Advanced Functions

Teacher Notes

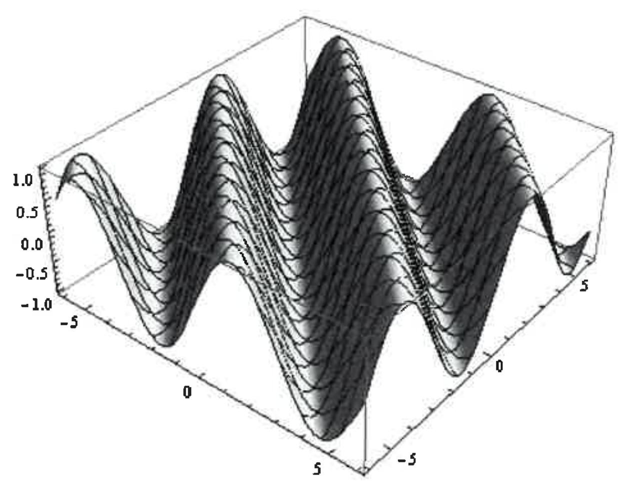
Course Notes

Unit 5 – Trigonometric Functions

Doing Trig with REAL Numbers

We will learn

- *about Radian Measure and its relationship to Degree Measure*
- *how to use Radian Measure with Trigonometric Functions*
- *about the connection between trigonometric ratios and the graphs of trigonometric functions*
- *how to apply our understanding of trigonometric functions to model and solve real world problems*



AooΩ
MATH@TD

Chapter 5 – Trigonometric Functions

Contents with suggested problems from the Nelson Textbook ***(Chapter 6)***

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

5.1 Radian Measure and Arc Length
Pg. 321 #2edfh, 3 – 9

5.2 Trigonometric Ratios and Special Triangles (Part 1)
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5.3 Trigonometric Ratios and Special Triangles (Part 2 – Exact Values)
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5.5 Sketching the Trigonometric Functions
Worksheet

5.6 Transformations of Trigonometric Functions
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5.1 Radian Measure and Arc Length

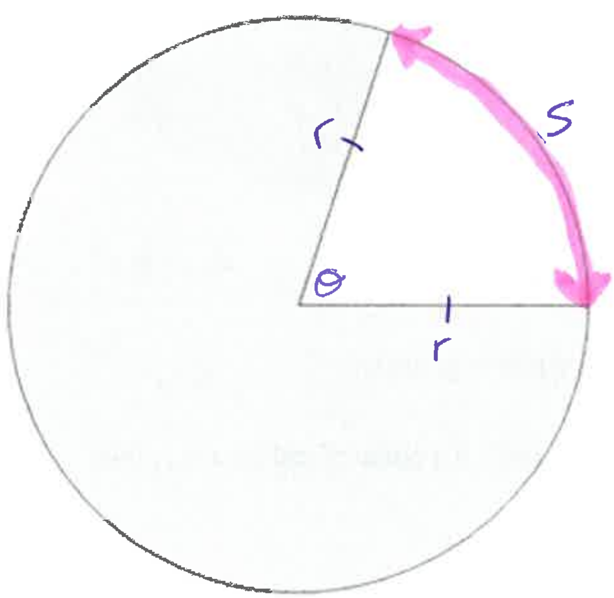
Learning Goal: We are learning to use radian measure to represent the size of an angle.

Radian Measure

We are familiar with measuring angles using "degrees", and now we will turn to another measure for angles: **Radians**.

Before getting to the notion of radians, we need to learn some notation.

Picture



θ = central angle

r = radius

s = arc length subtended by the central angle. Is a part of the circumference

There is a relationship between s, θ, and r.

Arc Length Formula: $s = r\theta$

θ must be in radians

Definition 5.1.1

In a circle of radius r, a central angle θ subtending an arc of length $s = r$ measures 1 radian.

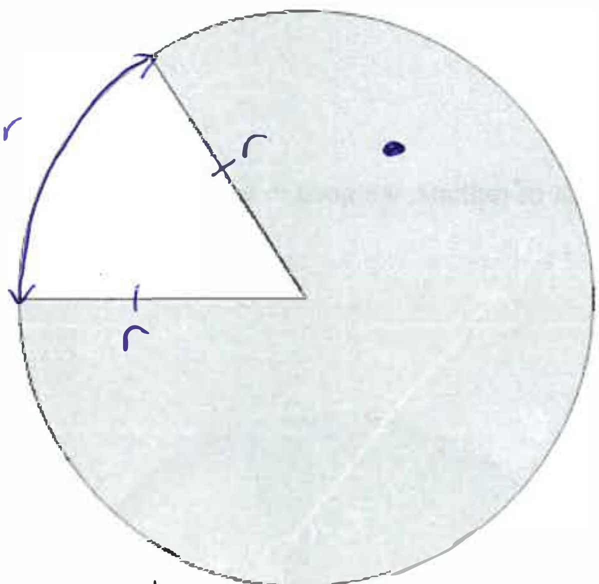
Picture

$s = r\theta$

$\frac{s}{r} = \frac{r}{r}\theta$

$1 = \theta$

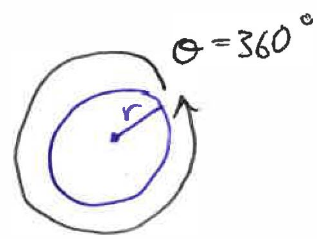
1 radian.



1 radian is the angle when $s = r$. Always the same on any size circle.

Note: The circumference of a circle is given by $C = 2\pi r$

So, for a central angle of $360°$, in a circle of radius $r = 1$, then



$s = 2\pi r$

$\cancel{r}\theta = 2\pi \cancel{r}$

$\theta = 2\pi$ radians

$360° = 2\pi$ radians

$\therefore \boxed{180° = \pi \text{ radians}}$ Conversion factor

$1° = \frac{\pi}{180}$ radians

Example 5.1.1

Convert the following to radians:

Cancel your units!

a) $30°\left(\frac{\pi}{180°}\right)$

$= \frac{30\pi}{180}$

$= \frac{\pi}{6}$ rad

b) $45°\left(\frac{\pi}{180°}\right)$

$= \frac{45\pi}{180}$

$= \frac{\pi}{4}$ rad

c) $120°\left(\frac{\pi}{180°}\right)$

$= \frac{12\cancel{0}\pi}{18\cancel{0}} = \frac{12\pi}{18}$

$= \frac{2\pi}{3}$

d) $\cancel{315}^{7}°\left(\frac{\pi}{\cancel{180°}_{4}}\right)$

$= \frac{7\pi}{4}$

e) $161.3°\left(\frac{\pi}{180°}\right)$

$= \frac{161.3°\cdot(3.14)}{180°}$

$= 2.81$ rad

Example 5.1.2

Convert the following to degrees (round to two decimal places where necessary)

cancel your units! flip the conversion Factor!

$\frac{180°}{\pi} = 1$ rad

a) $\frac{7\pi}{12}\ \boxed{\text{rad}}$ replace w/ conversion factor

$= \frac{7\cancel{\pi}}{12}\left(\frac{180°}{\cancel{\pi}}\right)$ Units of π radians cancel

$= \frac{7\cdot 180°}{12}$

$= 105°$

b) $\frac{10\pi}{9}$ rad

$= \frac{10\cancel{\pi}}{9}\left(\frac{180°}{\cancel{\pi}}\right)$

$= \frac{10\cdot 180°}{9}$

$= 200°$

c) 2.5 rad

$2.5\left(\frac{180°}{\pi}\right)$

$= \frac{450°}{\pi}$

$= 143.3°$

d) $\frac{\pi}{2}$ rad

$= \frac{\cancel{\pi}}{2}\left(\frac{180°}{\cancel{\pi}}\right)$

$= 90°$

e) $-\frac{\pi}{3}$ rad

$= \left(-\frac{\cancel{\pi}}{3}\right)\left(\frac{180°}{\cancel{\pi}}\right)$

$= \frac{-180°}{3}$

$= -60°$

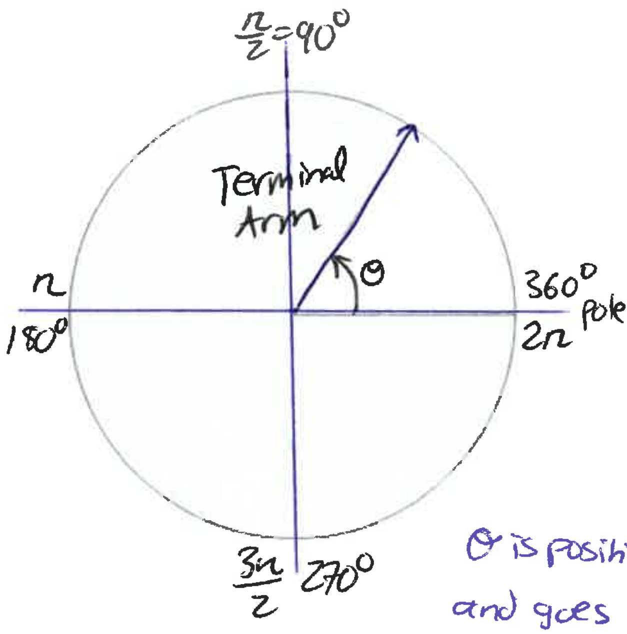
Q. What the rip is a negative degree?

Angles of Rotation

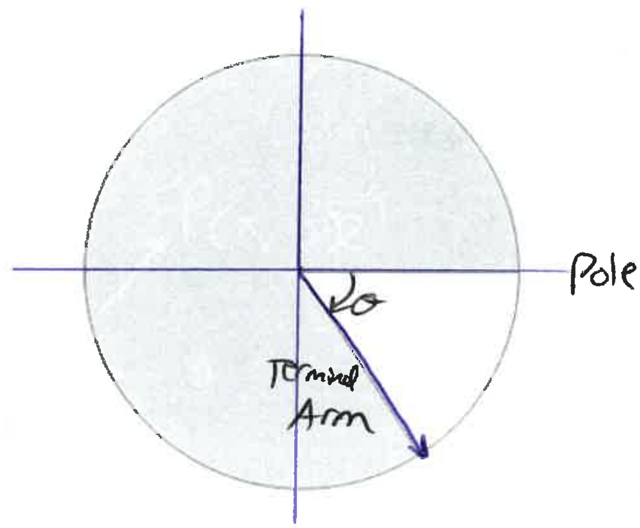
The sign on an angle can be thought of as the direction of rotation (around a circle).

Angles of rotation always begin at the pole (pos. x-axis) and end at the terminal arm.

Pictures



θ is positive and goes counter-clockwise



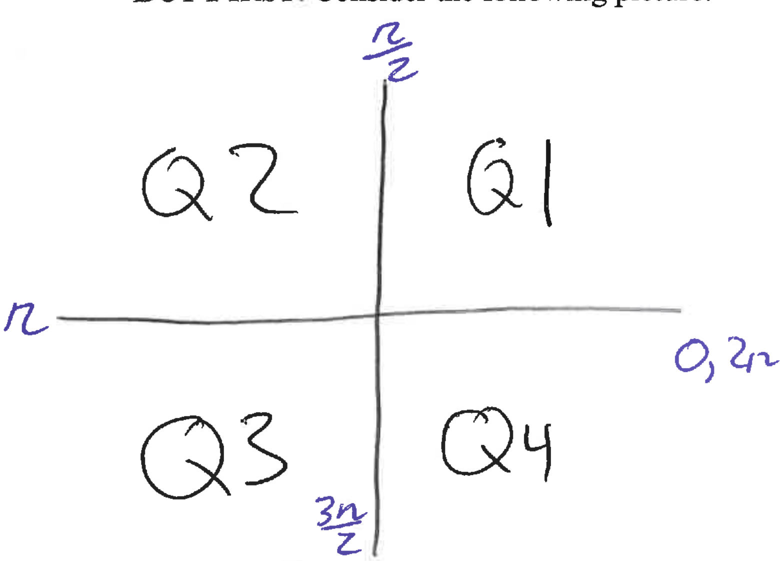
θ is negative + goes clockwise.

Example 5.1.3

Sketch the following angles of rotation:

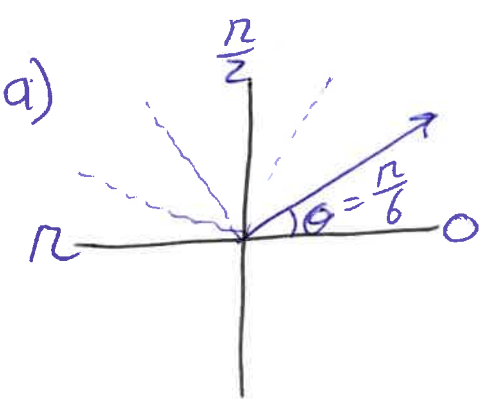
a) $\frac{\pi}{6}$ rad b) $\frac{2\pi}{3}$ rad c) $-\frac{3\pi}{4}$ rad d) $\frac{7\pi}{6}$

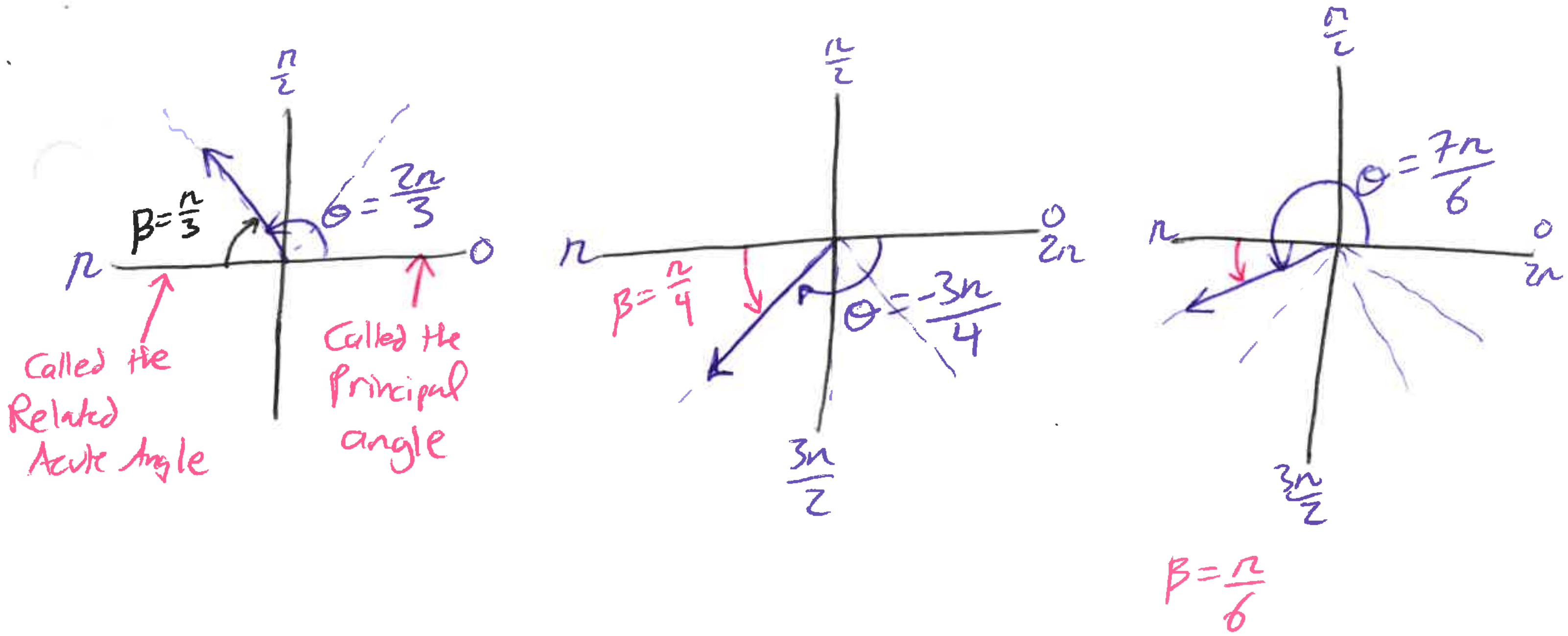
BUT FIRST: Consider the following picture:



(going counter-clockwise)

a)





Example 5.1.4

Determine the length of an arc, on a circle of radius $5cm$, subtended by an angle:

a) $\theta = 2.4$ rad

$S = r\theta$

$= (5)(2.4)$

$S = 12 \text{ cm}$

b) $\theta = 120°$

$\theta = 120°\left(\frac{\pi}{180°}\right)$

$\theta = \frac{2\pi}{3}$

$S = r\theta$

$= (5)\left(\frac{2\pi}{3}\right)$

$= \frac{10\pi}{3}$ ← Exact value

$S \approx 10.5 \text{ cm}$ ← Estimate

Success Criteria:

- I can understand that a radian is a real number
- I can convert from degrees to radians by multiplying by $\frac{\pi}{180^o}$
- I can convert from radians to degrees by multiplying by $\frac{180^o}{\pi}$

5.2 Trigonometric Ratios and Special Triangles

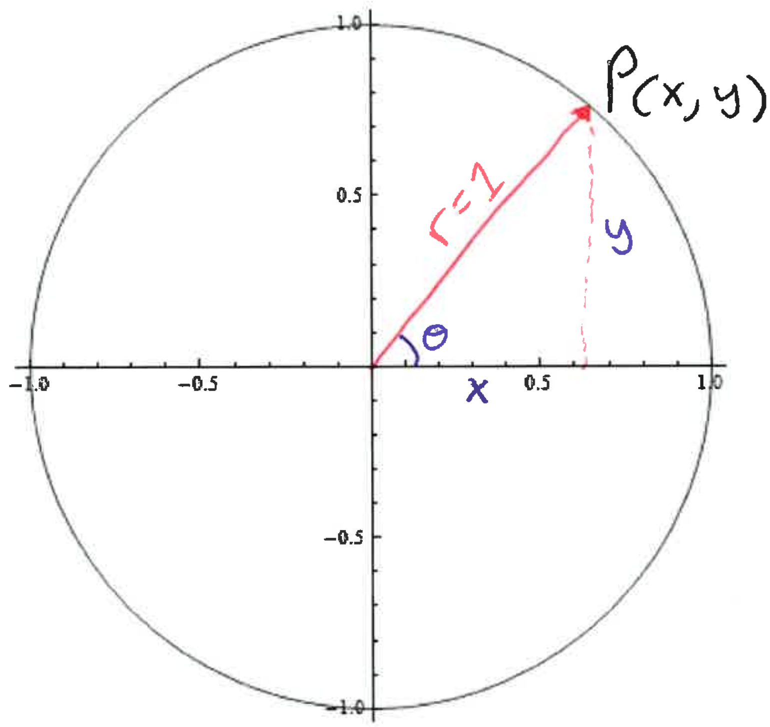
(*Part 1*) – Means NO CALCULATORS

Need Exact answers

Learning Goal: We are learning to use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

→ fractions & $\sqrt{\ }$

Consider the circle of radius 1:

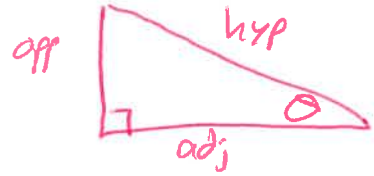


← Always based on x axis

Called a unit circle based on Pythagorean Theorem

$x^2 + y^2 = 1^2$

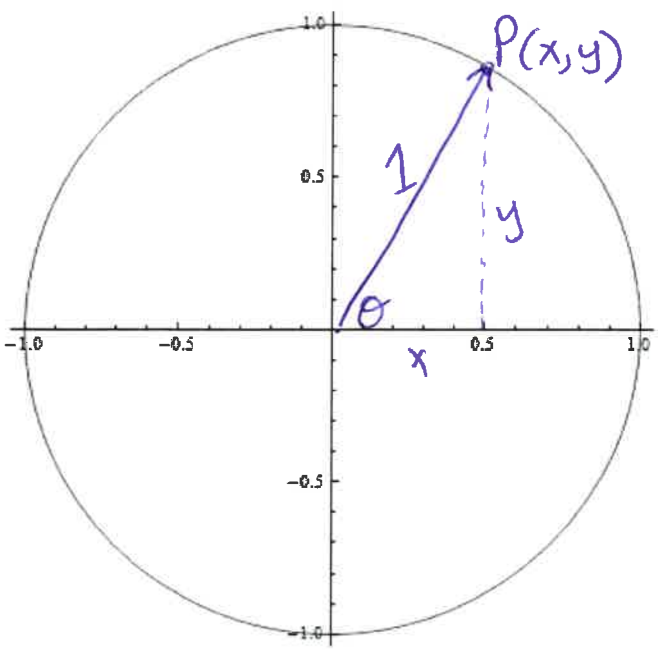
$\boxed{x^2 + y^2 = 1}$



Recall the six main Trigonometric Ratios:

Primary Trig Ratios	Reciprocal Trig Ratios
$\sin\theta = \frac{opp}{hyp}$	$\frac{1}{\sin\theta} = \csc\theta = \frac{hyp}{opp}$
$\cos\theta = \frac{adj}{hyp}$	$\frac{1}{\cos\theta} = \sec\theta = \frac{hyp}{adj}$
Always less than one	Always greater than one
$\tan\theta = \frac{opp}{adj}$	$\frac{1}{\tan\theta} = \cot\theta = \frac{adj}{opp}$

Consider again the circle of radius 1 (but now keeping in mind SOH CAH TOA)



$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{y}{x}$

$P(x,y)$ can be represented by $P(\cos\theta, \sin\theta)$

The Pythagorean Identity

well, $x^2+y^2=1$

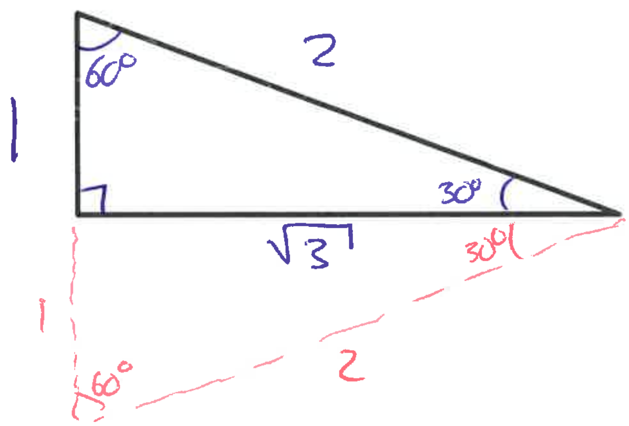
$\therefore (\cos\theta)^2+(\sin\theta)^2=1$

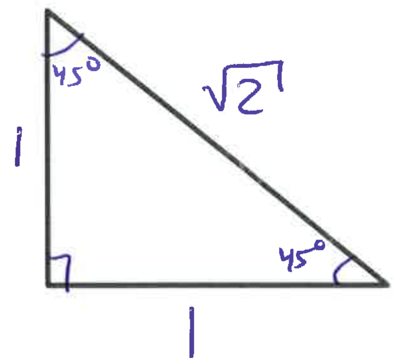
$\therefore \boxed{\cos^2\theta+\sin^2\theta=1}$ MEMORIZE

Special Triangles in Radians

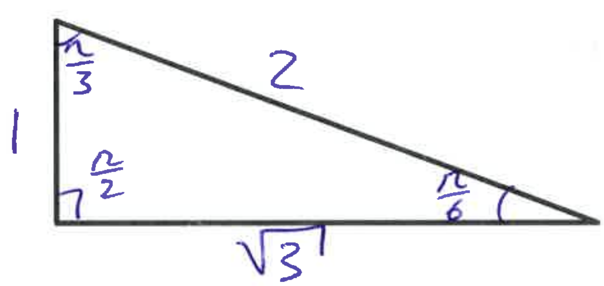
Recall: We have two "Special Triangles". In **degrees** they are:

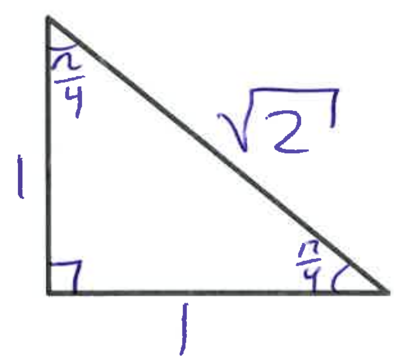
"1-60-2" "45"





In radians we have $60°\left(\frac{\pi}{180°}\right) = \frac{\pi}{3}$ $\quad 30°\left(\frac{\pi}{180°}\right) = \frac{\pi}{6}$ $\quad 45°\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$

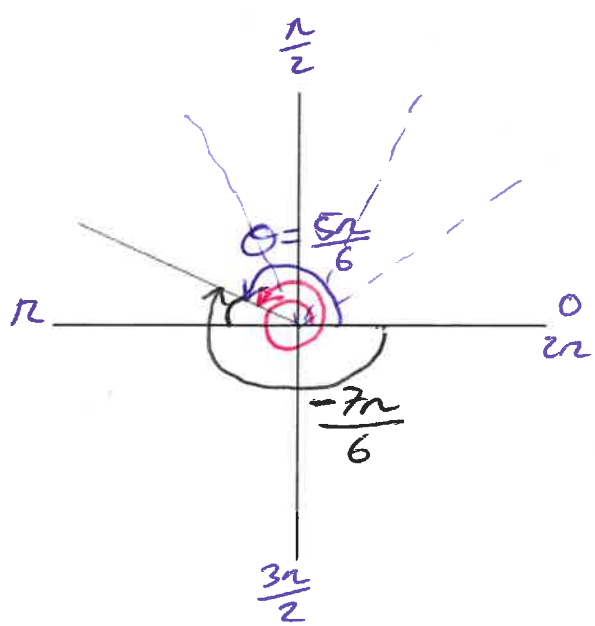




MEMORIZE THESE!

Angles of Rotations and Trig Ratios

Consider the following sketch of the angle of rotation $\theta = \frac{5\pi}{6}$: So, $\frac{5}{6}\pi$



Find two Identical angles

Let's Go backwards (clockwise)

– Related acute angle is $\frac{\pi}{6} = \beta$

$$\text{So, } \theta_2 = -\pi + -\frac{\pi}{6} = -\frac{6\pi}{6} + \frac{-\pi}{6} = -\frac{7\pi}{6}$$

Let's go around the circle once!

$$\theta_3 = 2\pi + \frac{5\pi}{6} = \frac{12\pi}{6} + \frac{5\pi}{6} = \frac{17\pi}{6}$$

There are infinitely many angles of rotation for each terminal arm.

In this Example, we call $\theta = \frac{5\pi}{6}$ the PRINCIPAL ANGLE, or the angle in Standard Position

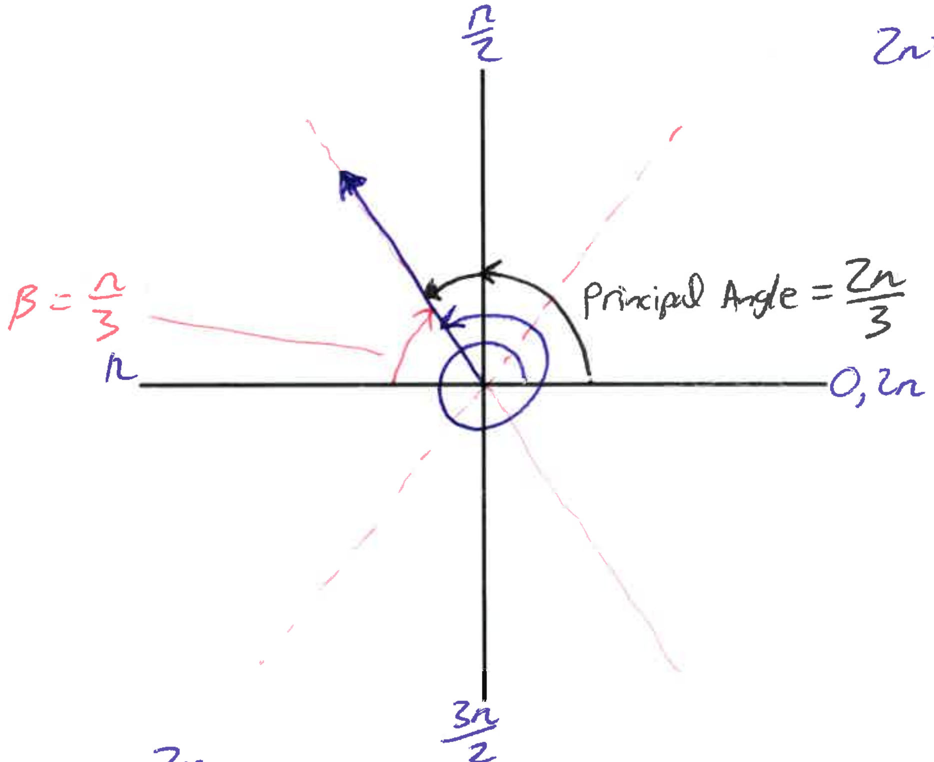
We take this to mean the "smallest positive angle of rotation"

[Note: All principal angles $\theta \in [0, 2\pi]$]

Example 5.2.1

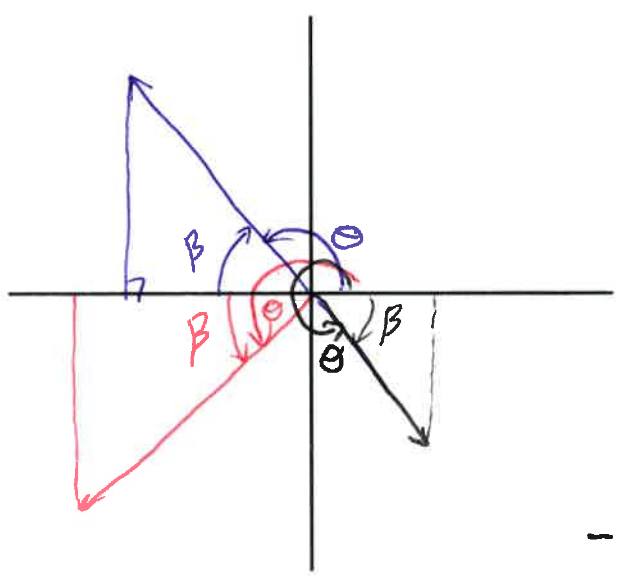
Sketch the angle of rotation $\theta = \dfrac{8\pi}{3}$ and determine the principal angle.

$2\pi = \frac{6\pi}{3}$ (once around the circle)



Principal Angle: $\frac{2\pi}{3}$

*Students draw one on each grid provided. They have 3.



- Principal angles are outside the triangle
- β is the related <u>acute</u> angle

$\therefore 0 < \beta \leq \frac{\pi}{2}$

I might call β a "reference angle".

- Now, in Q1,

$\theta = \beta$

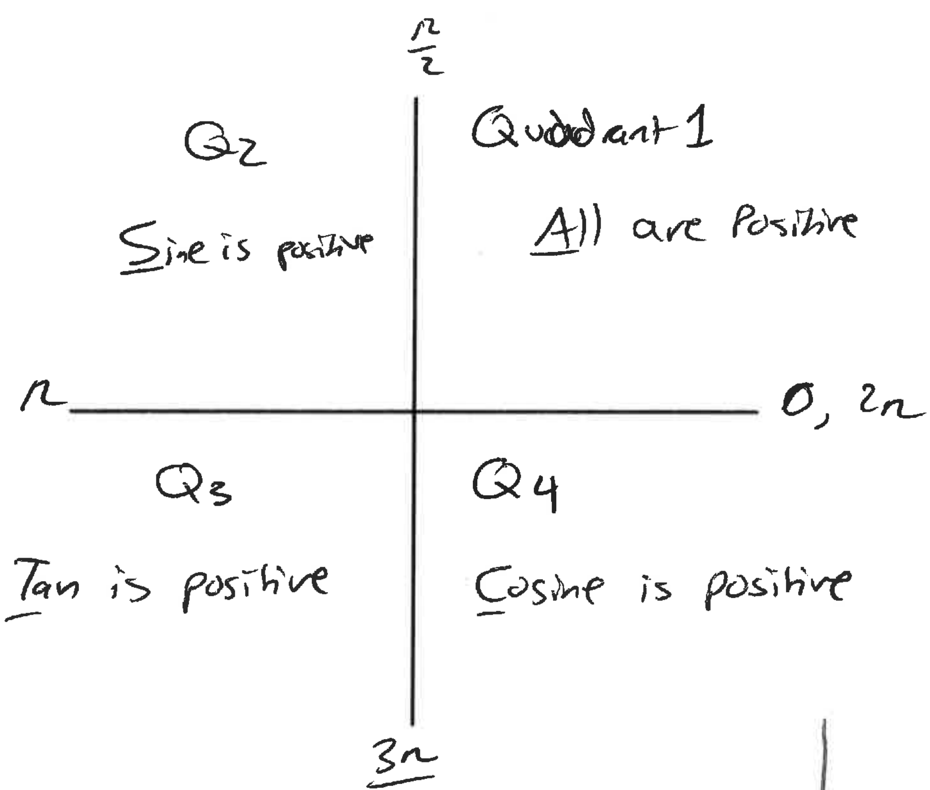
We now have enough tools to calculate the trigonometric ratios of any angle!

For any given angle θ (in radians from here on) we will:

1) Draw θ in standard position (i.e. draw the principal angle for θ)
2) Determine the related acute angle (between the terminal arm and the polar axis)
3) Use the related acute angle and the CAST RULE (and SOH CAH TOA) to determine the trig ratio in question

Recall the CAST RULE

Note: The CAST RULE determines the sign (+ or -) of the trig ratio



$\sin\theta = \frac{+y}{+r}$

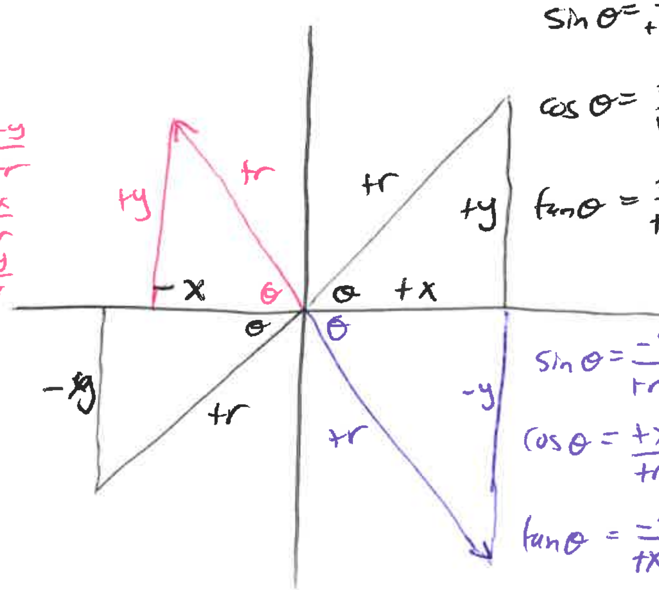
$\cos\theta = \frac{-x}{+r}$

$\tan\theta = \frac{+y}{-x}$

$\sin\theta = \frac{-y}{+r}$

$\cos\theta = \frac{-x}{+r}$

$\tan\theta = \frac{-y}{-x}$



$\sin\theta = \frac{+y}{+r}$

$\cos\theta = \frac{+x}{+r}$

$\tan\theta = \frac{+y}{+x}$

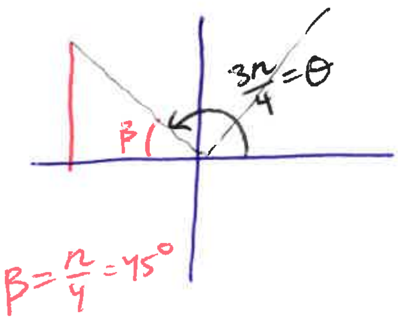
$\sin\theta = \frac{-y}{+r}$

$\cos\theta = \frac{+x}{+r}$

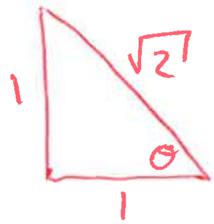
$\tan\theta = \frac{-y}{+x}$

Example 5.2.2

Determine the trig ratio $\sin\left(\frac{3\pi}{4}\right)$



$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$



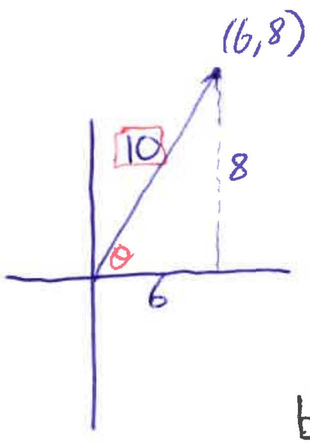
Example 5.2.3

The point $(6,8)$ lies on the terminal arm (of length r) of an angle of rotation. Sketch the angle of rotation.

Determine: a) the value of r

b) the primary trig ratios for the angle

c) the value of the angle of rotation in radians, to two decimal places



a) Pythag theorem

$$a^2 + b^2 = c^2$$
$$6^2 + 8^2 = c^2$$
$$36 + 64 = r^2$$
$$\sqrt{100} = r$$
$$10 = r$$

b) $\sin\theta = \frac{8}{10} = \frac{4}{5}$

$\cos\theta = \frac{6}{10} = \frac{3}{5}$

$\tan\theta = \frac{8}{6} = \frac{4}{3}$

c) Set calculator to radian mode

$\tan\theta = \frac{8}{6}$

$\theta = \tan^{-1}\left(\frac{8}{6}\right)$

$\boxed{= 0.93 \text{ rad}}$

Success Criteria

- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can write the trig ratios for any angle using "x, y, and r"
- I can use the CAST rule to determine where a ratio is positive or negative

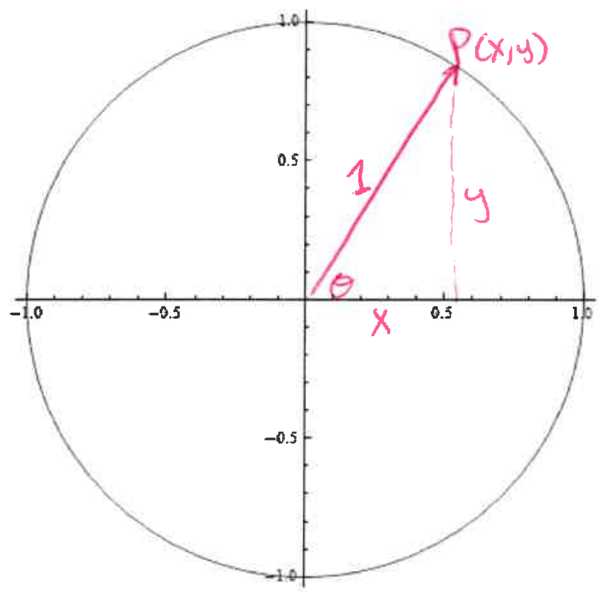
5.3 Trigonometric Ratios and Special Triangles

(*Part 2 – Exact Values*)

Learning Goal: We are learning to use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

Radius = 1

Recall the "Unit Circle" from yesterday:



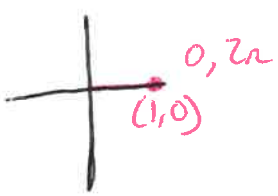
is $P(\cos\theta, \sin\theta)$

$\sin\theta = y$

$\cos\theta = x$

$\boxed{\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}}$

With this circle (and without a calculator!) we can evaluate EXACTLY the trig ratios for the angles (in radians) $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ radians.



$\sin 0 = 0$

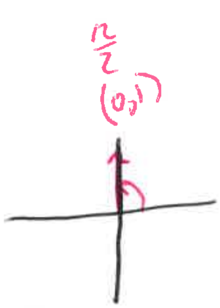
$\cos 0 = 1$

$\tan 0 = \frac{0}{1} = 0$

$\sin\pi = 0$

$\cos\pi = -1$

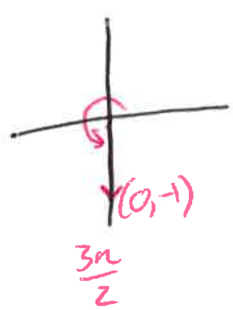
$\tan\pi = \frac{0}{-1} = 0$



$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2} = \frac{1}{0}$ = undefined



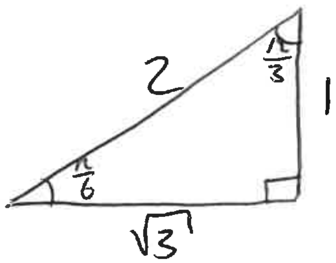
$\sin\frac{3\pi}{2} = -1$

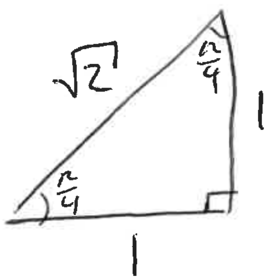
$\cos\frac{3\pi}{2} = 0$

$\tan\frac{3\pi}{2} = \frac{-1}{0}$

= undefined

Now, using **Special Triangles**, and **CAST** we can evaluate **EXACTLY** trig ratios for "special angles".





Note: A trig ratio is a NUMBER.
Numbers have 2 qualities

1) value
2) sign (+ or −)

Thus a trig ratio has a Value

(which we **evaluate** using the related acute angle and Special Triangles)

AND, a trig ratio has a Sign, which we get by using the Cast Rule or graphing it on the x-y axis

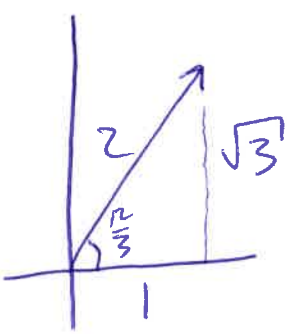
Example 5.3.1

Determine **Exactly** (i.e. the use of a calculator means **MARKS OFF**)

1) Figure out β
2) Get the value of the ratio
3) use the CAST rule for the Sign

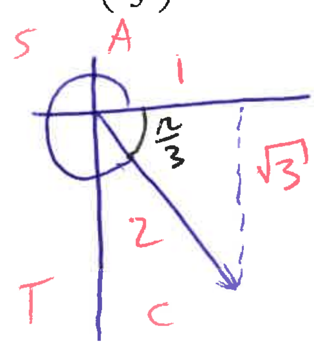
a) $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(check on calculator)



Create the special △

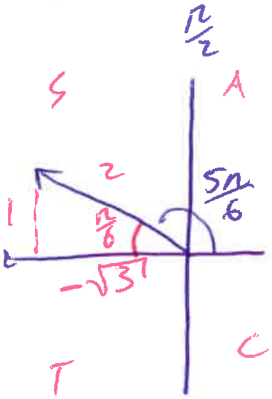
d) $\sec\left(\frac{5\pi}{3}\right)$



$\frac{3}{3} + \frac{2}{3}$

$\sec\left(\frac{5\pi}{3}\right) = \frac{hyp}{adj} = \frac{2}{1} = 2$

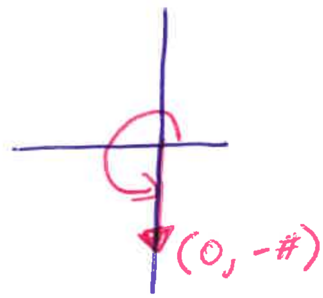
Since cos is + sec is + too



b) $\cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

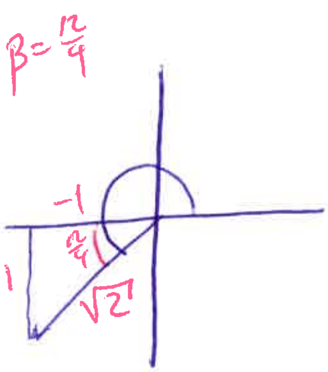
we can confirm ⊖
w/ CAST

e) $\tan\left(\frac{3\pi}{2}\right) = \frac{-\#}{0}$ = undefined

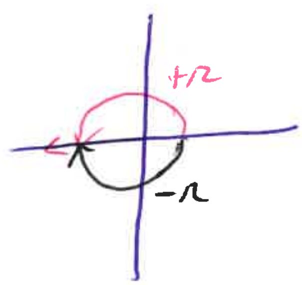


c) $\tan\left(\frac{5\pi}{4}\right) = \frac{-1}{-1}$

$= 1$



f) $\csc(-\pi)$ rewrite without - angle



$x = -\#$

$y = 0$

$r = \#$

$= \csc(\pi)$

$= \frac{1}{\sin \pi}$

$= \frac{1}{\frac{0}{1}}$ $\frac{y}{r}$

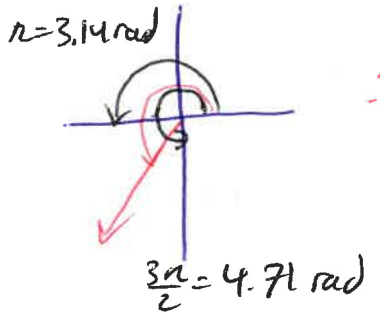
$= \frac{1}{0}$

= undefined

Example 5.3.2

Given sin(4) determine:

a) The quadrant $\theta = 4$ is in. b) The sign of sin(4) (no calculators!)



∴ In Quadrant 3

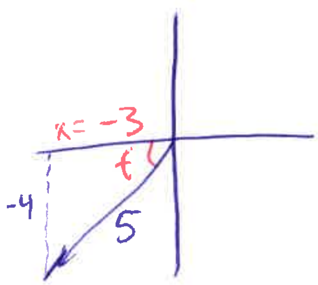
Only tan is +,

so sin(4) is neg.

Example 5.3.3

Given $\sin(t) = -\frac{4}{5}$, $\underbrace{\pi \le t \le \frac{3\pi}{2}}_{Q3}$, determine

a) $\cos(t)$ b) $\tan(t)$ c) t in radians, rounded to three decimal places.



$x^2 + (-4)^2 = 5^2$

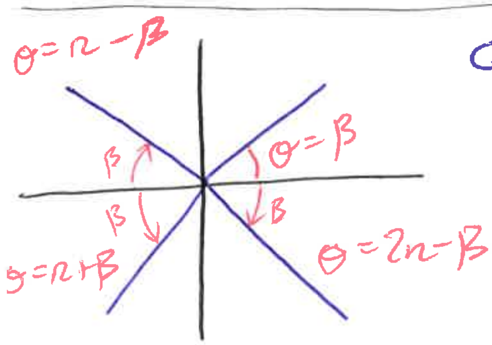
$x^2 = 25 - 16$

$x^2 = 9$

$x = 3$, well -3!

a) $\cos(t) = \frac{-3}{5}$

b) $\tan(t) = \frac{-4}{-3} = \frac{4}{3}$



c) t is an angle β, which we have to consider in standard position

$\tan\beta = \frac{4}{3}$

$\beta = \tan^{-1}\left(\frac{4}{3}\right)$

$\beta = 0.927$ rad

Now, the standard angle

$t = \pi + \beta$

$= 3.14.. + 0.927$

$t = 4.067$ rad

Success Criteria

- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can identify the trig ratios of the 4 axis angles
- I can write the trig ratios for any angle using “x, y, and r”
- I can use the CAST rule to determine where a ratio is positive or negative

5.4 Trigonometric Ratios and Special Triangles

(*Part 3 – Getting the Angles*)

Learning Goal: We are learning to determine the exact values for both angles between 0 and 2π, given a particular trig ratio.

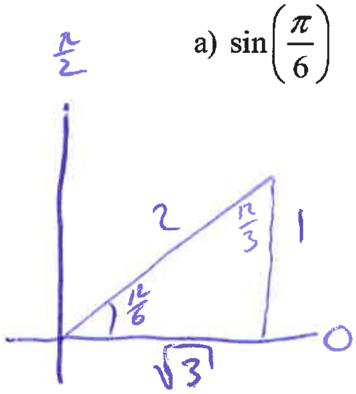
We have been looking at **evaluating exact values** for trigonometric ratios using special triangles and CAST, given an **angle of rotation.** We now turn our attention to the **inverse operation** – determining **angles of rotation** given a **trig ratio.**

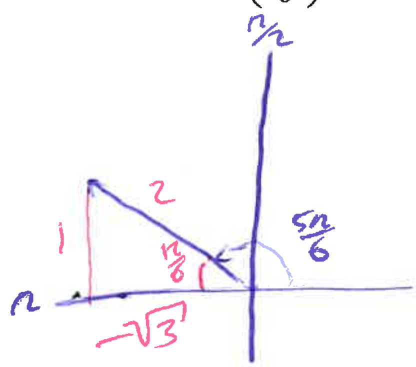
Example 5.4.1

Determine exactly:

a) $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

b) $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$





Note: Every trig ratio has two angles of rotation in $[0, 2\pi]$. Both give the same answer.

Exception is some axis angles.

Utilize CAST RULE

Example 5.4.2

Determine BOTH angles of rotation, θ, for $0 \le \theta \le 2\pi$ given

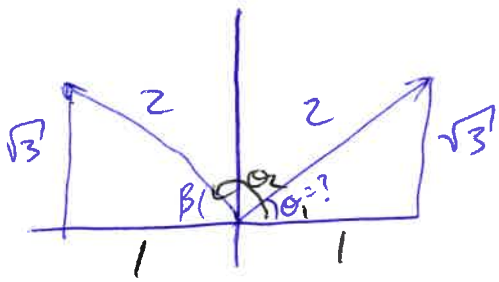
S | A
T | C

a) $\sin(\theta) = \frac{\sqrt{3}}{2}$

sin ⊕, so Quad 1 & 2

Procedure

1) Determine the quadrants θ is in.
2) Draw the angles of rotation.
3) Determine the related acute angle φ and construct the appropriate special triangles.
4) Determine the angles of rotation.



Special Δ, $\theta_1 = 60^\circ$ or $\frac{\pi}{3}$

$\theta_2 = \pi - \frac{\pi}{3}$

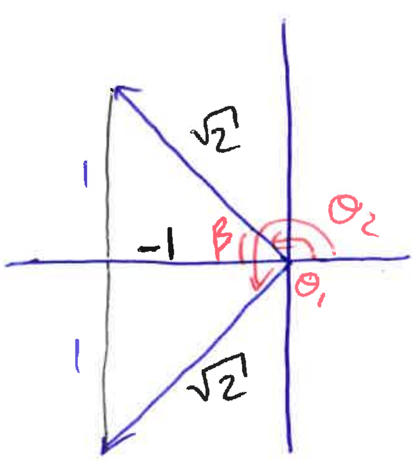
$= \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore \theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

b) $\cos(\theta) = -\frac{1}{\sqrt{2}}$

Quad 2 & 3

Special Δ, $\beta = 45^\circ$ or $\frac{\pi}{4}$



$\theta_1 = \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4}$

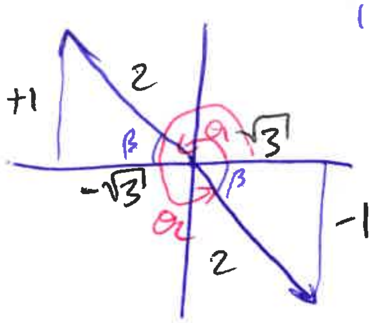
$\theta_2 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\therefore \theta = \frac{3\pi}{4}$ and $\frac{5\pi}{4}$

Follows tan (negative) $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$

c) $\cot(\theta) = -\sqrt{3}$ $= \frac{-\sqrt{3}}{1} = \frac{a}{o}$

Special Δ, $\beta = \frac{\pi}{6}$



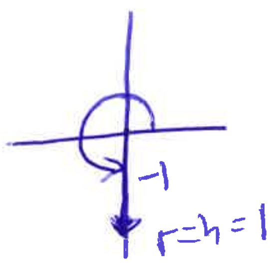
$\theta_1 = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$

$\theta_2 = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$

$\therefore \theta = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$

d) $\sin(\theta) = -1$

$\boxed{\text{Recall : } \sin\theta = y, \ \cos\theta = x}$

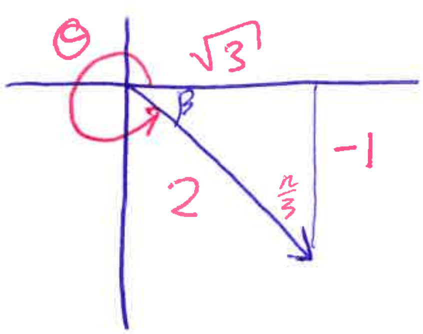


$\theta = \frac{3\pi}{2}$

Example 5.4.3

Determine θ where $\underbrace{\frac{3\pi}{2} \le \theta \le 2\pi}_{Q4}$ for $\csc(\theta) = -2$ $= \frac{-2}{1} = \frac{h}{o}$

Sin + csc opposite



$\beta = \frac{\pi}{6}$

$\theta = 2\pi - \frac{\pi}{6}$
$= \frac{12\pi}{6} - \frac{\pi}{6}$
$= \frac{11\pi}{6}$

Example 5.4.4

Determine θ when $\cos\theta = -0.8213$.

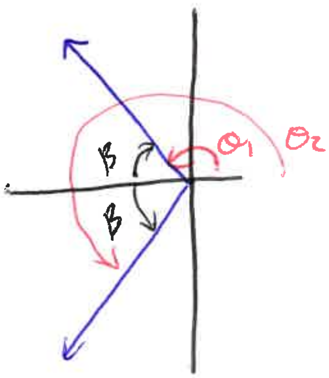
Not a special Δ!

Find β using positive acute angle

$\cos\beta = 0.8213$

$\beta = \cos^{-1}(0.8213$

$= 0.61$ rad



$\theta_1 = \pi - 0.61 = 3.14 - 0.61 \boxed{= 2.53}$

$\theta_2 = 3.14 + 0.61 \boxed{= 3.75}$

Practice Problems (Homework)

Determine the angles of rotation, θ, for $0 \le \theta \le 2\pi$:

a) $\sin(\theta) = -\frac{\sqrt{3}}{2}$ b) $\sec(\theta) = \sqrt{2}$

c) $\tan(\theta) = \frac{1}{\sqrt{3}}$ d) $\cot(\theta) = -1$

e) $\csc(\theta) = \frac{2}{\sqrt{3}}$ f) $\cos(\theta) = 0$

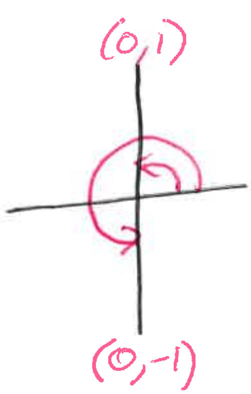
g) $\sin(\theta) = 1$ h) $\sqrt{3}\cos(\theta) - 2\cos(\theta) \cdot \sin(\theta) = 0$

h) Tricky! Factor out like terms

$$(\cos\theta)(\sqrt{3} - 2\sin\theta) = 0$$

we have two zeros to test!

Solve: $\cos\theta = 0$ Remember ($\cos\theta = x$)

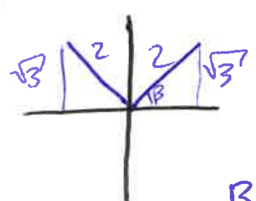


$\theta_1 = \frac{\pi}{2}$

$\theta_2 = \frac{3\pi}{2}$

Solve: $\sqrt{3} - 2\sin\theta = 0$

$\sqrt{3} = 2\sin\theta$



$\frac{o}{h}$ $\frac{\sqrt{3}}{2} = \sin\theta$

$\beta = \frac{\pi}{3}$

$\theta_1 = \frac{\pi}{3}$ $\theta_2 = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$

4 Solutions $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$

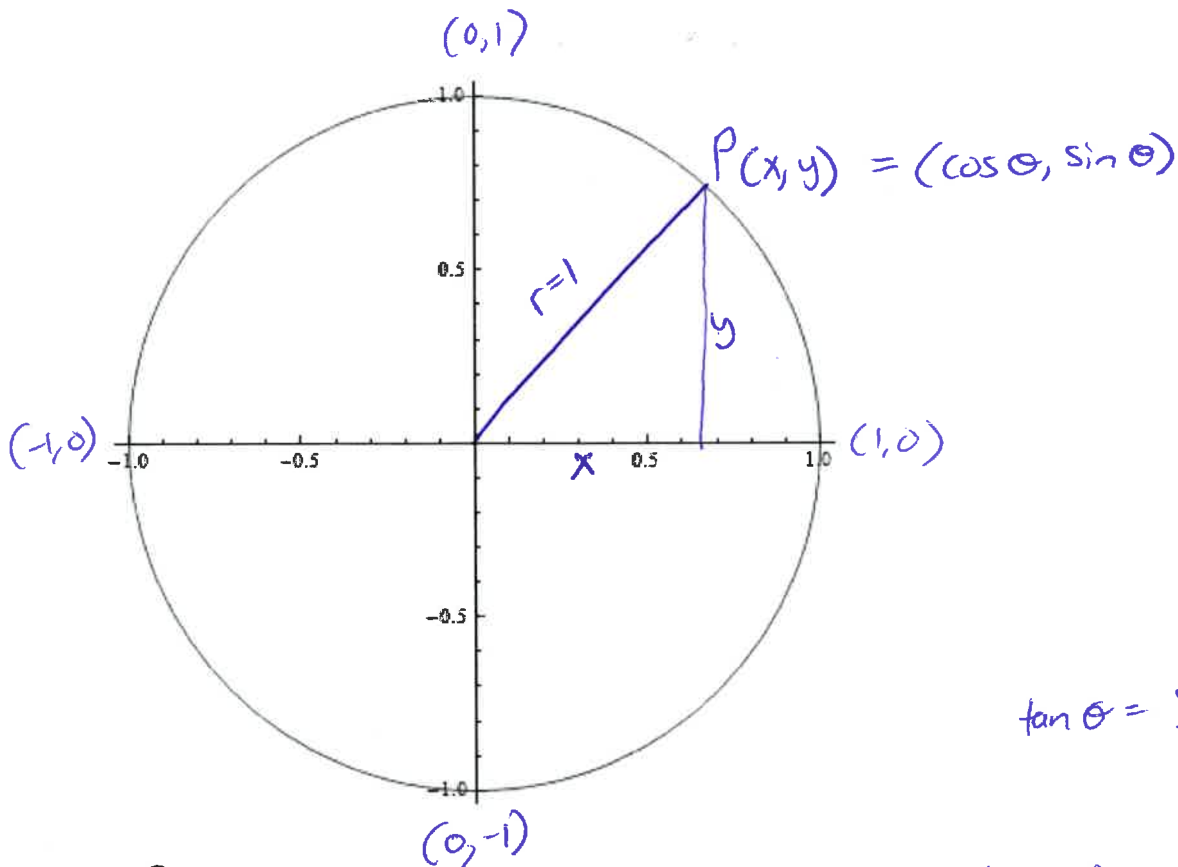
Success Criteria

- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can, given a trig ratio, determine the exact values for both angles between 0 and 2π
- I can identify the trig ratios of the 4 axis angles
- I can write the trig ratios for any angle using "x, y, and r"
- I can use the CAST rule to determine where a ratio is positive or negative

5.5 Sketching the Trigonometric Functions

Learning Goal: We are learning to sketch the graphs of 6 trigonometric functions.

Before beginning the sketches, recall the diagram of the unit circle that we have been using to explore the basic ideas in trigonometry:

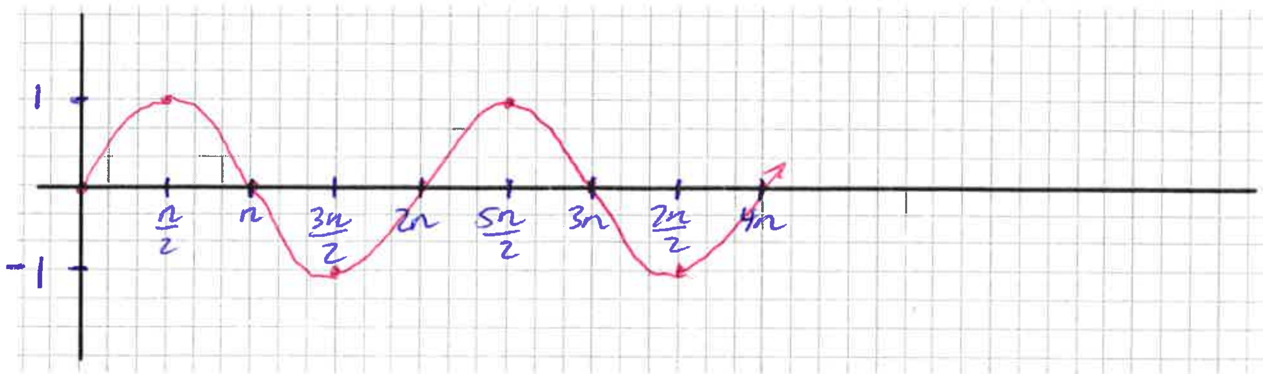


$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

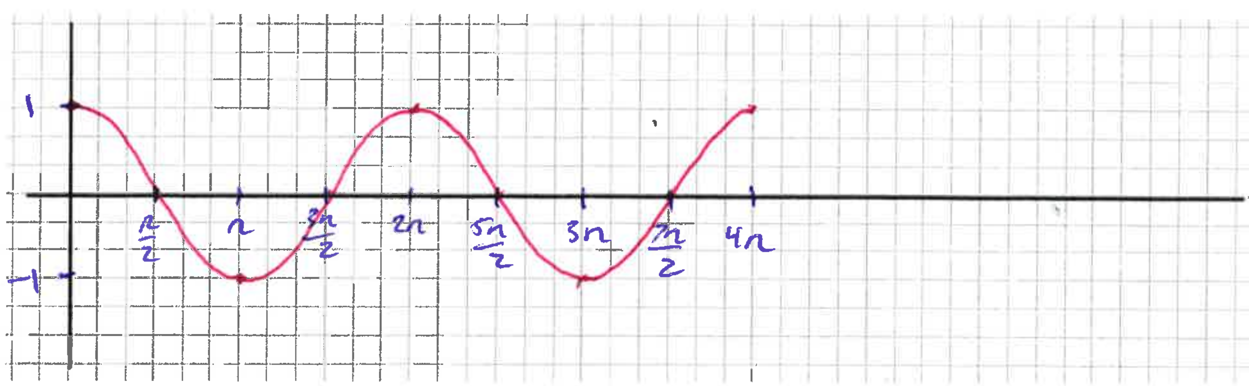
$\sin(0) = 0$	$\cos(0) = 1$	$\tan(0) = 0$
$\sin(\frac{\pi}{2}) = 1$	$\cos(\frac{\pi}{2}) = 0$	$\tan(\frac{\pi}{2})$ = undefined
$\sin(\pi) = 0$	$\cos(\pi) = -1$	$\tan(\pi) = 0$
$\sin(\frac{3\pi}{2}) = -1$	$\cos(\frac{3\pi}{2}) = 0$	$\tan(\frac{3\pi}{2})$ = undefined
$\sin(2\pi) = 0$	$\cos(2\pi) = 1$	$\tan(2\pi) = 0$

The Primary Trigonometric Functions

$f(\theta) = \sin(\theta), \quad \theta \in [0, 4\pi]$



$g(\theta) = \cos(\theta)$

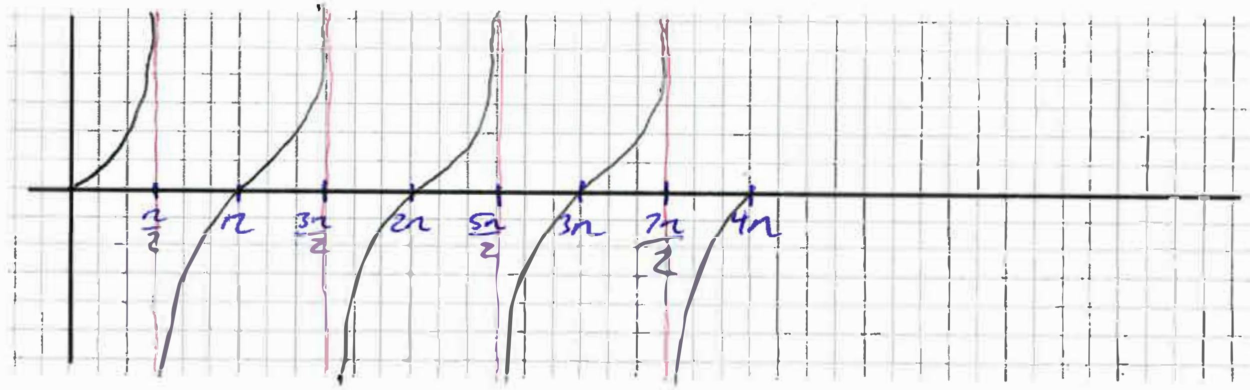


Notice: $\cos\theta$ is just $\sin\theta$ shifted to the left by $\frac{\pi}{2}$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have

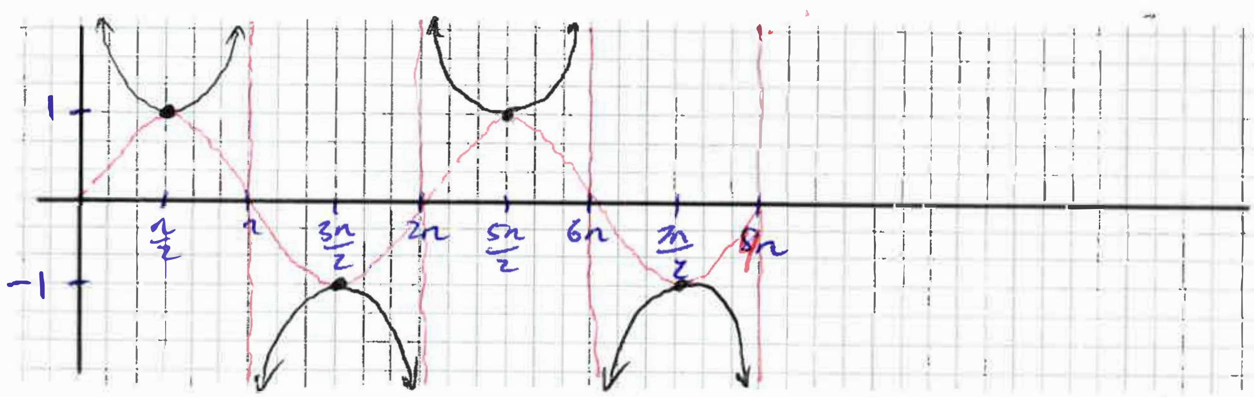
VA's whenever $\cos\theta = 0$

$h(\theta) = \tan(\theta)$

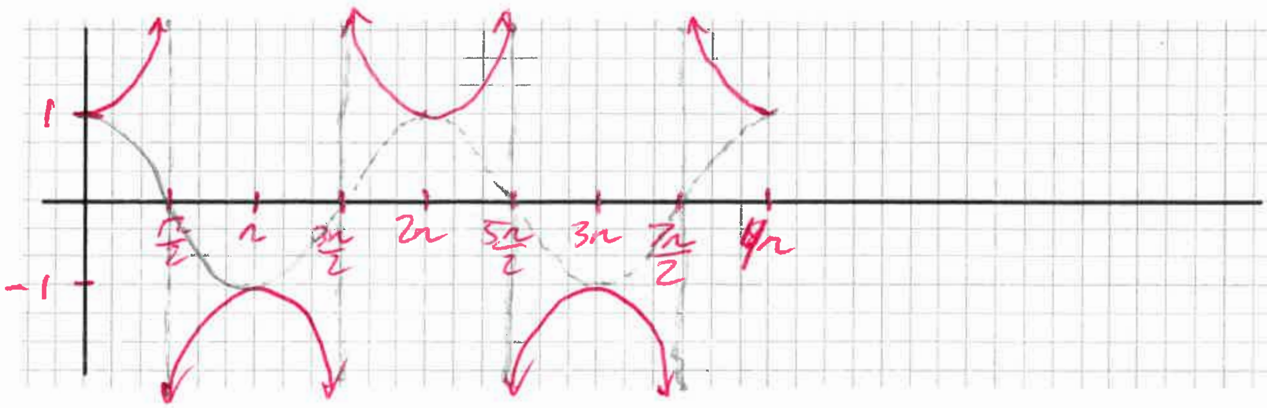


The Reciprocal Trig Functions

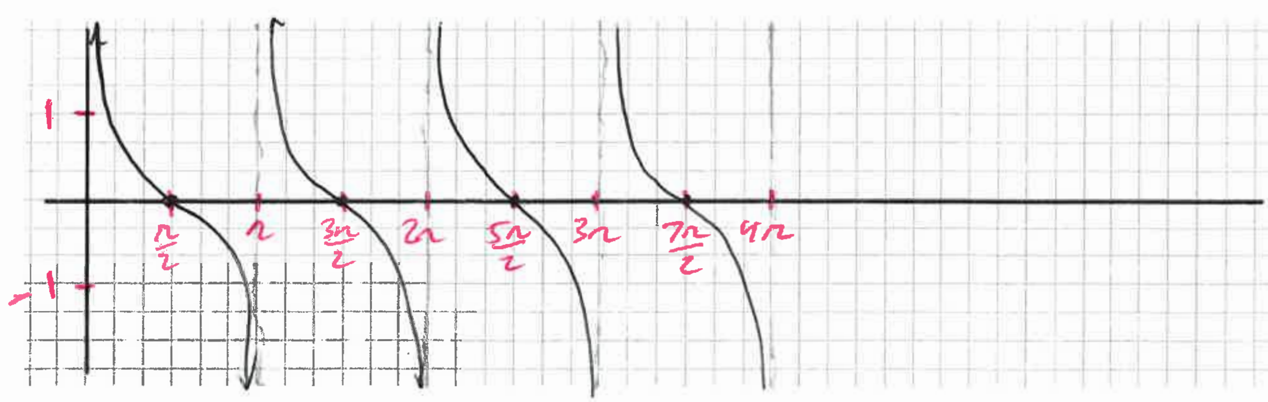
$f(\theta) = \csc(\theta) = \frac{1}{\sin\theta}$; $\therefore$ V.A's whenever $\sin\theta = 0$



$g(\theta) = \sec(\theta) = \frac{1}{\cos\theta}$ $\therefore$ VA's whenever $\cos\theta = 0$!



$h(\theta) = \cot(\theta) = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ $\therefore$ VA's whenever $\sin\theta = 0$



Success Criteria:

- I can recognize the graphs of sin, cos, tan, csc, sec, and cot

5.6 Transformations of Trigonometric Functions

Learning Goal: We are learning to use transformations to sketch the graphs of trigonometric functions in radians.

By this point in your illustrious High School careers, you have a solid understanding of Transformations of Functions in general. In terms of the trig functions Sine and Cosine in particular, the concepts are as you expect, but the transformations have specific meanings relating to nature of the sinusoidal "wave".

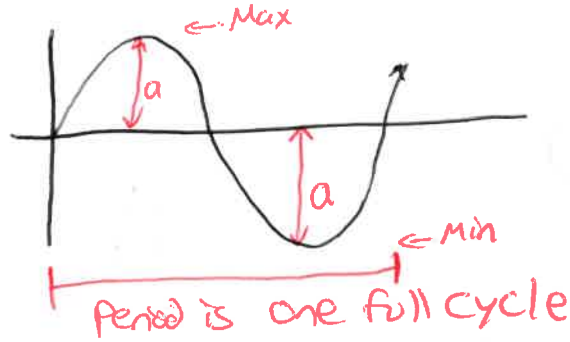
General Form of the Sine and Cosine Functions

$f(\theta)=a\sin(k(\theta-d))+c$ $\qquad$ $g(\theta)=a\cos(k(\theta-d))+c$

$|a|$ = Amplitude

peak trough

$a=\dfrac{\max-\min}{2}$



k = Period Factor

Period = $\dfrac{2\pi}{|k|}$

$\therefore k=\dfrac{2\pi}{\text{period}}$

d = Phase Shift
"Starting point when Graphing"

Note: To determine d you MUST isolate the θ or x.

c = Central Axis
→ the middle

$c=\dfrac{\max+\min}{2}$

Example 5.6.1

Determine the amplitude, period, phase shift and the equation of the central axis for:

a) $f(\theta)=2\sin\left(\theta+\frac{\pi}{3}\right)+1$

$a = \text{amplitude} = 2$

$k=1$, $\therefore$ period $=\frac{2\pi}{1}=2\pi$ (one full cycle)

Phase Shift $=-\frac{\pi}{3}$ (left $\frac{\pi}{3}$)

Central axis: $y=1$

b) $g(\theta)=3\cos\left(2\theta-\frac{\pi}{2}\right)+0$ $\quad 2\left(\theta-\frac{\pi}{4}\right)$

amplitude $=3$

$k=2$ $\therefore$ Period $=\frac{2\pi}{2}=\pi$ (compression)

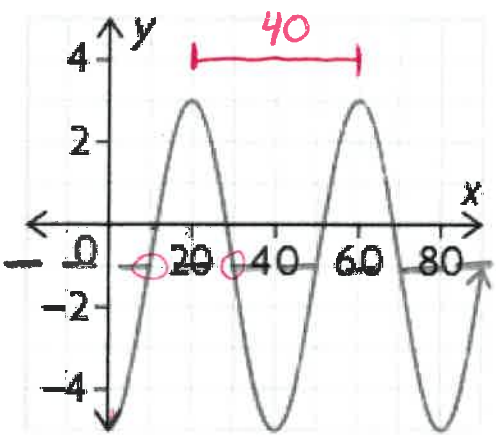
Phase shift: $d=\frac{\pi}{4}$ (right $\frac{\pi}{4}$)

Central Axis: $y=0$

Example 5.6.2

From your text: Pg. 346 #14c

Determine a ***sinusoidal*** function for the given sketch of a graph (sin OR cos!)



Amplitude: $a=4$ (can count) | OR $a=\frac{3-(-5)}{2}=\frac{8}{2}=4$

Period: 40 $\quad \therefore k=\frac{2\pi}{40}=\frac{\pi}{20}$

Phase Shift: As Cosine → peak or trough

Peak $d=20$ Trough $d=0$

As Sine → middle

$d=10\nearrow$ or $d=30\searrow$

Equation of Central Axis: $y=-1$

Equation as a Cosine Wave

Using Peak: $f(\theta)=4\cos\left(\frac{\pi}{20}(\theta-20)\right)-1$

Using trough:

$f(\theta)=-4\cos\left(\frac{\pi}{20}\theta\right)-1$

Need an upside-Down cosine

Equation as a Sine Wave

Using ~~Peak~~ Center increasing

$f(\theta)=4\sin\left(\frac{\pi}{20}(\theta-10)\right)-1$

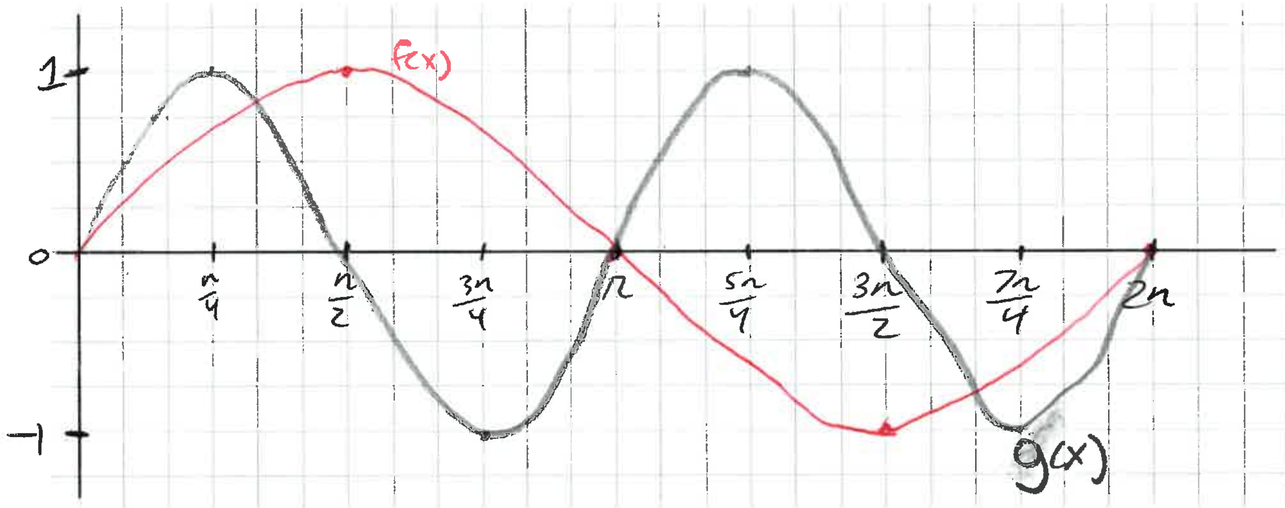
Using center decreasing

$f(\theta)=-4\sin\left(\frac{\pi}{20}(\theta-30)\right)-1$

parent equation $\qquad$ $k=2 \therefore \text{period} = \frac{2\pi}{2} = \pi$ (compression)

Example 5.6.3

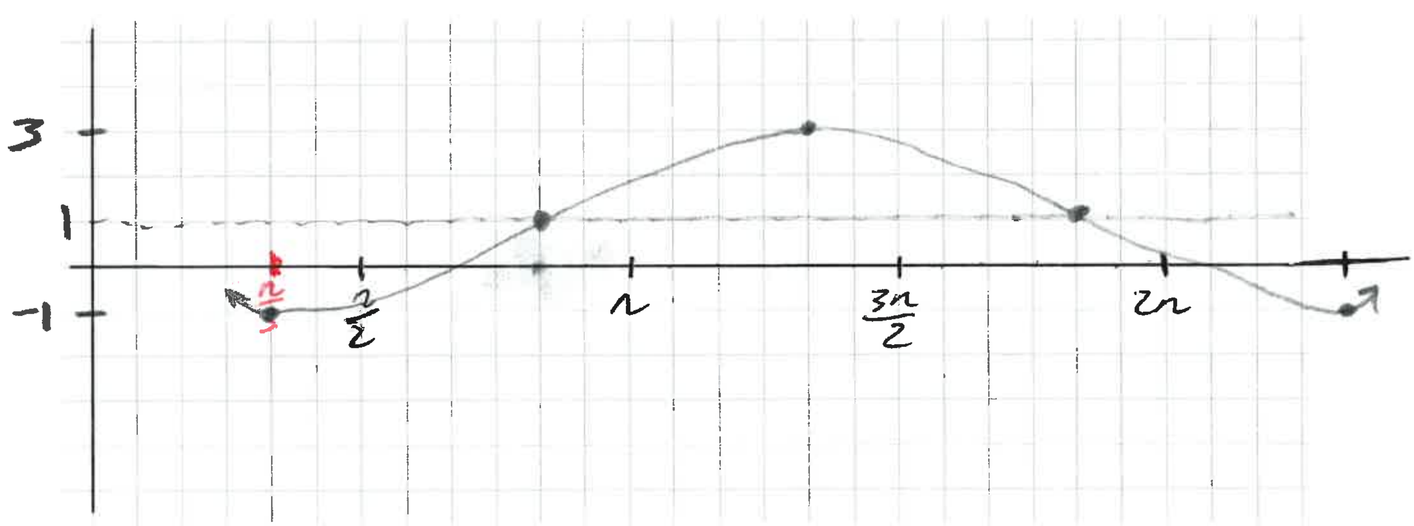
Sketch $f(x)=\sin(x)$ and $g(x)=\sin(2x)$ for $0 \le x \le 2\pi$ on the same set of axes.



Start @ trough $\qquad$ Amplitude $\qquad$ Phase shift Right $\frac{\pi}{3}$ $\qquad$ ← center line

Example 5.6.4

Sketch $f(\theta)=-2\cos\left(\theta-\frac{\pi}{3}\right)+1$ on $0 \le \theta \le 2\pi$



① center line $y=1$

② Amplitude is 2 $\quad$ Start @ $\left(\frac{\pi}{3}, -1\right)$ $\quad$ Plot on scale.

③ Phase shift zeros, peaks, troughs.

Amplitude Begins in middle Center line $y=-1$

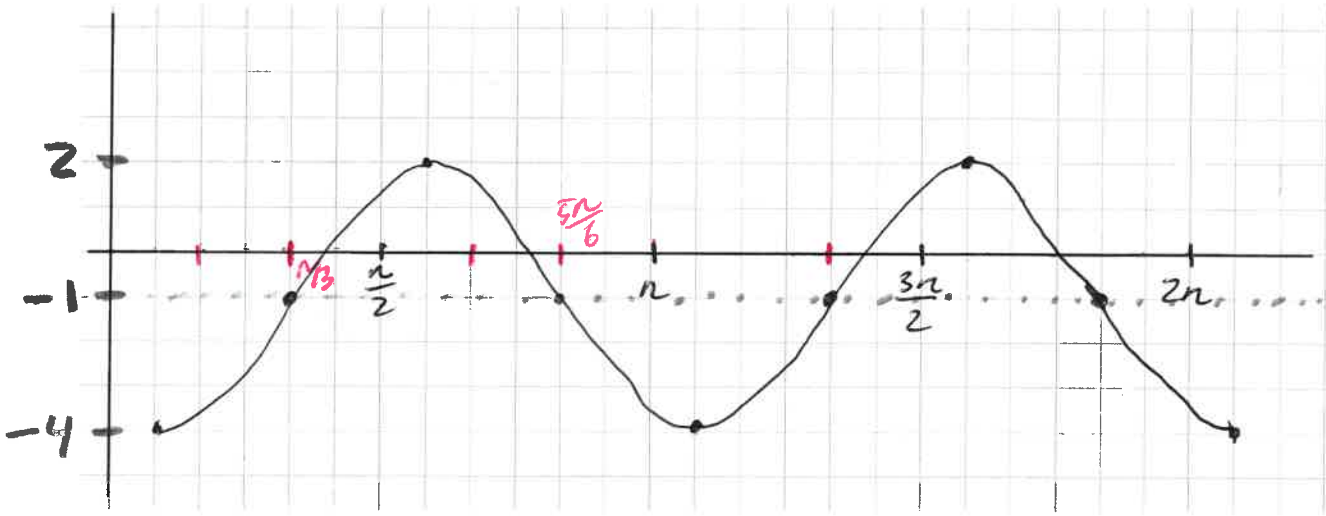
$k=2 \therefore \text{period} = \frac{2\pi}{2} = \pi$

Right $\frac{\pi}{3}$

Example 5.6.5

Sketch $f(\theta)=3\sin\left(2\theta-\frac{2\pi}{3}\right)-1$

$\left(2\left(\theta-\frac{\pi}{3}\right)\right)$



zeros, peaks, troughs

① Start Point is $\left(\frac{\pi}{3}, -1\right)$

② one complete cycle in π, not 2π

- End @ $\frac{4\pi}{3}$. $\boxed{\text{Zeros every } \frac{\pi}{2}}$
- Third zero @ $\frac{5\pi}{6}$

For Sin
- plot zeros w/ compression
- peaks + troughs occur halfway between.

Success Criteria:

- I can sketch the graph of a trigonometric function that has undergone transformations
- I can recognize the properties (amplitude, central axis, maximum, minimum, period, and phase shift) of a trigonometric function from its graph or equation

5.7 Applications of Trigonometric Functions

Learning Goal: We are learning to solve real-world problems that can be modeled with a trigonometric function.

For any phenomenon in the real world which has a periodically repeating behaviour, Trigonometric Functions can be used to describe and analyze that behaviour. There are a myriad of such phenomena. From the rise and fall of tides to computer gaming habits, Trigonometric Functions have a say.

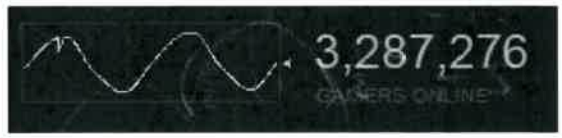


Figure 5.7.1 A periodic rise and fall in online gamers

Example 5.7.1

From your text: Pg. 345 #9

9. Each person's blood pressure is different, but there is a range o pressure values that is considered healthy. The function $P(t) = -20\cos\frac{5\pi}{3}t + 100$ models the blood pressure, p, in millimetres of mercury, at time t, in seconds, of a person at res

a) What is the period of the function? What does the period represent for an individual?

b) How many times does this person's heart beat each minute

c) Sketch the graph of $y = P(t)$ for $0 \le t \le 6$.

d) What is the range of the function? Explain the meaning of range in terms of a person's blood pressure.

a) $k = \frac{5\pi}{3}$

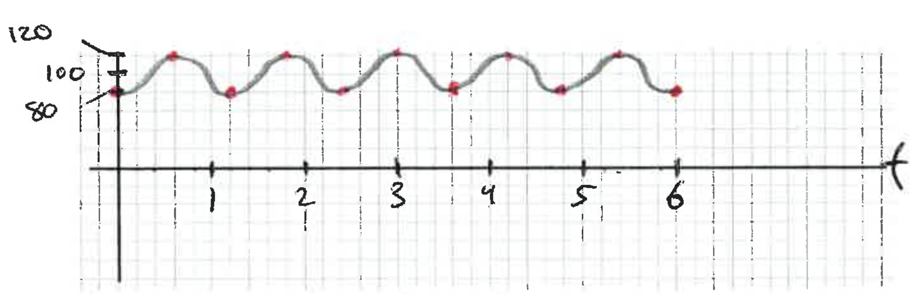
$\therefore$ period $= \frac{2\pi}{\frac{5\pi}{3}}$

$= \frac{2\cancel{\pi}}{1} \times \frac{3}{5\cancel{\pi}}$

$\boxed{\text{Period} = \frac{6}{5} \text{ seconds for every beat}}$

(1.2 sec / beat)

b) $\frac{5 \text{ beat}}{\cancel{6}\ 1 \cancel{\text{sec}}} \times \frac{\cancel{60}\ 10 \cancel{\text{sec}}}{1\text{min}} = 50 \text{ beats/min}$



1.2 sec/beat means cycle repeats Here!

* Notice 5 complete cycles in 6 seconds

d) $R = P(t) \in [80, 120]$

Example 5.7.2

From your text Pg. 361 #7

7. A person who was listening to a siren reported that the frequency of the sound fluctuated with time, measured in seconds. The minimum frequency that the person heard was 500 Hz, and the maximum frequency was 1000 Hz. The maximum frequency occurred at $t = 0$ and $t = 15$. The person also reported that, in 15, she heard the maximum frequency 6 times (including the times at $t = 0$ and $t = 15$). What is the equation of the cosine function that describes the frequency of this siren?

min = 500

Max = 1000

Period = $\frac{15 \text{ seconds}}{5 \text{ times}}$ $\boxed{= 3}$

3 seconds for one max

$a = \frac{max - min}{2}$

$= \frac{1000 - 500}{2}$

$a = 250$

Half ~~Average~~ of the two

$c = \frac{max + min}{2}$

$C = \frac{1000 + 500}{2}$

$C = 750$

Average of the two

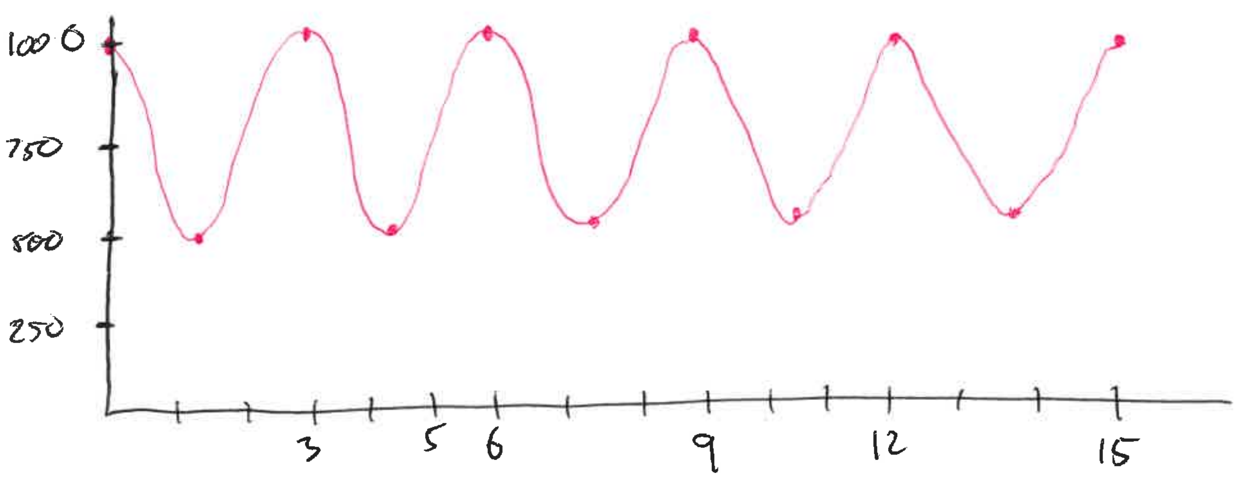
$d = 0$

At $t=0$, there is a max

∴ + cosine

$k = \frac{2\pi}{3 \text{ (period)}}$

$$\therefore f(t) = 250 \cos\left(\frac{2\pi}{3}\theta\right) + 750$$



← 6 max frequencies

Success Criteria:

- I can model a real-world situation using a trigonometric function
- I can use the trigonometric model to solve problems