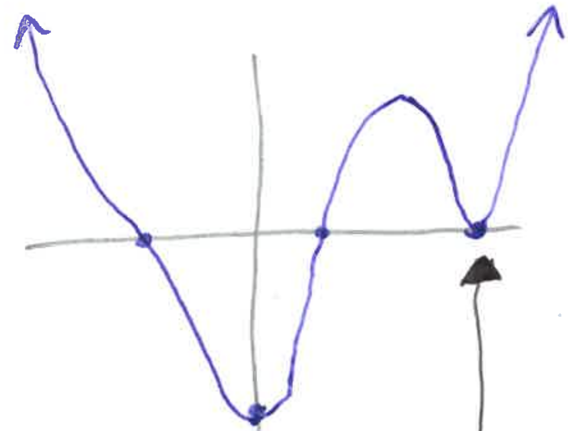
Now, consider the polynomial function $g(x)=(x-3)^2(x-1)(x+2)$

Observations:

1. Leading Term: $(x)^2(x)(x)=x^4$
   $\hookrightarrow$ order is 4 (even). L.C. is positive, so opens up
   $\therefore x \to \pm\infty,\ g(x) \to \infty$

2. G(x) has 3 zeros at $x=3,1,-2$

3. y-int: $g(0)=(-3)^2(-1)(2)$
   $=-18$



Geometric Perspective on Repeated Roots (zeros) of order **2**

Consider the quadratic in factored form: $f(x)=(x-1)^2$

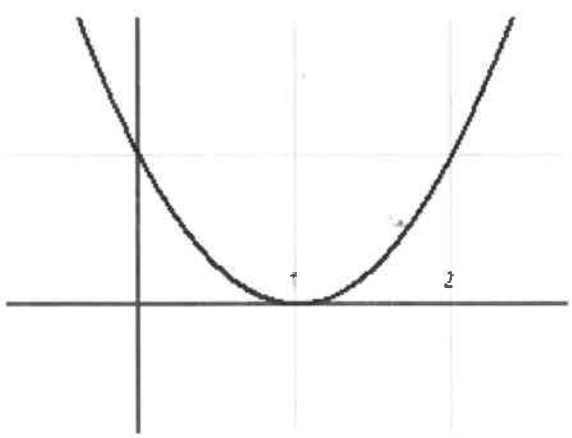


Figure 2.3.1

All order 2 zeros act like parabolas

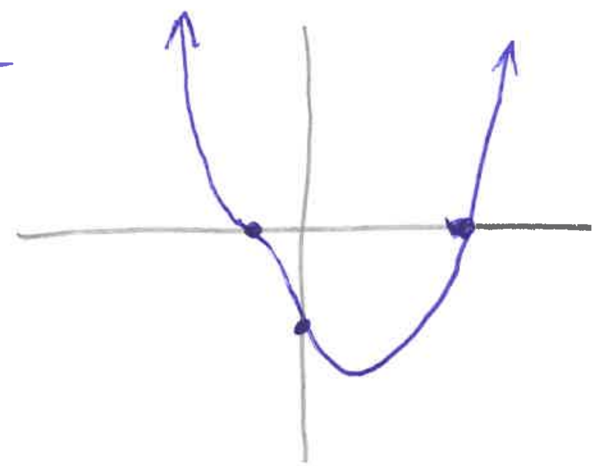
Consider the polynomial function in factored form: $h(t) = (t+1)^3(2t-5)$

Observations:

1. Leading Term is $(t)^3(2t) = 2t^4$
   ↳ order is 4 (even), opens up

2. $h(t)$ has $\underline{2}$ zeros, $\boxed{t=-1}$ + $t = \frac{5}{2}$
   order 3

3. y-int = $h(0) = (1)^3(-5) = -5$



Geometric Perspective on Repeated Roots (zeros) of order 3

Consider the function $f(x) = (x-1)^3$

All order 3 zeros
Zig-zag through the zero

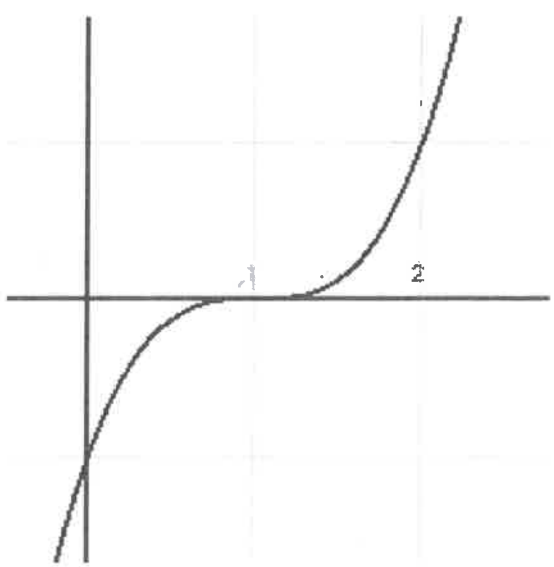
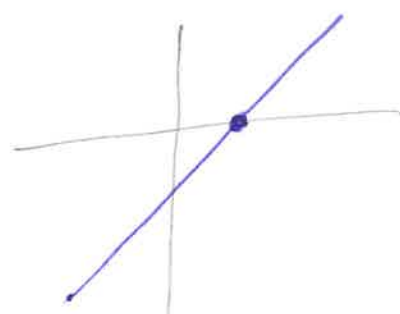


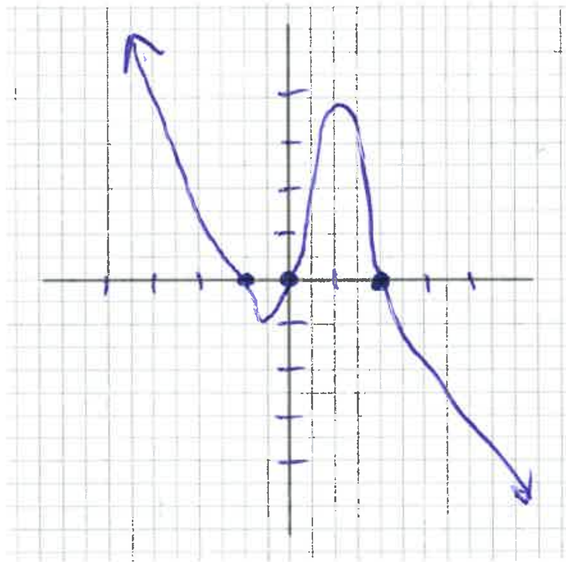
Figure 2.3.2

Order 1 Go straight through
(or small zig-zag)



Example 2.3.1

Sketch a **(possible)** graph of $f(x) = -2x(x+1)(x-2)$



Leading term: $(-2x)(x)(x)$

$= -2x^3$ — odd

negative

$x \to -\infty,\ f(x) \to \infty$

$x \to \infty,\ f(x) \to -\infty$

Zeros: $x=-1, x=2, x=0$

y-int: $f(0)=0$

Families of Functions

Polynomial functions which share the same order are "broadly related" (e.g. **all** quadratics are in the "order 2 family").

Polynomial Functions which share the same order + zeros are more tightly related.

Polynomial Functions which share the same order, zeros, + end behaviours are like siblings.

Example 2.3.2

The family of functions of order 4, with zeros $x = -1, 0, 3, 5$ can be expressed as:

$f(x) = a(x+1)(x+0)(x-3)(x-5)$

The L.C. distinguishes from family members.

Example 2.3.3

Sketch a graph of $g(x) = 4x^4 - 16x^2$

Factor First

$g(x) = 4x^2(x^2-4)$

$= 4x^2(x+2)(x-2)$

. L.T. of $4x^4$

↳ positive & even order

(opens up) $\therefore x \to \pm\infty, g(x) \to \infty$

2. zeros

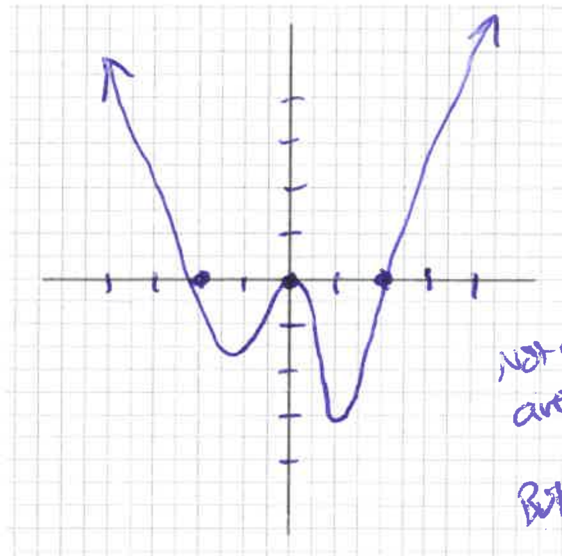
$x = 0$ (order 2)

$x = 2$

$x = -2$

3. y-int

$g(0) = 0$



Not all functions are symmetric. But you could check.

Example 2.3.4

Sketch a (possible) graph of $h(t) = (t-1)^3(t+2)^2$

L.T. is $(t)^3(t)^2 = t^5$

↳ positive & odd

$\therefore t \to -\infty, h(t) \to -\infty$

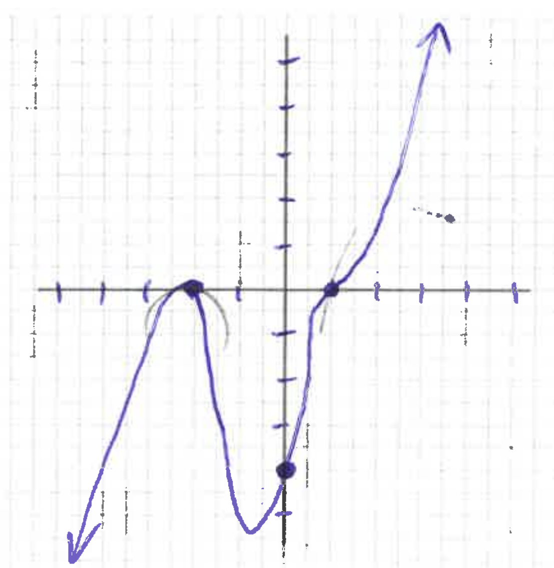
$t \to \infty, h(t) \to \infty$

2. zeros at $t = 1$ (order 3)

$t = -2$ (order 2)

3. y-int $h(0) = (-1)^3(2)^2$

$= -4$



Example 2.3.5

Determine the quartic function, $f(x)$, with zeros at $x=-2, 0, 1, 3$, if $f(-1)=-2$.

Start w/ zeros

$$f(x) = \underbrace{a}_{\text{Remember this makes the function unique}}(x+2)(x+0)(x-1)(x-3)$$

Now, find a.

$$(-2) = a((-1)+2)((-1))((-1)-1)((-1)-3)$$

$$-2 = a(1)(-1)(-2)(-4)$$

$$\frac{-2}{-8} = \frac{-8a}{-8}$$

$$\frac{1}{4} = a$$

$$\therefore f(x) = \frac{1}{4}x(x+2)(x-1)(x-3)$$

Success Criteria:

- I can determine the equation of a polynomial function in factored form
- I can determine the behaviour of a zero based on the order/exponent of that factor

2.4a Dividing a Polynomial by a Polynomial

(The Hunt for Factors)

Learning Goal: We are learning to divide a polynomial by a polynomial using long division

Note: In this course we will almost always be dividing a polynomial by a ~~monomial~~

linear divisor $(x+1)$ or $(2x-5)$

Before embarking, we should consider some "basic" terms (and notation):

the thing you are dividing

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

the stuff left over

the thing you are dividing by

the "answer"

The Division Statement

Dividend = (Quotient)(Divisor) + Remainder

Note: The Divisor and the Quotient will both be FACTORS

IF

the remainder is zero!

Example 2.4.1

Use **LONG DIVISION** for the following division problem:

$$\frac{5x^4+3x^3-2x^2+6x-7}{x-2}$$

Please read Example 1 (Part A) on Pgs. 162 – 163 in your textbook.

$$\begin{array}{r} 5x^3+13x^2+24x+54 \\ x-2\,\overline{)\,5x^4+3x^3-2x^2+6x-7} \\ -(5x^4-10x^3) \\ \hline 13x^3 \\ -(13x^3-26x^2) \\ \hline 24x^2 \\ -(24x^2-48x) \\ \hline 54x-7 \\ -(54x-108) \\ \hline +101 \end{array}$$

① x times ___ is $5x^4$?

$(x)(5x^3) = 5x^4$

② place $5x^3$ in cube column

③ Multiply $5x^3$ by x and -2, then put in the appropriate column.

④ Subtract

⑤ Repeat

$(x)(13x^2) = 13x^3$

$(x)(24x) = 24x^2$

$(x)(54) = 54x$

$\therefore\ 5x^4+3x^3-2x^2+6x-7 = (5x^3+13x^2+24x+54)(x-2)+101$

KEY OBSERVATION:

$(x-2)$ is NOT a factor of the polynomial

Caution: Always include all x-terms, even if they are not there

Example 2.4.2

Using Long Division, divide $\dfrac{2x^5 \overset{0x^4}{\downarrow} +3x^3 \overset{0x^2}{\downarrow} -4x-1}{x-1}$.

$$
\begin{array}{r}
2x^4 + 2x^3 + 5x^2 + 5x + 1 \\
x-1 \,\big)\overline{\,2x^5 + 0x^4 + 3x^3 + 0x^2 - 4x - 1} \\
-(2x^5 - 2x^4) \downarrow \\
\hline
2x^4 + 3x^3 \\
-(2x^4 - 2x^3) \downarrow \\
\hline
5x^3 + 0x^2 \\
-(5x^3 - 5x^2) \downarrow \\
\hline
5x^2 - 4x \\
-(5x^2 - 5x) \downarrow \\
\hline
x - 1 \\
-(x - 1) \\
\hline
0
\end{array}
$$

$\therefore 2x^5 + 3x^3 - 4x - 1 = (x-1)(2x^4 + 2x^3 + 5x^2 + 5x + 1)$

KEY OBSERVATION:

Classwork: Pg. 169 #5 (Yep, that's it for today)

Success Criteria:

- I can use long division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after long division, there is no remainder

2.4b Dividing a Polynomial by a Polynomial

(The Hunt for Factors – Part 2)

Learning Goal: We are learning to divide a polynomial by a polynomial using synthetic division

Here we will examine an alternative form of polynomial division called **Synthetic Division**. Don't be fooled! This is not "fake division". You're thinking with the wrong meaning for "synthetic". (Do a search online and see if you can come up with the meaning I am taking!)

In Synthetic Division we concern ourselves with the coefficients of the dividend and the zero of the divisor.

Synthetic Division uses

numbers, no variables, and 3 operations

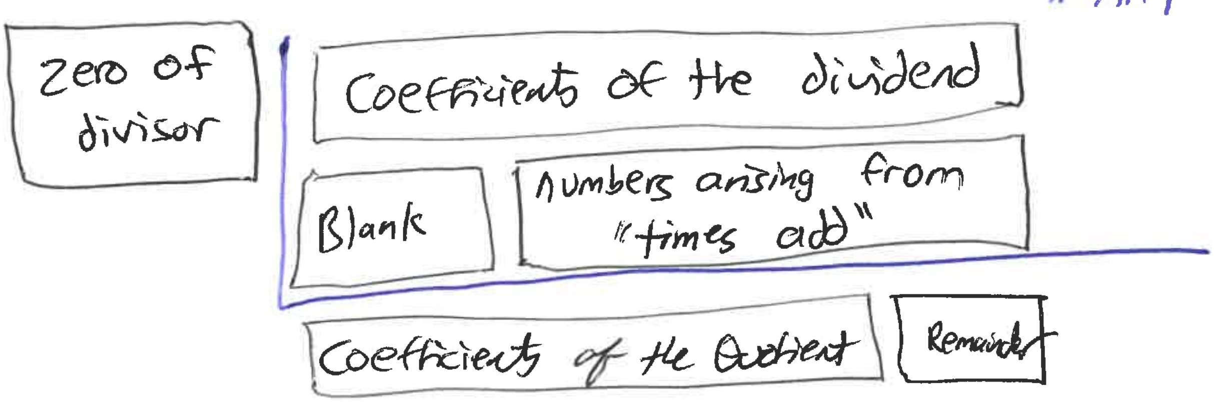
→ bring down, times, and add

Note: synthetic divison only uses linear divisors

$2x-5$, $x+3$, $\cancel{x^2-4}$

The Set-up

* All powers must be present



zero $x=2$

$(x-\boxed{(+2)})$

Example 2.4.3

Divide using synthetic division:

$$(4x^3-5x^2+2x-1)\div(x-2)$$

① Bring Down
② Times
③ Add

times

$$\begin{array}{r|rrrr} 2 & 4 & -5 & 2 & -1 \\ & \downarrow & +8 & +6 & +16 \\ \hline & 4 & 3 & 8 & 15 \end{array}$$

15 Remainder

x^2 x^1 x^0 ← Degree of quotient is 1 less than the start.

$\therefore\ 4x^3-5x^2+2x-1=(x-2)(4x^2+3x+8)+15$

zero $x=-1$

Example 2.4.4

Divide using synthetic division: $0x^3$

$$\frac{4x^4+3x^2-2x+1}{x+1}$$

$(x-\boxed{(-1)})$

$$\begin{array}{r|rrrrr} -1 & 4 & 0 & 3 & -2 & 1 \\ & \downarrow & +(-4) & +4 & -7 & 9 \\ \hline & 4 & -4 & 7 & -9 & 10 \end{array}$$

remainder

$\therefore\ 4x^4+3x^2-2x+1=(x+1)(4x^3-4x^2+7x-9)+10$

$x = \frac{3}{2}$ (zero)

Example 2.4.5

Divide using your choice of method (*and you choose synthetic division…amen*)

$$(2x^3 - 9x^2 + x + 12) \div (2x - 3)$$

$\frac{3}{2} \cdot \frac{2}{1} = 3$

$\frac{3}{2} \cdot \frac{-6}{1} = -9$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -9 & 1 & 12 \\ & \downarrow & 3 & -9 & -12 \\ \hline & 2 & -6 & -8 & 0 \end{array}$$

(no remainder)

BUT, synthetic division only works when the coefficient is 1. Divide answer by 2. → 1 −3 −4

$$\therefore 2x^3 - 9x^2 + x + 12 = (2x-3)(x^2-3x-4)$$
$$= (2x-3)(x-4)(x+1)$$

Must be descending

Example 2.4.6

Is $3x-1$ a factor of the function $f(x) = 6x - x^3 + 2 + 3x^4$? $= 3x^4 - x^3 + 0x^2 + 6x + 2$

$$\begin{array}{r|rrrrr} \frac{1}{3} & 3 & -1 & 0 & 6 & 2 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & 3 & 0 & 0 & 6 & 4 \end{array}$$

$\therefore$ $3x-1$ is not a factor because there is a remainder.

Example 2.4.7 (OK…this is a lot of examples!)

Consider again (from **Example 2.4.6**) $f(x)=3x^4-x^3+6x+2$, and calculate $f\left(\frac{1}{3}\right)$.

$$f\left(\tfrac{1}{3}\right) = 3\left(\tfrac{1}{3}\right)^4 - \left(\tfrac{1}{3}\right)^3 + 6\left(\tfrac{1}{3}\right) + 2$$

$$= \cancel{3}^{1}\left(\frac{1}{\cancel{81}\,27}\right) - \left(\frac{1}{27}\right) + 2 + 2$$

$$= 4$$

Wait!!! This is the same remainder when dividing $3x-1$!!

Example 2.4.8

Consider **Example 2.4.5**. Let $g(x)=2x^3-9x^2+x+12$, and calculate $g\left(\frac{3}{2}\right)$.

$$g\left(\tfrac{3}{2}\right) = 2\left(\tfrac{3}{2}\right)^3 - 9\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right) + 12$$

$$= \cancel{2}^{1}\left(\frac{27}{\cancel{8}\,4}\right) - 9\left(\frac{9}{4}\right) + \frac{3}{2} + 12$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{6}{4} + \frac{48}{4}$$

$$= \frac{0}{4} = 0 \,!$$

The Remainder Theorem

Given a polynomial function, $f(x)$, divided by a linear binomial, $x-k$, then the remainder of the division is the value $f(k)$

Proof of the Remainder Theorem

Consider $f(x) \div (x-k)$

Then the division statement is:

some quotient

$$f(x) = (x-k) \cdot q(x) + r$$

Now, $f(k) = (\overset{(0)}{k-k}) \cdot q(k) + r$

$$f(k) = r$$

Example 2.4.9

Determine the remainder of $\dfrac{5x^4 - 3x^3 - 50}{x-2}$.

WAIT!!!! We MUST have a FUNCTION

Let $f(x) = 5x^4 - 3x^3 - 50$

divisor is $x-2$

Calculate $f(2) = 5(2)^4 - 3(2)^3 - 50$

$= 80 - 24 - 50$

$= 6$

$\therefore$ Remainder is 6

Success Criteria:

- I can appreciate that synthetic division is "da bomb"
- I can use synthetic division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after synthetic division, there is no remainder (The Remainder Theorem)

2.5 The Factor Theorem

(Factors have been FOUND)

Learning Goal: We are learning the connections between a polynomial function and its remainder when divided by a binomial

The Factor Theorem

Given a polynomial function, $f(x)$, then $x-a$ is a factor of $f(x)$ if and only if

$f(a) = 0$

Example 2.5.1

Use the Factor Theorem to factor x^3+2x^2-5x-6.

WAIT!!!! We need a FUNCTION

$f(x) = x^3 + 2x^2 - 5x - 6$

First consider how the function is made:

$f(x) = (x-a)(x-b)(x-c)$

$(a)(b)(c) = -6$

$\therefore$ the factors must divide -6 !!

Try $x - 1$, $\therefore x = +1$

$f(1) = (1)^3 + 2(1)^2 - 5(1) - 6$

$= -8$ NOPE

Try $x+1$, $\therefore x = -1$

$f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$= -1 + 2 + 5 - 6 = 0!$

So, $(x+1)$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & \downarrow & -1 & -1 & 6 \\ \hline & 1x^2 & 1x^1 & -6x^0 & 0 \end{array}$$

$\therefore x^3 + 2x^2 - 5x - 6 = (x+1)(x^2+x-6)$

$= (x+1)(x+3)(x-2)$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Example 2.5.2

Factor **fully** $x^4 - x^3 - 16x^2 + 4x + 48$

① Try $(x+2)$, $x = -2$

$$f(-2) = (-2)^4 - (-2)^3 - 16(-2)^2 + 4(-2) + 48$$
$$= 16 + 8 - 64 - 8 + 48$$
$$= 0$$

②

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -16 & 4 & 48 \\ & \downarrow & -2 & +6 & 20 & -48 \\ \hline & 1 & -3 & -10 & 24 & 0 \end{array}$$

$g(x) = x^3 - 3x^2 - 10x + 24$

③ Try $(x-2)$, $x = 2$

$$g(2) = (2)^3 - 3(2)^2 - 10(2) + 24$$
$$= 8 - 12 - 20 + 24$$
$$= 0$$

④

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & & + & + & + \\ & \downarrow & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

So, $x^4 - x^3 - 16x^2 + 4x + 48 = (x+2)(x-2)(x^2 - x - 12)$

$$\boxed{= (x+2)(x-2)(x-4)(x+3)}$$

Example 2.5.3 (*Pg 177 #6c in your text*)

Factor fully $x^4 + 8x^3 + 4x^2 - 48x$

Note x $f(x) = (x)(x^3 + 8x^2 + 4x - 48)$

Now Try $(x-2)$ $x = 2$

$f(2) = (2)^3 + 8(2)^2 + 4(2) - 48$

$= 8 + 32 + 8 - 48 = 0!$

$$\begin{array}{c|cccc} 2 & 1 & 8 & 4 & -48 \\ & \downarrow & 2 & 20 & 48 \\ \hline & 1 & 10 & 24 & 0 \end{array}$$

$\therefore f(x) = (x)(x-2)(x^2 + 10x + 24)$

$= (x)(x-2)(x+4)(x+6)$

Example 2.5.4 (*Pg 177 #10*)

When $ax^3 - x^2 + 2x + b$ is divided by $\boxed{x-1 \text{ the remainder is } 10.}$ When it is divided by $\boxed{x-2 \text{ the remainder is } 51.}$ Find *a* and *b*.

Use the remainder theorem!

This problem is very instructive.

For $(x-1)$, $x = 1$

$f(1) \Rightarrow a(1)^3 - (1)^2 + 2(1) + b = 10$

$a - 1 + 2 + b = 10$

$a + b = 9$

For $(x-2)$, $x = 2$

$f(2) \Rightarrow a(2)^3 - (2)^2 + 2(2) + b = 51$

$8a \cancel{-4} \cancel{+4} + b = 51$

$8a + b = 51$

Now solve the system of equations

$$\begin{array}{r} 8a + b = 51 \\ - \quad a + b = 9 \\ \hline 7a \quad = 42 \end{array}$$

$\therefore a = 6$

$(6) + b = 9$

$\therefore b = 3$

Success Criteria:

- I can use test values to find the factors of a polynomial function
- I can factor a polynomial of degree three or greater by using the factor theorem
- I can recognize when a polynomial function is not factorable

2.6 Factoring Sums and Differences of Cubes

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Knowing how to factor a sum or difference of cubes is a simple matter of remembering patterns.

Learning Goal: We are learning to factor a sum or difference of cubes.

Example 2.6.1 (*Recalling the pattern for factoring a Difference of Squares*)

Factor $4x^2 - 25$

$= (2x-5)(2x+5)$

Note: Sums of Squares DO NOT factor!!

e.g. Simplify $x^2 + 4 \neq (x+2)(x+2)$

$(8x^3 \pm 27)$

Differences of Cubes

SOAP

Pattern — Same — Opposite — Always Positive

$(cube_1 - cube_2) = (cuberoot_1 - cuberoot_2)(cuberoot_1^2 + cuberoot_1 \times cuberoot_2 + cuberoot_2^2)$

$8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)$

TWO POSITIVES and ONE NEGATIVE

Sums of Cubes (These DO factor!!)

Pattern

$(cube_1 + cube_2) = (cuberoot_1 + cuberoot_2)(cuberoot_1^2 - cuberoot_1 \times cuberoot_2 + cuberoot_2^2)$

$8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$

$$2\,\big|\;\begin{array}{cccc} 1 & 0 & 0 & -8 \\ \downarrow & 2 & 4 & 8 \\ \hline 1 & 2 & 4 & 0 \end{array}$$

Example 2.6.2 SOAP

Factor $x^3 - 8$

$= (x-2)(x^2 + 2x + 4)$

Example 2.6.3

Factor $27x^3 + 125y^3$ $= (3x + 5y)(9x^2 - 15xy + 25y^2)$

$3x + 5y$

Example 2.6.4

Factor $1 - 64z^3$ $= (1-4z)(1 + 4z + 16z^2)$

$1 + 4z$

Example 2.6.5

Factor $1000x^3 + 27$

$10x + 3$ $= (10x+3)(100x^2 - 30x + 9)$

Example 2.6.6

Factor $x^6 - 729$

$\sqrt[3]{x^6}$

$= (x^6)^{1/3}$

$= x^2$

$\sqrt[3]{729}$

$= 9$

$= (x^2 - 9)(x^4 + 9x^2 + 81)$

$(x^2)^2$

BUT WAIT!

$= (x+3)(x-3)(x^4 + 9x^2 + 81$

Success Criteria:

- I can use patterns to factor a sum of cubes
- I can use patterns to factor a difference of cubes