Advanced Functions

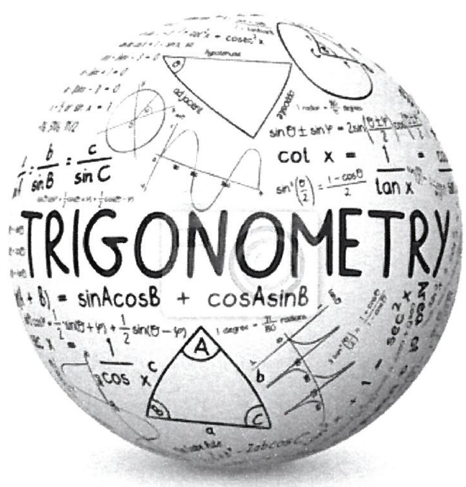
Teacher Notes

Course Notes

Unit 6 – Trigonometric Identities and Equations

We will learn

- *about Equivalent Trigonometric Relationships*
- *how to use compound angle formulas to determine exact values for trig ratios which DON'T involve the two special triangles*
- *techniques for proving trigonometric identities*
- *how to solve linear and quadratic trigonometric equations using a variety of strategies*



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Chapter 6 – Trigonometric Identities and Equations

Contents with suggested problems from the Nelson Textbook ***(Chapter** 7)*

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

6.1 Basic Trigonometric Equivalencies
Pg. 392 – 393 #3cdef, 5cdef

6.2 Compound Angle Formulae
Pg. 400 – 401 #3 – 6, 8 – 10, 13

6.3 Double Angle Formulae
Pg. 407 – 408 Finish #2, 4, 12 – Do # 6, 7

6.4 Trigonometric Identities
Pg. 417 – 418 #8 – 11

6.5 Linear Trigonometric Equations
Pg. 427 – 428 #6, 7def, 8, 9abc

6.6 Quadratic Trigonometric Equations
Pg. 436 - 437 #4ade, 5acef, 6ac, 7 - 9

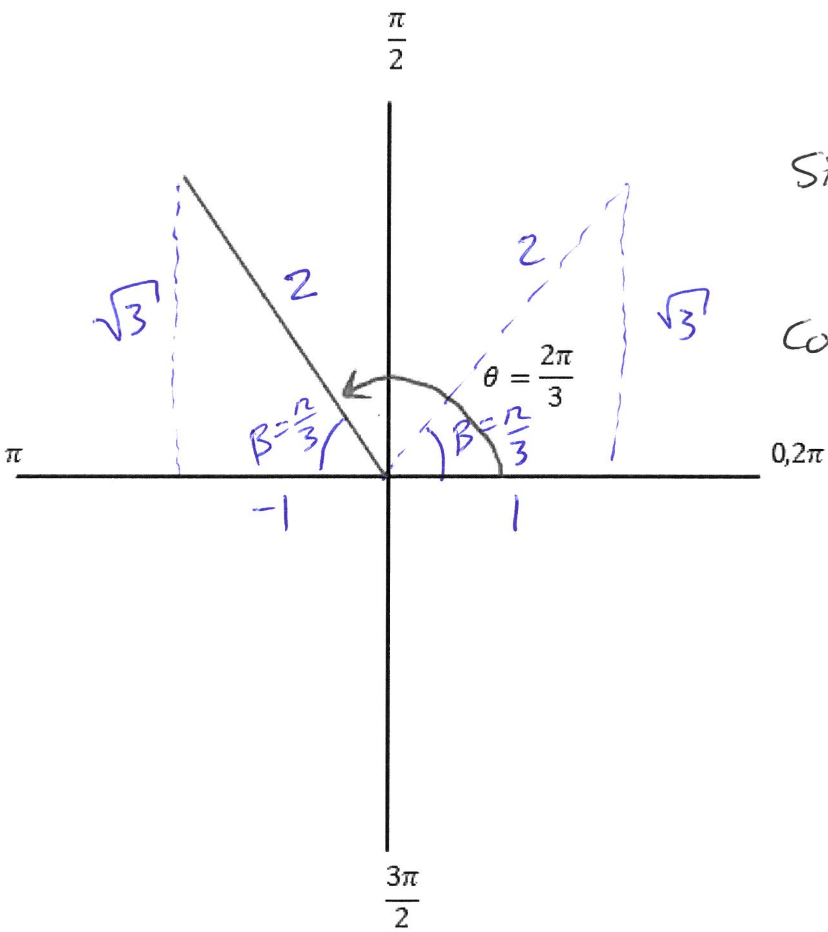
6.1 Basic Trigonometric Equivalencies

Learning Goal: We are learning to identify equivalent trigonometric relationships.

We have already seen a very basic trigonometric equivalency when we considered angles of rotation. For example, consider the angle of rotation for $\theta = \dfrac{2\pi}{3}$:

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

$$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$$

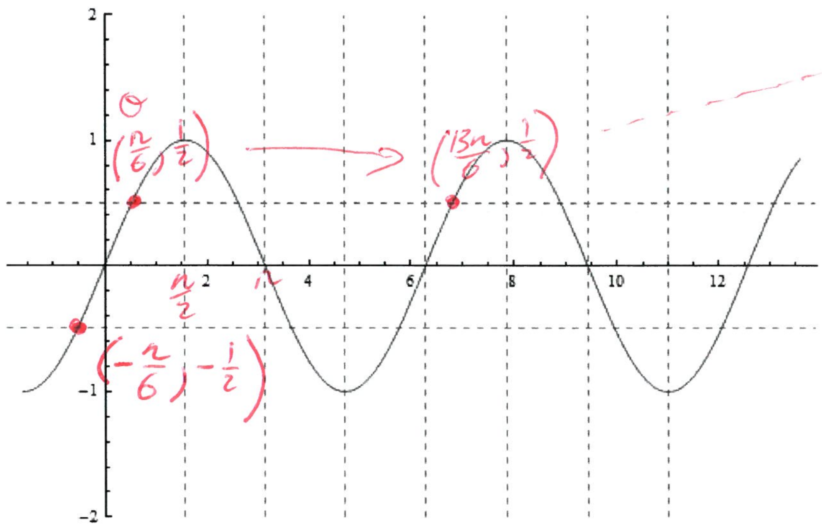
Periodic Equivalencies

Period is 2π

$\therefore \sin\theta$ is 2π periodic

Example 6.1.1

Consider the sketch of the function $f(\theta) = \sin(\theta)$



$\sin\theta = \sin(\theta + 2\pi)$

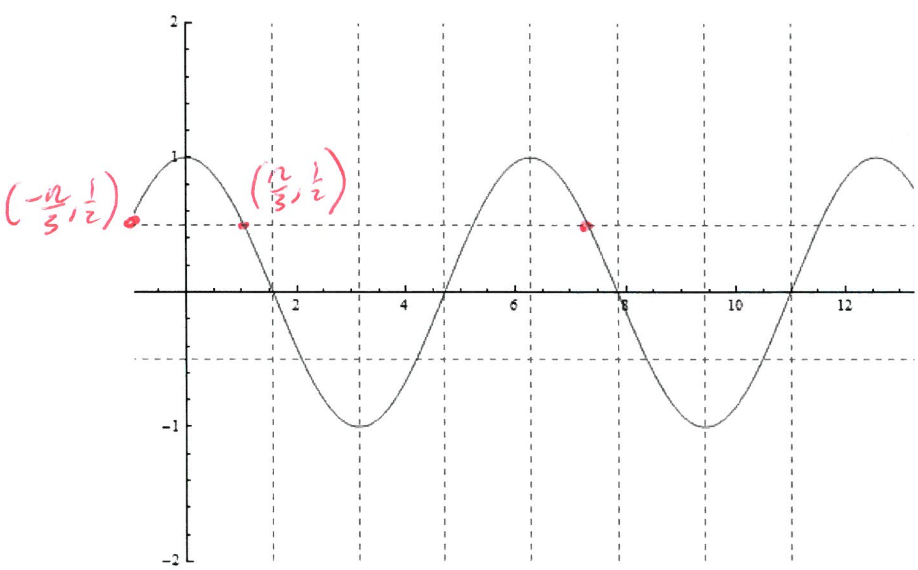
$\sin\theta = -\sin(-\theta)$

$\therefore$ Sine is an odd function (rotational symmetry)

Example 6.1.2

Consider $g(x) = \cos(x)$

Cosine is 2π periodic



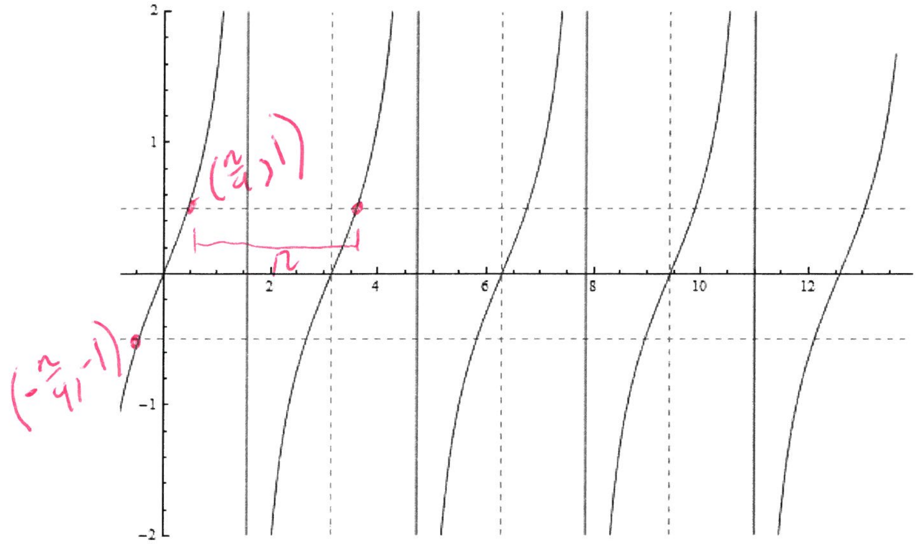
$\cos\theta = \cos(\theta + 2\pi)$

$\cos\theta = \cos(-\theta)$

$\therefore$ cosine is an even function

Example 6.1.3

Consider $h(\theta) = \tan(\theta)$



tan is π periodic

$\tan\theta = \tan(\theta + \pi)$

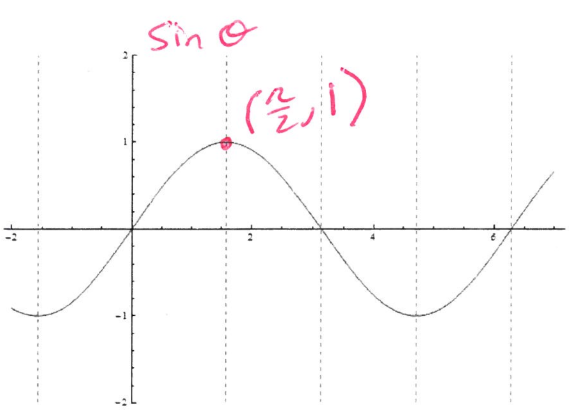
$\tan\theta = -\tan(-\theta)$

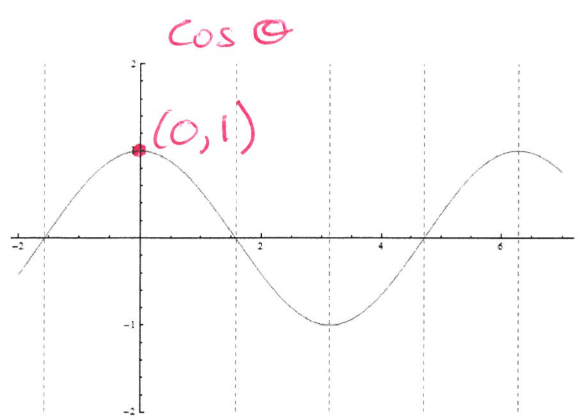
$\therefore$ tan is an odd function

Shift Equivalencies

Example 6.1.4

Consider the sketches of the graphs for $f(\theta) = \sin(\theta)$ and $g(\theta) = \cos(\theta)$





$\sin\theta = \cos(\theta - \frac{\pi}{2})$ || $\cos\theta = \sin(\theta + \frac{\pi}{2})$

shift right ↗ ← shift left

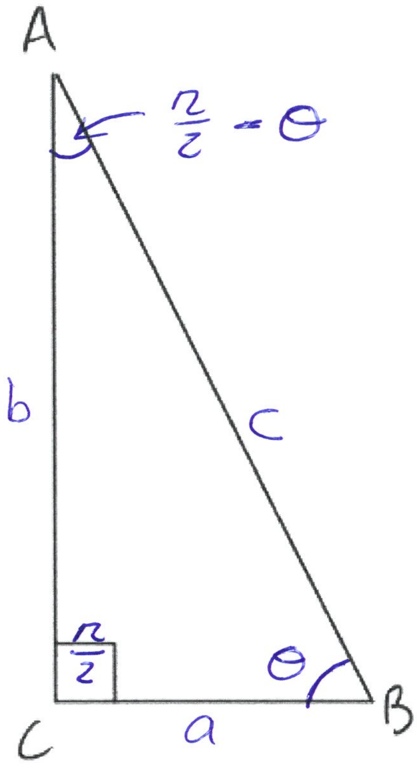
Test: $\theta = 406$ radians

$\sin(406) = -0.6702515...$

$\cos(406 - \frac{\pi}{2}) = \cos(404.429) = -0.6702515...$

Cofunction Identities

Consider the right angle triangle



$$\sin\theta = \frac{b}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$$

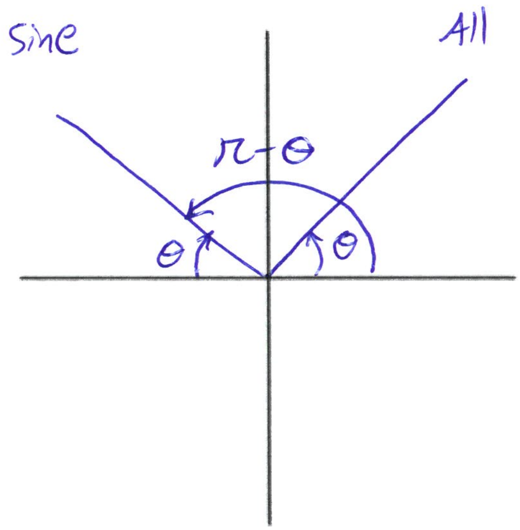
$$\cos\theta = \frac{a}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \frac{b}{a} = \cot\left(\frac{\pi}{2} - \theta\right)$$

Related Acute Angle Equivalencies

Using CAST, relating angles of rotation to π and 2π

Compare Q1 and Q2

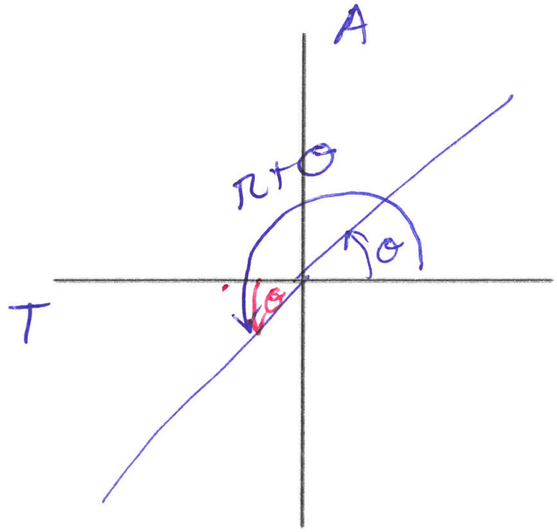


$$\sin\theta = \sin(\pi - \theta)$$

$$\cos\theta = -\cos(\pi - \theta)$$

$$\tan\theta = -\tan(\pi - \theta)$$

Compare Q1 and Q3

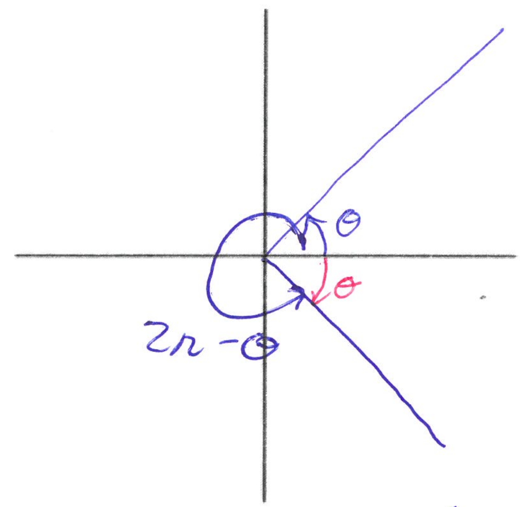


$\sin\theta = -\sin(\pi+\theta)$

$\cos\theta = -\cos(\pi+\theta)$

$\tan\theta = \tan(\pi+\theta)$

Compare Q1 and Q4



$\sin\theta = -\sin(2\pi-\theta)$

$\cos\theta = \cos(2\pi-\theta)$

$\tan\theta = -\tan(2\pi-\theta)$

Example 6.1.5

From your text: Pg. 392 #3

Use a cofunction identity to find an equivalency:

a) $\sin\left(\frac{\pi}{6}\right)$ $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

(θ) $= \cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{6}-\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

d) $\cos\left(\frac{5\pi}{16}\right) = \sin\left(\frac{\pi}{2}-\frac{5\pi}{16}\right)$

(θ)

$= \sin\left(\frac{8\pi}{16}-\frac{5\pi}{16}\right)$

$= \sin\left(\frac{3\pi}{16}\right)$

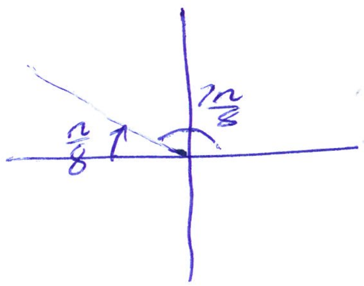
These depend on your quadrant!

Example 6.1.6

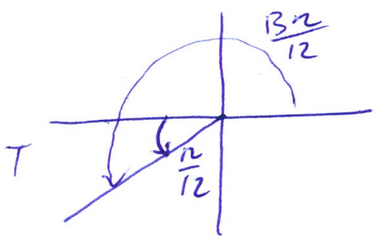
From your text: Pg. 393 #5

Using the related acute angle, find an equivalent expression:

a) $\sin\left(\frac{7\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$



b) $\cos\left(\frac{13\pi}{12}\right) = -\cos\left(\frac{\pi}{12}\right)$



Success Criteria:

- I can recognize that there are many equivalent trigonometric expressions due to their periodic nature
- I can recognize several types of equivalencies
    - Shifting sin/cos by 2π OR $\pi/2$
    - Cos has even symmetry, Sin and Tan have odd symmetry
    - Cofunction equivalencies
    - Equivalencies based on the quadrant a principal angle is in

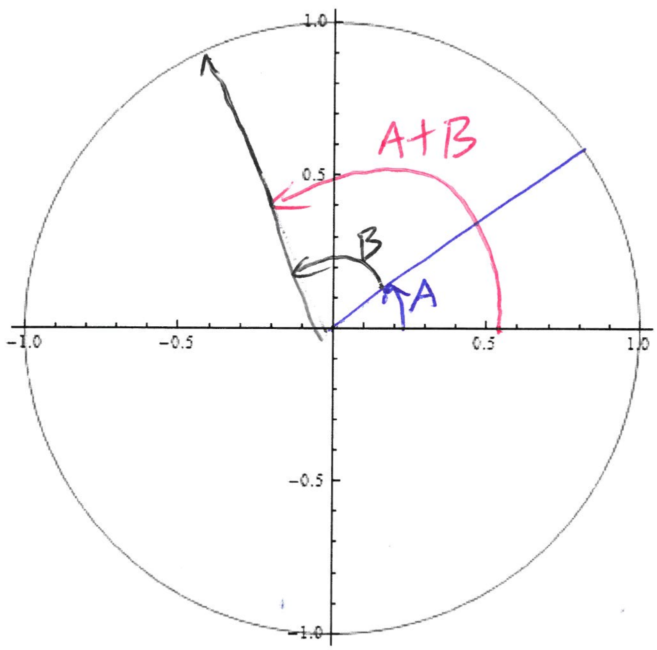
6.2 Compound Angle Formulae

Learning Goal: We are learning to use the compound angle formulas

Here we learn to find **exact trig ratios for non-special angles!**

NO CALCULATORS!

Consider the picture:



Can we find $\sin(A+B)$ if we know $\sin(A)$ and $\sin(B)$?

or $\cos(A+B)$?

or $\tan(A+B)$?

Yes!

Your text has a nice proof of one of the six compound angle formulas (there are six of them!...see Pg. 394)

Namely your text proves that $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

Using some trig equivalencies (from 6.1) we will find the other 5 compound angle formulae.

Example 6.2.1

Find a formula for $\cos(A+B)$

Consider, $\cos(A-(-B)) = \cos A \underbrace{\cos(-B)} + \sin A \underbrace{\sin(-B)}$

Cosine is even.
$\therefore \cos(-B) = \cos B$

Sine is odd
$\therefore \sin\theta = -\sin(-\theta)$
$-\sin\theta = \sin(-\theta)$

we do this b.c. we can't evaluate negative angles.

$= \cos A \cos B + \sin A(-\sin B)$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\boxed{\therefore \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B}$$

Example 6.2.2

Determine a compound angle formula for $\sin(A+B)$ using a **cofunction identity** and a **cosine compound angle formula.**

$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

$$\sin\underbrace{(A+B)}_{\theta} = \cos\left(\frac{\pi}{2}-(A+B)\right)$$

$$= \cos\left(\underbrace{\frac{\pi}{2}-A}_{\text{Consider this A}}-B\right)$$

Use $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \underbrace{\cos\left(\frac{\pi}{2}-A\right)}_{\text{This is } \sin A}\cos B + \underbrace{\sin\left(\frac{\pi}{2}-A\right)}_{\text{This is } \cos A}\sin B$$

Cofunction Identity

So, $\sin(A+B) = \sin A\cos B + \cos A\sin B$

In the same way, $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$\boxed{\therefore \sin(A\pm B) = \sin A\cos B \pm \cos A\sin B}$$

Example 6.2.3

Using the fact that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ determine a compound angle formula for

$\tan(A+B))$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Use our latest identities!

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Through some math magic

$$\boxed{\therefore \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}}$$

Example 6.2.4

From your text: Pg. 400 #3acd

Express each given angle as a compound angle using a pair of special triangle angles

30-60 or 45

a) $75^{\circ} = 30^{\circ} + 45^{\circ}$

$\frac{\pi}{6}, \frac{2\pi}{6}\left(\frac{\pi}{3}\right)$

c) $-\frac{\pi}{6} = \frac{\pi}{6} - \frac{2\pi}{6} = \frac{\pi}{6} - \frac{\pi}{3}$

$\frac{\pi}{3} = \frac{4\pi}{12}$

$\frac{\pi}{4} = \frac{3\pi}{12}$

$\frac{\pi}{6} = \frac{2\pi}{12}$

d) $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

OR

$= \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$

Example 6.2.5

From your text: Pg. 400 #4ac, 8bd

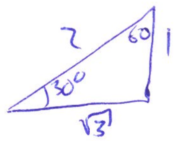
Determine the **EXACT** value of the trig ratio

$\frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$

a) $\sin\left(75^{\circ}\right)$

$= \sin(\underset{A}{30} + \underset{B}{45})$

$= \sin 30 \cos 45 + \sin 45 \cos 30$



$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$ Valid test answer

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

b) $\tan\left(\frac{5\pi}{12}\right)$

$= \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)}$

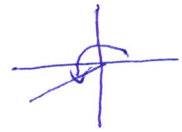
$= \frac{\frac{1}{\sqrt{3}} + 1\left(\frac{\sqrt{3}}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3}}{\sqrt{3}}\right)1 - \left(\frac{1}{\sqrt{3}}\right)(1)}$

$= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \frac{1+\sqrt{3}}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}-1}$

$= \frac{1+\sqrt{3}}{\sqrt{3}-1}$

Stop Here



30 − 45 or 45 − 60

#8b) $\tan(-15^\circ)$

$= \tan(30-45)$

$$= \frac{\tan 30 - \tan 45}{1 + \tan 30 \tan 45}$$

$$= \frac{\frac{1}{\sqrt{3}} - 1\left(\frac{\sqrt{3}}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3}}{\sqrt{3}}\right)1 + \frac{1}{\sqrt{3}}(1)}$$

$$= \frac{\frac{1-\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$= \frac{1-\sqrt{3}}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}+1}$$

$$= \frac{1-\sqrt{3}}{\sqrt{3}+1}\left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$ Stop here. If you rationalize...

FOIL

$$= \frac{\sqrt{3}-1-3+\sqrt{3}}{3-1}$$

$$= \frac{2\sqrt{3}-4}{2} = \boxed{\sqrt{3}-2}$$

d) $\sin\left(\frac{13\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right)$

$$= -\left[\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right]$$

$$= -\left[\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right]$$

$$= -\left[\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\right]$$

$$= -\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

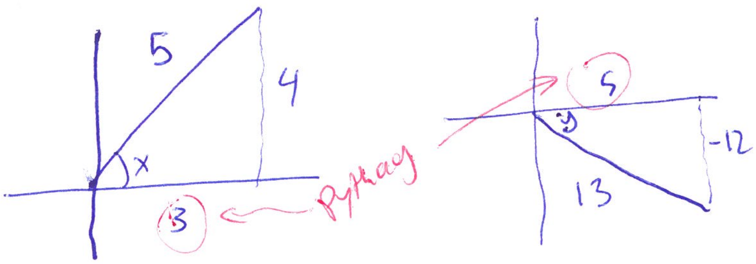
$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

Example 6.2.6

From your text: Pg. 401 #9a

Q1 Q4

If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, where $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le y \le 2\pi$ evaluate $\cos(x+y)$.



$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Don't cancel →

$$= \left(\frac{3}{\cancel{5}}\right)\left(\frac{\cancel{5}}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$$

$$= \frac{15}{65} + \frac{48}{65}$$

$$= \frac{63}{65}$$

Success Criteria:

- I can identify SIX compound angle formulas
- I can use compound angle formulas to obtain exact values for trigonometric ratios
- I can use compound angle formulas to show that some trigonometric expressions are equivalent

6.3 Double Angle Formulae

This is a nice extension of the compound angle formulae from section 6.2.

Learning Goal: We are learning to use double angle formulas

Recall the compound Angle Formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

"B"

1) $\sin(2A) = \sin(A + A)$

$= \sin A \cos A + \sin A \cos A$

$= 2\sin A \cos A$

Recall (write after ①)

$\sin^2\theta + \cos^2\theta = 1$

$\rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\rightarrow \cos^2\theta = 1 - \sin^2\theta$

2) $\cos(2A) = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

① $\cos(2A) = \cos^2 A - \sin^2 A$

② $\cos(2A) = 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

③ $\cos(2A) = \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

3) $\tan(2A) = \tan(A+A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

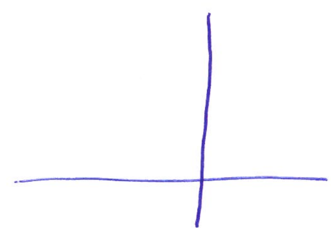
$$= \frac{2\tan A}{1-\tan^2 A}$$

Example 6.3.1

From your text: Pg. 407 #2ae

Express as a single trig ratio and evaluate:

a) $2\sin\left(\underset{A}{45^\circ}\right)\cos\left(\underset{A}{45^\circ}\right)$

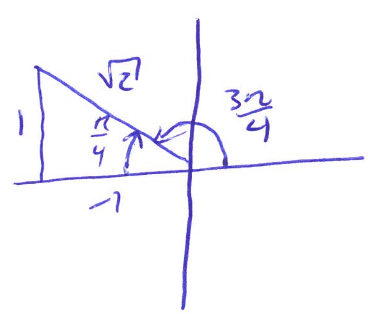


$= \sin(2(45))$

$= \sin(90) = 1$

e) $1-2\sin^2\left(\underset{A}{\frac{3\pi}{8}}\right)$

$= \cos\left(\cancel{2}\left(\frac{3\pi}{\cancel{8}_4}\right)\right)$



$= \cos\left(\frac{3\pi}{4}\right)$

$= \frac{-1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$ stop here

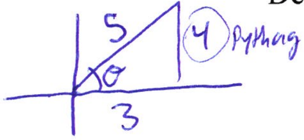
$= \frac{-\sqrt{2}}{2}$

Example 6.3.2

From your text: Pg. 407 #4

Q1

Determine the values of $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$ given $\cos(\theta)=\frac{3}{5}$, $0 \le \theta \le \frac{\pi}{2}$



$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{2}{1}\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$

$= \frac{24}{25}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} - \frac{16}{25}$

$= \frac{-7}{25}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\left(\frac{4}{3}\right)}{\frac{1}{1} - \left(\frac{4}{3}\right)^2}$

$= \frac{\frac{8}{3}}{\frac{9}{9} - \frac{16}{9}} = \frac{\frac{8}{3}}{\frac{-7}{9}}$

$= \frac{8}{3} \cdot \frac{9}{-7}$

$= \frac{-24}{7}$

Example 6.3.3

From your text: Pg. 408 #12

Use the appropriate angle and double angle formulae to determine a formula for:

a) $\sin(3\theta)$ ← using only sine in the end

$\sin(A+B)$

$= \sin A\cos B + \sin B\cos A$

$\sin 2\theta$, see above

$\cos 2\theta$, see ~~above~~ pg 14

$= 1 - 2\sin^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$= \sin(\overset{A}{2\theta} + \overset{B}{\theta})$

$= \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$

$= (2\sin\theta\cos\theta)\cos\theta + \sin\theta(1 - 2\sin^2\theta)$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Example 6.3.4

From your text: Pg. 407 #8

Let's isolate to start.

Determine the value of a in the following:

$$2\tan(x)-\tan(2x)+2a=1-\tan(2x)\cdot\tan^2(x)$$

$$2a = 1-\tan(2x)\tan^2 x - 2\tan(x) + \tan(2x)$$

$$= \tan(2x) - \tan(2x)\tan^2 x + 1 - 2\tan x$$

$$= \tan(2x)\left(1-\tan^2 x\right) + 1 - 2\tan x$$

$$= \left(\frac{2\tan x}{\cancel{1-\tan^2 x}}\right)\left(\frac{\cancel{1-\tan^2 x}}{1}\right) + 1 - 2\tan x$$

$$= \cancel{2\tan x} + 1 - \cancel{2\tan x}$$

$$2a = 1$$

$$a = \frac{1}{2}$$

Success Criteria:

- I can identify FIVE double angle formulas
- I can use the double angle formulas to simplify expressions and to calculate exact values
- I can use the double angle formulas to develop other equivalent formulas

6.4 Trigonometric Identities

Learning Goal: We are learning to prove trigonometric identities.

Proving Trigonometric Identities is so much fun, it's plainly ridiculous. I should be paid extra for letting you play with these proofs! We will be using ALGEBRA (remember the rules?). Inside our algebra we will be using the following tools:

Reciprocal Identities

e.g. $\csc(\theta) = \dfrac{1}{\sin(\theta)}$ $\sin\theta = \dfrac{1}{\csc\theta}$

Quotient Identities

e.g. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, or $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

The Pythagorean Trig Identities

$\sin^2(\theta) + \cos^2(\theta) = 1$ $1 + \tan^2(\theta) = \sec^2(\theta)$ $1 + \cot^2(\theta) = \csc^2(\theta)$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$ $\Rightarrow \tan^2\theta = \sec^2\theta - 1$ $\Rightarrow \cot^2\theta = \csc^2\theta - 1$

or $\cos^2\theta = 1 - \sin^2\theta$ or $1 = \sec^2\theta - \tan^2\theta$ or $1 = \csc^2\theta - \cot^2\theta$

The Compound Angle Formulae

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \begin{cases} \cos(2\theta) = 1 - 2\sin^2\theta \\ \cos(2\theta) = 2\cos^2\theta - 1 \end{cases}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

(And let's not forget our friends, "The Trig Equivalencies" such as the Cofunction Identities!)

A General Rule of Thumb

Convert everything to sine and cosine

Example 6.4.1

Prove $1+\tan^2(x) = \sec^2(x)$

$$L.S. = \frac{1}{1} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \checkmark$$

$\therefore LS = RS$

Example 6.4.2

Prove $\sin(x+y)\cdot\sin(x-y)=\cos^2(y)-\cos^2(x)$

$$L.S.: = (\sin x\cos y + \sin y\cos x)(\sin x\cos y - \sin y\cos x)$$

FOIL

$$= \sin^2 x\cos^2 y - \sin^2 y\cos^2 x$$

Rewrite as cos to match RS

$\boxed{\sin^2\theta = 1-\cos^2\theta}$

$$=(1-\cos^2 x)(\cos^2 y) - (1-\cos^2 y)(\cos^2 x)$$

$$=\cos^2 y - \cos^2 x\cos^2 y - (\cos^2 x - \cos^2 x\cos^2 y)$$

$$=\cos^2 y - \cancel{\cos^2 x\cos^2 y} - \cos^2 x + \cancel{\cos^2 x\cos^2 y}$$

$$=\cos^2 y - \cos^2 x$$

$LS = RS$

Example 6.4.3

change

Prove $\sin(\theta)\cdot\tan(\theta)=\sec(\theta)-\cos(\theta)$

$$R.S. = \sec(\theta) - \cos(\theta)$$

$$= \frac{1}{\cos\theta} - \frac{\cos\theta}{1}\left(\frac{\cos\theta}{\cos\theta}\right)$$

$$= \frac{1-\cos^2\theta}{\cos\theta}$$

see above

$$= \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\sin\theta\,\sin\theta}{\cos\theta}$$

tan θ

$$= \sin\theta\tan\theta \quad ✓$$

$LS = RS$

ugly !

Example 6.4.4

Prove $\tan(x)\cdot\tan(y)=\dfrac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}$

R.S. $\dfrac{\tan x+\tan y}{\cot x+\cot y}$

$$=\frac{\tan x+\tan y}{\frac{(\tan y)\,1}{(\tan y)\tan x}+\frac{1\,(\tan x)}{\tan y\,(\tan x)}}$$

common den.

$$=\frac{\tan x+\tan y}{\frac{\tan y+\tan x}{(\tan x)(\tan y)}}$$

$$=\frac{\cancel{\tan x+\tan y}}{1}\times\frac{\tan x\ \tan y}{\cancel{\tan y+\tan x}}$$

$=(\tan x)(\tan y)$ ✓

LS = RS

Example 6.4.5

From your text: Pg. 417 #9a

Difference of squares.

Prove $\underbrace{\frac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin(\theta)\cos(\theta)}}=1-\tan(\theta)$

Common factor.

... let's start here...

$$L.S. = \frac{\cancel{(\cos\theta+\sin\theta)}(\cos\theta-\sin\theta)}{\cos\theta\,\cancel{(\cos\theta+\sin\theta)}}$$

$$= \frac{\cos\theta-\sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= 1 - \tan\theta$$

Not too bad!

Success Criteria:

- I can prove trigonometric identities using a variety of strategies
- I can recognize the proper form to proving trigonometric identities

6.5 Linear Trigonometric Equations

Learning Goal: We are learning to solve linear trigonometric equations.

By this time, asking you to solve a "linear equation" is almost an insult to your intelligence. BUT it is never an insult to ask you to solve problems with math. Instead it is a special treat to be able to spend time thinking mathematically. And so, **you're very welcome**.

→ inverse operation

You $\boxed{\text{Undo}}$ the operation acting on the "x".

e.g. Solve the linear equation

$$3x-4=9$$

$$\frac{3x}{3}=\frac{13}{3}$$

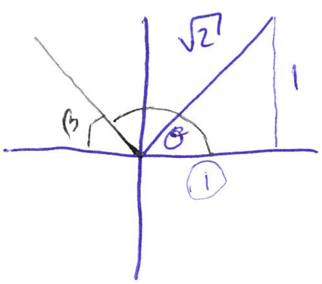
$$x=\frac{13}{3}$$

Example 6.5.1

From your text: Pg. 427 #6

For $\theta \in [0, 2\pi]$, solve the linear trigonometric equation

a) $\sin(\theta)=\frac{1}{\sqrt{2}}$ exactly, and using a calculator.



$\theta_1=\frac{\pi}{4}$

$\theta_2=\pi-\beta$

$=\pi-\frac{\pi}{4}$

$=\frac{3\pi}{4}$

Calculator

$\sin\theta=\frac{1}{\sqrt{2}}$

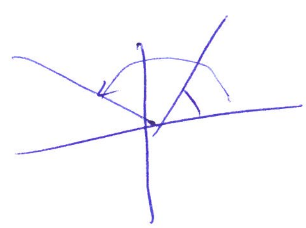
$\theta=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta_1=0.7854=\beta$

$\theta_2=\pi-\beta$

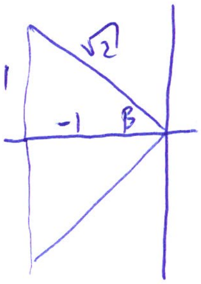
$=\pi-0.7854$

$=2.35$



Q3+Q4

e) $\cos(\theta) = -\frac{1}{\sqrt{2}}$ exactly and using a calculator.



$\beta = \frac{\pi}{4}$

$\therefore \theta_1 = \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

$\theta_2 = \pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$

Q3+Q4

$\cos\beta = \frac{1}{\sqrt{2}}$

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\beta = 0.79$

$\theta_1 = 3.14 - 0.79 = 2.35$

$\theta_2 = 3.14 + 0.79 = 3.93$

Example 6.5.2

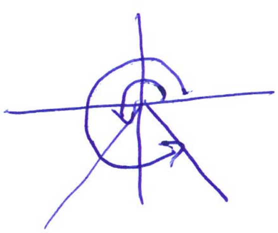
From your text: Pg. 427 #7

Using a calculator, determine solutions for $0^\circ \le \theta \le 360^\circ$

a) $2\sin(\theta) = -1$

Note: Our Domain is in Degrees!!

Degree MODE!



$\sin\theta = -\frac{1}{2}$

Ignore the negative. It only tells us the Quadrant

$\sin\beta = \frac{1}{2}$

$\beta = \sin^{-1}\left(\frac{1}{2}\right)$

$\beta = 30^\circ$

$\therefore \theta_1 = 180 + 30$

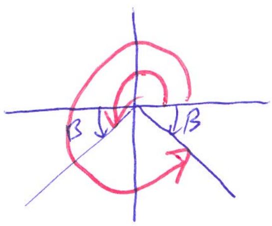
$= 210^\circ$

$\theta_2 = 360 - 30$

$= 330^\circ$

d) $-3\sin(\theta)-1=1$ (correct to one decimal place)

$+1 \quad +1$



$-3\sin\theta = 2$

$\sin\theta = \frac{2}{-3}$

$\sin\beta = \frac{2}{3}$

$\beta = \sin^{-1}\left(\frac{2}{3}\right)$

$= 41.8^\circ$

$\theta_1 = 180 + 41.8$

$= 221.8^\circ$

$\theta_2 = 360 - 41.8$

$= 318.2^\circ$

Example 6.5.3

From your text: Pg. 427 #8

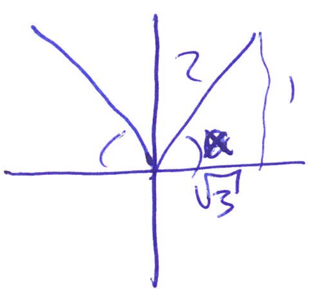
Determine solutions to the equations for $0 \le x \le 2\pi$. Radians again !!

a) $3\sin(x) = \sin(x)+1$

$-\sin x \quad -\sin x$

$2\sin(x) = 1$

$\sin(x) = \frac{1}{2}$



$\therefore \beta = \frac{\pi}{6}$

$x_1 = \frac{\pi}{6} \qquad x_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

Example 6.5.4

From your text: Pg. 427 #9f

Solve for $x \in [0, 2\pi]$

$$8 + 4\cot(x) = 10$$

$$-8 \qquad -8$$

$$4\cot x = 2$$

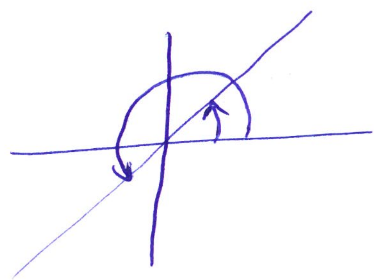
Flip $\cot x = \frac{2}{4} = \frac{1}{2}$ Flip

$$\boxed{\tan x = \frac{2}{1}}$$

Calculator: $\tan x = 2$

$\beta = \tan^{-1}(2)$

$\beta = 1.11$



$\therefore x_1 = 1.11$

$x_2 = 3.14 + 1.11$

$= 4.25$

$$\boxed{\therefore x = 1.11, 4.25}$$

Success Criteria:

- I can solve a linear trigonometric equation using: special triangles, a calculator, a sketch of the graph, and/or the CAST rule
- I can recognize that because of their periodic nature, there are infinite solutions. We normally want solutions within a specified interval.

6.6 Quadratic Trigonometric Equations

Learning Goal: We are learning to solve quadratic trigonometric equations.

Before moving on to Quadratic Trigonometric Equations, we need to consider a mind stretching problem, because it's good stretch from time to time *(in Baseball parlance, this would be the Lesson 6 Stretch)*.

Example 6.6.1

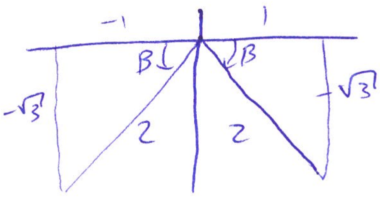
Don't be afraid of the 3! (though it does give one some concern…)

Our period is cut in a third.

Instead of 2 solutions, there are 6!

Period: $\frac{2\pi}{3}$ after transformation.

Solve $\sin(3x) = -\frac{\sqrt{3}}{2}$ exactly on $x \in [0, 2\pi]$



$\sin\beta = \frac{\sqrt{3}}{2}$

$\beta = \frac{\pi}{3}$

So, $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ $\qquad$ $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

These are the original 2 answers. Must compress + find others. $\theta = 3x$!

	$3x = \frac{4\pi}{3}$ $x = \frac{4\pi}{9}$ ①	$3x = \frac{5\pi}{3}$ $x = \frac{5\pi}{9}$ ②	Are solutions in $[0, \frac{2\pi}{3}]$
Add the period	$x = \frac{4\pi}{9} + \frac{2\pi}{3}$ $= \frac{10\pi}{9}$ ③	$x = \frac{5\pi}{9} + \frac{2\pi}{3}$ $= \frac{11\pi}{9}$ ④	Solutions in $[\frac{2\pi}{3}, \frac{4\pi}{3}]$
Add the period	$x = \frac{10\pi}{9} + \frac{2\pi}{3}$ $= \frac{16\pi}{9}$ ⑤	$x = \frac{11\pi}{9} + \frac{2\pi}{3}$ $= \frac{17\pi}{9}$ ⑥	Solutions in $[\frac{4\pi}{3}, 2\pi]$

$\therefore x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$

In Quadratic Trigonometric Functions the highest power on the trig 'factor' will be 2.

Example 6.6.2

From your text: Pg. 436 #4: Solve, to the nearest degree, for $0^{\circ} \le \theta \le 360^{\circ}$

b) $\sqrt{\cos^2(\theta)} = \sqrt{1}$

$\cos\theta = \pm 1$ (Axis angles)

$\cos\theta = 1$	$\cos\theta = -1$
$\theta = 0, 2\pi$	$\theta = -\pi$
$\theta = 0^{\circ}, 360^{\circ}$	$\theta = 180^{\circ}$

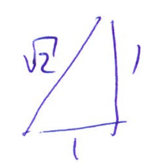
$\theta = 0, \pi, 2\pi$

$= 0^{\circ}, 180^{\circ}, 360^{\circ}$

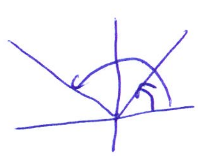
f) $2\sin^2(\theta) = 1$

$\sqrt{\sin^2\theta} = \sqrt{\frac{1}{2}}$

$\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$



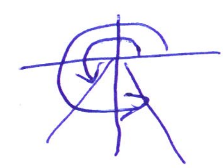
$\sin\theta = \pm\frac{1}{\sqrt{2}}$



$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\sin\theta = \frac{-1}{\sqrt{2}}$



$\theta = \pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$

$\theta = 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

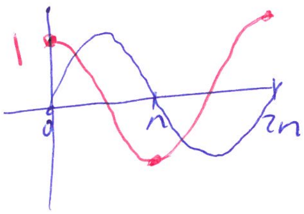
$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Example 6.6.3

From your text: Pg. 436 #5: Solve for $0^\circ \le x \le 360^\circ$

b) $\underbrace{\sin(x)}_{=0}\underbrace{(\cos(x)-1)}_{=0}=0$ Solve when each Factor $= 0$

$\sin x = 0$



$x = 0,\ 0^\circ$

$= \pi,\ 180^\circ$

$= 2\pi,\ 360^\circ$

$\cos x - 1 = 0$

$\cos x = 1$

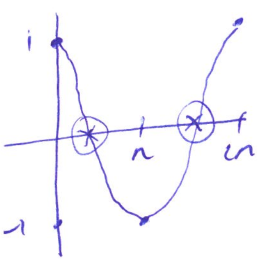
$x = 0,\ 2\pi$

$0^\circ,\ 36^\circ$

$\therefore x = 0, \pi, 2\pi$ or $0^\circ, 180^\circ, 360^\circ$

d) $\underbrace{\cos(x)}_{=0}\underbrace{\left(2\sin(x)-\sqrt{3}\right)}_{=0}=0$

$\cos x = 0$



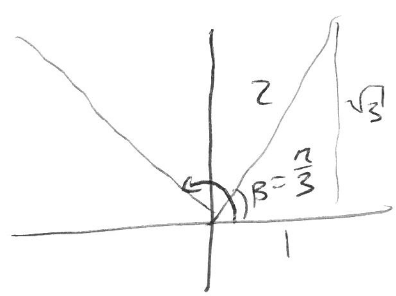
$x = \frac{\pi}{2},\ \frac{3\pi}{2}$

$90^\circ,\ 270^\circ$

$2\sin x - \sqrt{3} = 0$

$2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$



$x = \frac{\pi}{3},\ \frac{2\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$

$= 60^\circ, 90^\circ, 120^\circ, 270^\circ$

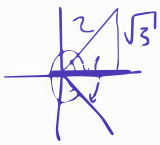
Example 6.6.4

From your text: Pg. 436 #6: Solve for $0 \le x \le 2\pi$

d) $\left(2\cos(x)-1\right)\left(2\sin(x)+\sqrt{3}\right)=0$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

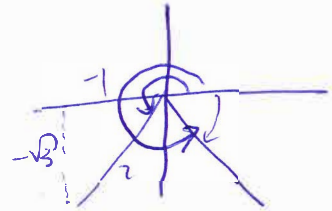


$x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

$2\sin x + \sqrt{3} = 0$

$\sin x = \frac{-\sqrt{3}}{2}$

$x = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$



$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Example 6.6.5

From your text: Pg. 436 #7: Solve for $0 \le \theta \le 2\pi$ to the nearest hundredth (if necessary).

a) $2\cos^2(\theta)+\cos(\theta)-1=0$

Let $x = \cos\theta$ $\therefore 2x^2 + x - 1 = 0$

Quadratic!

$\therefore (2x-1)(x+1) = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$

See above

$x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

$\cos\theta = -1$

$\theta = \pi$

$\therefore \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

e) $3\tan^2(\theta)-2\tan(\theta)=1$

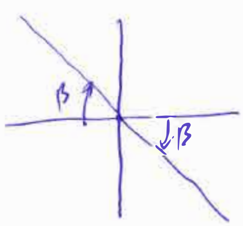
Let $x = \tan\theta$ $\quad 3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$3\tan^2\theta - 2\tan\theta - 1 = 0$

$(3\tan\theta + 1)(\tan\theta - 1) = 0$

$\tan\theta = \frac{-1}{3}$



$\tan\beta = \frac{1}{3}$

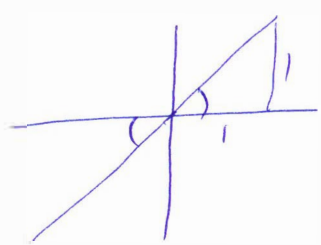
$\beta = \tan^{-1}\left(\frac{1}{3}\right)$

$= 0.32$

$\theta_1 = \pi - 0.32 = 2.82$

$\theta_2 = 2\pi - 0.32 = 5.96$

$\tan\theta = 1$



$\theta_3 = \frac{\pi}{4}$

$\theta_4 = \frac{5\pi}{4}$

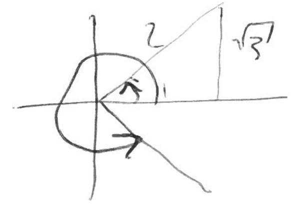
$\therefore \theta = \frac{\pi}{4}, 2.82, \frac{5\pi}{4}, 5.96$

Example 6.6.6 (decimals are between the sixes!)

From your text: Pg. 436 #8: Solve for $x \in [0, 2\pi]$

a) $\sec(x) \cdot \csc(x) - 2\csc(x) = 0$

Factor! $(\csc(x))(\sec(x) - 2) = 0$



$\csc x = \frac{0}{1}$ (flip)

$\sin x = \frac{1}{0}$

$\therefore$ no solution

$\sec x = \frac{2}{1}$ (flip)

$\cos x = \frac{1}{2}$

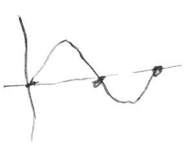
$\therefore x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

c) $2\sin(x) \cdot \sec(x) - 2\sqrt{3}\sin(x) = 0$

$2 \sin x (\sec x - \sqrt{3}) = 0$

$2 \sin x = 0$

$\sin x = 0$



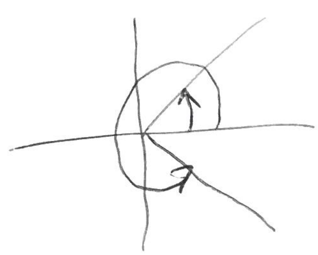
$\therefore x = 0, \pi, 2\pi$

$\sec x = \frac{\sqrt{3}}{1}$ (flip)

$\cos x = \frac{1}{\sqrt{3}}$ (not special)

$\cos \beta = \frac{1}{\sqrt{3}}$

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.96$



$\therefore x = 0.96$

$x = 5.32$
$(2\pi - 0.96)$

$\therefore x = 0, 0.96, \pi, 5.32, 2\pi$

$\cos(2x) = 2\cos^2 x - 1$

Example 6.6.7

From your text: Pg. 437 #9: Solve for $x \in [0, 2\pi]$. Round to two decimal places.

a) $5\cos(2x) - \cos(x) + 3 = 0$

$5(2\cos^2 x - 1) - \cos x + 3 = 0$

$10\cos^2 x - 5 - \cos x + 3 = 0$

$10\cos^2 x - \cos x - 2 = 0$

$(5\cos x + 2)(2\cos x - 1) = 0$

Let $x = \cos x$

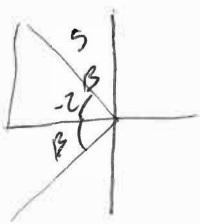
$10x^2 - x - 2 = 0$

$\underbrace{10x^2 - 5x}_{5x} \underbrace{+4x - 2}_{2} = 0$

$(5x + 2)(2x - 1) = 0$

$5\cos x + 2 = 0$

$\cos x = \frac{-2}{5}$



$\cos \beta = \frac{2}{5}$

$\beta = \cos^{-1}\left(\frac{2}{5}\right)$

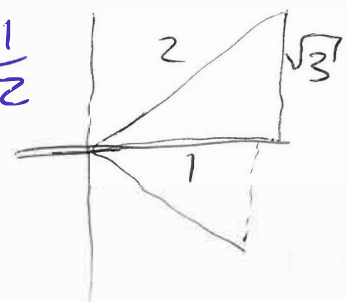
$\beta = 1.16$

$\therefore x_1 = \pi - 1.16 = 1.98$

$x_2 = \pi + 1.16 = 4.3$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$



$\therefore x_3 = \frac{\pi}{3}$

$x_4 = \frac{5\pi}{3}$

$\therefore x = \frac{\pi}{3}, 1.98, 4.3, \frac{5\pi}{3}$

Success Criteria:

- I can solve quadratic trigonometric equations by factoring, or using the quadratic formula
- I can recognize when I must use other trigonometric identities to create a quadratic equation with only a single trigonometric function
- I can recognize when I need to use special triangles VS a calculator to solve quadratic trigonometric equations