Advanced Functions

Teacher Notes

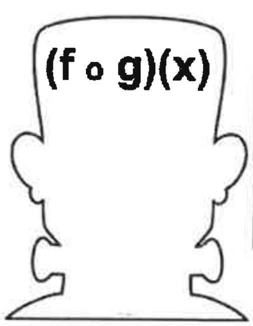
Course Notes

Unit 8 – Combinations of Functions

Mad Mathematicians Creating New Functions!!!

We will learn

- *how to use basic arithmetic to construct new functions from given functions*
- *how to describe the composition of two functions numerically, graphically and algebraically*
- *key characteristics of the newly created functions*



A∞Ω
MATH@TD

Chapter 8 – Combinations of Functions

Contents with suggested problems from the Nelson Textbook ***(Chapter 9)***

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

8.1 Sums and Differences of Functions

Pg. 451 #1c, 4, Finish 5 – 7, 9, 11

8.2 Product and Quotient Combinations

Pg. 537 - 539 #1bd,3,8bd,10,15

Pg. 542 # 1aef, 2 (for #1aef)

8.3 Composition of Functions

READ Ex. 4 Pg550 making sure you understand the concept of a function composed with its inverse.

Pg. 552 – 553 #1abf, 2bdf, 5bcdf (use tech to graph), 6bcd, 7bcdef, 13

8.1 Sums and Differences of Functions

Learning Goal: We are learning to combine functions through addition and subtraction.

Definition 8.1.1

Given two functions $f(x)$, and $g(x)$ with domains D_f and D_g respectively, then we can **construct** new functions:

Capital letters for the new functions

$$F(x)=(f+g)(x) \qquad\qquad G(x)=(f-g)(x)$$

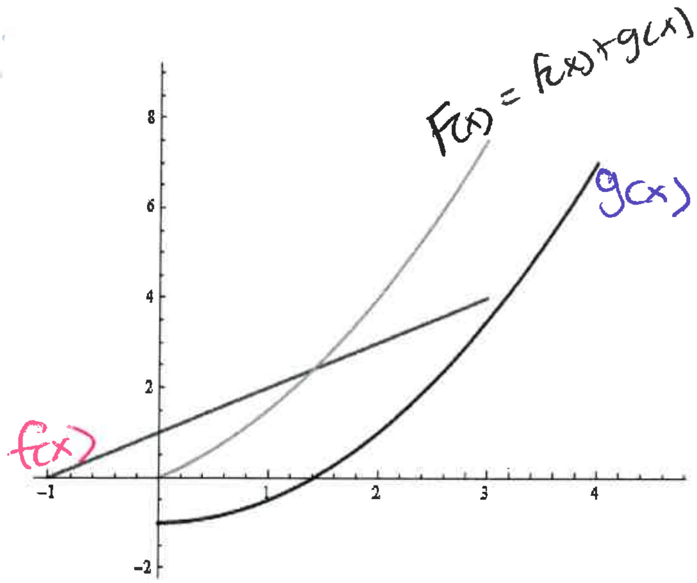
where the **meaning** of the notation for the "sum" and "difference" functions is as follows:

$$F(x) = f(x) + g(x) \qquad\qquad G(x) = f(x) - g(x)$$

we add/subtract functional values (y) (for f(x) + g(x)) at a single domain value (x)

Example 8.1.1

Consider the sketch (so that we can get at the domain of sum and difference functions):



$D_f = [-1, 3]$

$D_g = [0, 4]$

$D_F = [0, 3]$

$D_F = D_f \cap D_g$

"intersect" what values the two domains share

$$F(-0.5) = f(-0.5) + g(-0.5)$$
$$= 0.5 + \text{D.N.E.}$$
$$= \text{D.N.E.}$$

In general, given functions $f(x)$ and $g(x)$ with domains D_f and D_g, then the combined function

$$F(x)=(f\pm g)(x)$$

$$= f(x) \pm g(x)$$

and has the domain $D_F = D_f \cap D_g$

Example 8.1.2

Determine the domain of $F(x)=(f-g)(x)$ for $f(x)=\sqrt{x}$, and $g(x)=\log(-(x-2))$.

$F(x) = \sqrt{x} - \log(-(x-2))$

$D_f = [0,\infty)$

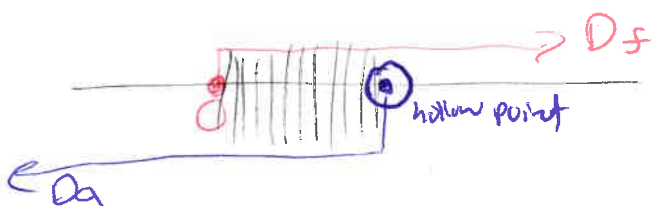
logs must be +

$-(x-2) > 0$

$x-2 < 0$

$x < 2$

$\therefore D_g(-\infty, 2)$



So, $D_F = D_f \cap D_g = x \in [0,2)$

Example 8.1.3

Given $f(x)=x^3-4x+1$, D_f=[-4,5] and $g(x)=2x^2-1$, $D_g=(0,6]$,

Determine: a) the order of $F(x)=(g-f)(x)$

b) D_F

c) an algebraic representation for $F(x)$

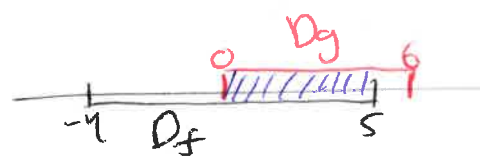
c) $F(x) = g(x) - f(x)$

$= (2x^2-1)-(x^3-4x+1)$

$= 2x^2 - 1 - x^3 + 4x - 1$

$= -x^3 + 2x^2 + 4x - 2$

b)



$D_F = D_f \cap D_g$

$= (0, 5]$

a) order of $F(x)$ is 3

Example 8.1.4

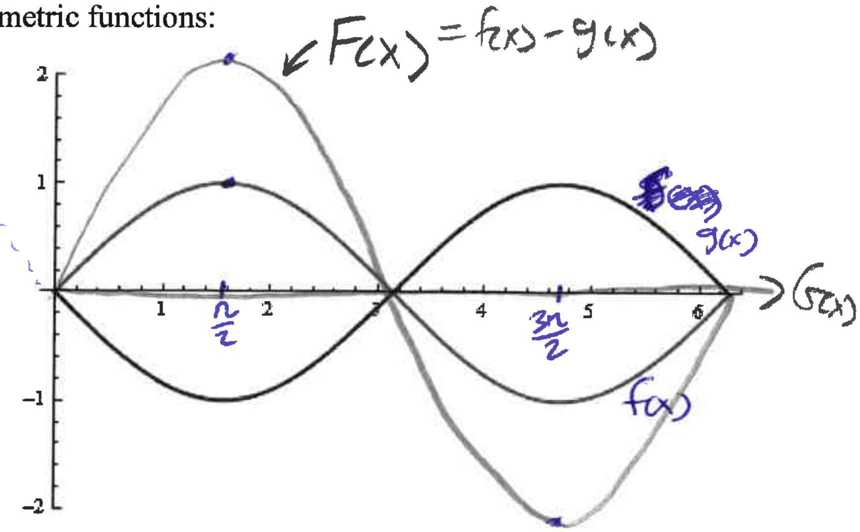
Consider the sketches of the trigonometric functions:

$f(x) = \sin(x),\ D_f = [0, 2\pi]$

$g(x) = \cos\left(x + \frac{\pi}{2}\right),\ D_g = [0, 2\pi]$

Sketch a) $F(x) = (f - g)(x)$

b) $G(x) = (f + g)(x)$



a) $F(x) = f(x) - g(x)$

= subtract pairs of y-values

$\frac{\pi}{2}: 1 - (-1) = 2$

$\frac{3\pi}{2}: -1 - (1) = -2$

b) $G(x) = f(x) + g(x)$

In this case, the y-values cancel each other out

$\frac{\pi}{2} = (1) + (-1) = 0$

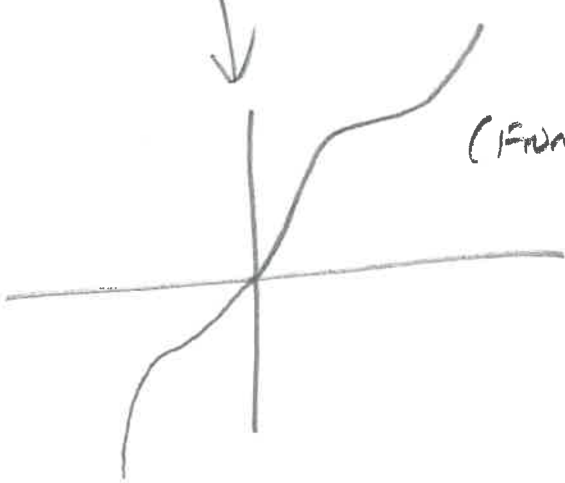
$\frac{3\pi}{2} = (-1) + (1) = 0$

Example 8.1.5

Consider the functions $f(x) = x,\ g(x) = \sin(x),\ x \in$ ¡ .

What does $F(x) = (f+g)(x)$ look like? (see Ex. 3 Pg. 526)

We should see some linearness + some sinusoidalness.



Both properties contribute to the structure of $F(x)$

(From Desmos

Add the y-values of corresponding domain values. only shared domains occur.

Example 8.1.6

From your text: Pg. 528 #1ae

Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$.

Determine: a) $(f+g)(x)$ e) $(f+f)(x)$

a) $(f+g)(x) = \{(-4,6), (-2,5), (1,5), (4,10)\}$

b) $(f+f)(x) = \{(-4,8), (-2,8), (1,6), (3,10), (4,12)\}$

Example 8.1.7

From your text: Pg. 529 #7ab

Given $f(x) = \dfrac{1}{3x-4}$ and $g(x) = \dfrac{1}{x-2}$, determine $(f+g)(x)$ and $D_{(f+g)}$

$$(f+g)(x) = \left(\frac{1}{3x-4}\right)^{(x-2)} + \left(\frac{1}{x-2}\right)^{(3x-4)}$$

$$= \frac{(x-2)+(3x-4)}{(3x+4)(x-2)}$$

$$= \frac{4x-6}{(3x+4)(x-2)}$$

D_f: $x \neq \frac{4}{3}$, $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$

D_g: $x \neq 2$, $(-\infty, 2) \cup (2, \infty)$

$\therefore D_{f+g} = (-\infty, \frac{4}{3}) \cup (\frac{4}{3}, 2) \cup (2, \infty)$

Success Criteria:

- I can combine functions by adding or subtracting them
- I can determine the domain and range of these combined functions

8.2 Product and Quotient Combinations

Learning Goal: We are learning to combine functions through multiplication and division.

Definition 8.2.1

Given two functions $f(x)$, and $g(x)$ with domains D_f and D_g respectively, then we can **construct** new functions:

$$F(x)=(f\cdot g)(x) = f(x)\cdot g(x)$$

$$G(x)=\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$D_{f\cdot g} = D_f \cap D_g$ — takes the common elements

$D_{\frac{f}{g}} = D_f \cap D_g$, $g(x) \neq 0$!

Example 8.2.1

Determine $(f\cdot g)(x)$ given $f(x)=\{(-2,3),(-1,5),(0,3),(1,-3),(2,-5)\}$ and $g(x)=\{(-1,4),(0,-2),(1,7),(2,-2),(3,2)\}$

$(f\cdot g)(x) = \{(-1,20),(0,-6),(1,-21),(2,10)\}$

Multiply the corresponding y-values

Example 8.2.2

From your text: Pg. 537 #2 for #1e

Sketch the given pair of function on the same set of axes. State their domains. Use your sketch to draw $(f\cdot g)(x)$. State $(f\cdot g)(x)$ and $D_{f\cdot g}$.

$f(x)=x+2,\quad g(x)=x^2-2x+1$

linear

$g(x) = (x-1)(x-1)$
$= (x-1)^2$
$= a(x-h)^2+k$

$(f\cdot g)(x) = (x+2)(x^2-2x+1)$ — leave factored
$= (x+2)(x-1)^2$

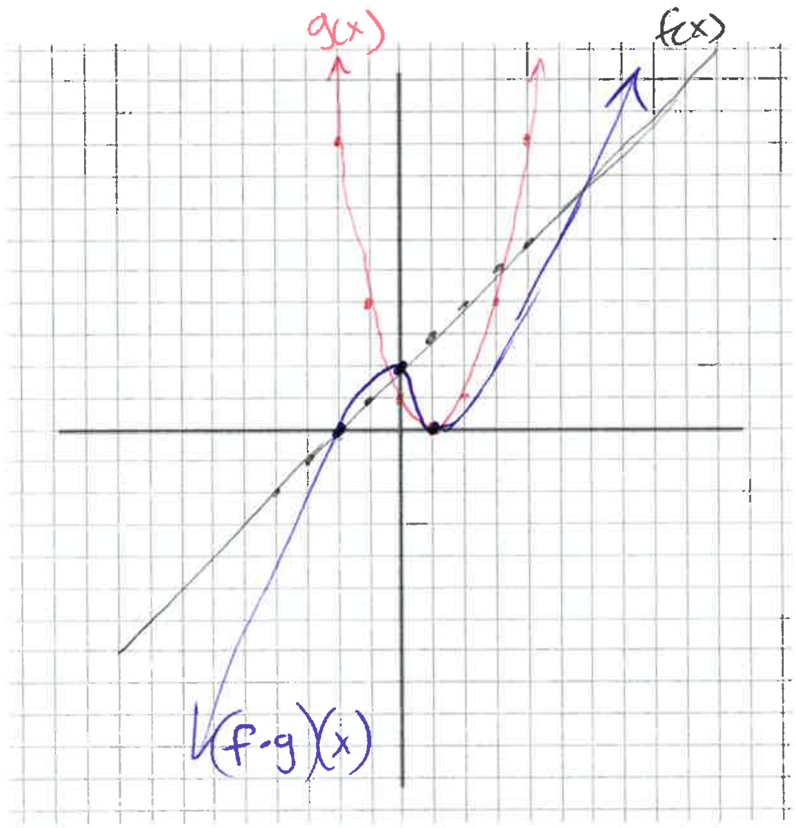
Degree = 3, odd

leading coeff = 1 $\therefore$ +

Zeros: $x=-2$, $\boxed{x=1}$ order 2

y-int: $(0,2)$

②



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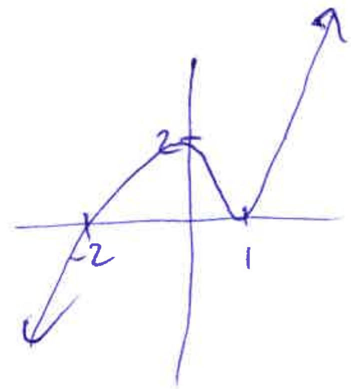


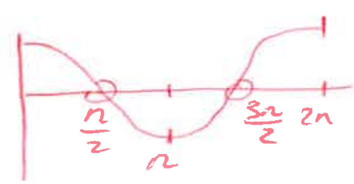
Example 8.2.3

Determine the domain of $D_{f\cdot g}$ and $D_{f\div g}$ given $f(x)=\sqrt{\underbrace{2x+3}}$, and $g(x)=\sec(x)$ $= \dfrac{1}{\cos x}$

$2x+3=0$
$x = \dfrac{-3}{2}$

$D_f: \left[-\frac{3}{2}, \infty\right)$

$D_g: \left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots \right\}$



$D_{f\cdot g}: \left\{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, \ x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \right\}$

Note: sec(x) never equals zero, but it does have restrictions on the domain.

$D_{\frac{f}{g}}: \left\{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, \ x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \right\}$

$\therefore D_{f\cdot g} = D_{\frac{f}{g}}$!

Fun to pop up the graph on desmos afterwards

Example 8.2.4

From your text: Pg. 542 #2 for 1d.

Sketch the given pair of function on the same set of axes. State their domains. Use your sketch to draw $\left(\frac{f}{g}\right)(x)$. State $\left(\frac{f}{g}\right)(x)$ and $D_{\frac{f}{g}}$.

$$f(x) = x + 2 \text{ and } g(x) = \sqrt{x-2}$$

$D_f : (-\infty, \infty)$

$D_g : [2, \infty)$

$\therefore D_{\frac{f}{g}} = (2, \infty)$

(only need + axis)

$\therefore$ VA @ 2

$\therefore \left(\frac{f}{g}\right)(x) = \frac{x+2}{\sqrt{x-2}}$

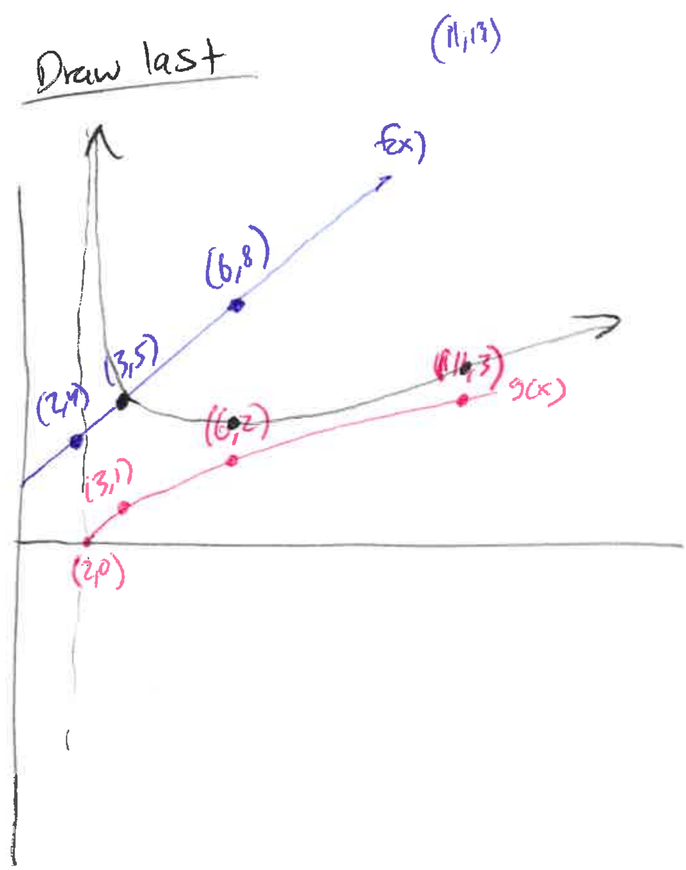
Draw last

$\frac{4}{0} = \infty$/D.N.E.

$\frac{5}{1} = 5$

$\frac{8}{2} = 4$

$\frac{13}{3} = 4.\overline{3}$



Success Criteria:

- I can combine functions by multiplying or dividing them
- I can determine the domain and range of these combined functions
  - The domain is the intersection of the domains of f and g

8.3 Composition of Functions

Learning Goal: We are learning to combine functions by inserting one into the other.

In sections 8.1 and 8.2 we examined how to combine functions (constructing new functions) through the standard algebraic operations of addition, subtraction, multiplication and division. Here we will learn another method for combining functions, but **we won't be using a standard algebraic operation.**

The concept we define as **Composition of Functions** is very **useful for Calculus** (among other things) as some of you will see next semester.

The basic idea is that given two functions $f(x)$ and $g(x)$, we can define the composition of the two by

inserting one function into the other (like a mad scientist!)

The "algebraic" notation may seem a little weird, but don't make fun. Math has feelings too.

Definition 8.3.1

Given two functions $f(x)$ and $g(x)$ we write the composition of $f(x)$ and $g(x)$ as

$$(f \circ g)(x)$$

We can also write the composition of $g(x)$ and $f(x)$ as

$$(g \circ f)(x)$$

The "Algebraic Meaning" of Composition

outer fn

$(f \circ g)(x) = f(g(x))$

composed with

$(g \circ f)(x) = g(f(x))$

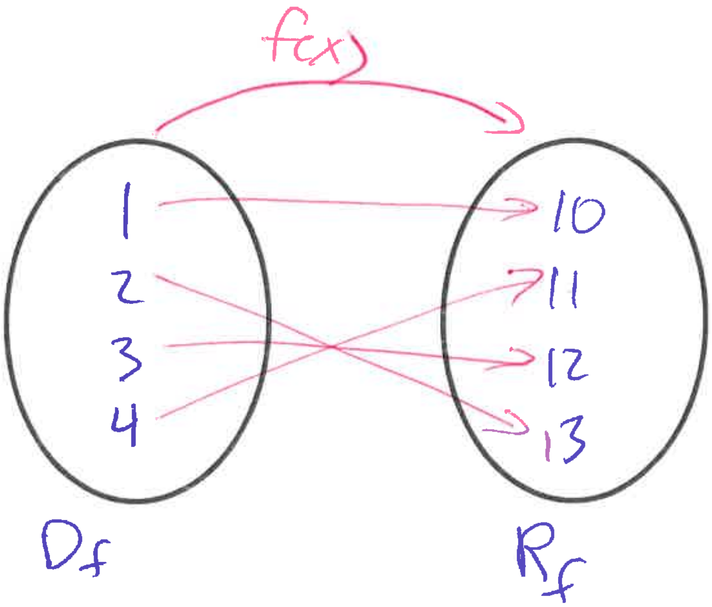
Inner fn

Note: It is very **helpful** to keep in mind the distinction between the **inner and outer functions**

The Domain of a Composition of Functions

Recall the basic "**machinery**" of any function: **"Plug a (domain) number into the function, and get a (range) number out."**

Picture



Recall further that many functions cannot claim "all real numbers" as their (natural) domain.

e.g. Determine the domain of $f(x)=\sqrt{x+1}$ $D_f: [-1, \infty)$

$x+1 \geq 0$

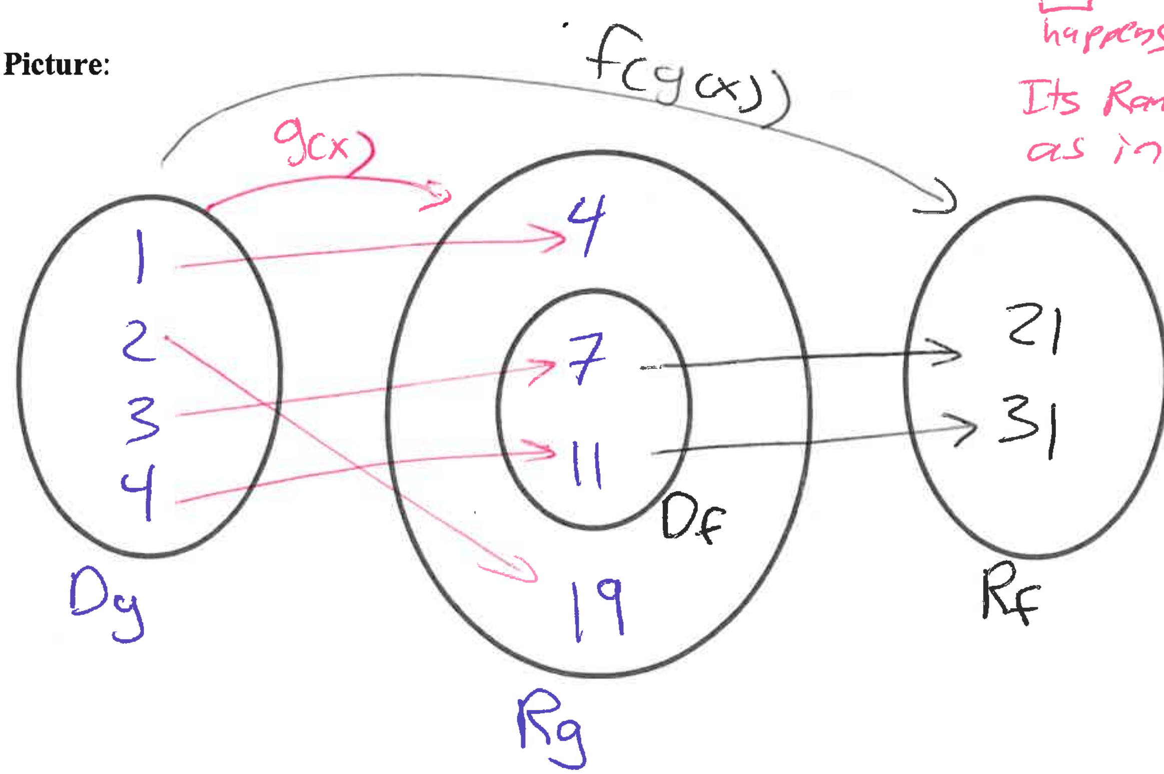
$x \geq -1$

Now we consider the "machinery" for the composition function $(f \circ g)(x)=f(\underbrace{g(x)})$

happens first.

Its Range is used as inputs for f

Picture:



Algebraic Definition of the Domain of a Composition of Two Functions

Given two functions $f(x)$ and $g(x)$ with domains D_f and D_g respectively, then the domain of the composition of $f(x)$ and $g(x)$ is given by:

Range of g(x)

$$D_{(f \circ g)} = \{x \in \mathbb{R} \mid x \in D_g \text{ such that } g(x) \in D_f\}$$

Using words we might write that the **domain of a composition of functions** $(f \circ g)(x) = f(g(x))$ **is the set of all** x **values which belong to** the domain of the

Inner function which have range values which are in the domain of the

Outer function.

Example 8.3.1

Given $f(x) = 3x+1$ and $g(x) = x^3 - 1$ determine:

a) $(f \circ g)(0)$ b) $g(f(0))$ c) $(f \circ f)(1)$

a)
$= f(g(0))$
$= f(0^3 - 1)$
$= f(-1)$
$= 3(-1)+1$
$\boxed{=-2}$
$\therefore (0,-2)$

b)
$= g(3(0)+1)$
$= g(1)$
$= (1)^3 - 1$
$= 0$
$\therefore (0,0)$

c)
$= f(f(1))$
$= f(3(1)+1)$
$= f(4)$
$= 3(4)+1$
$= 13$
$\therefore (1,13)$

Example 8.3.2

From your text: Pg. 552 #2 ac"g"

Given $f(x) = \{(0,1),(1,2),(2,5),(3,10)\}$ and $g(x) = \{(2,0),(3,1),(4,2),(5,3),(6,4)\}$

determine: a) $(g \circ f)(2)$ c) $f(g(5))$ g) $g(f(3))$

a)
$= g(f(2))$
$= g(5)$
$= 3$
$\therefore (2,3)$

c)
$= f(3)$
$= 10$
$\therefore (5,10)$

g)
$= g(10)$
$=$ DNE Finite set!
$x = 3 \notin D_{g \circ f}$

Something Silly but Entirely Serious

Given $f(x) = 2x^2 - 1$ determine:

a) $f(2)$ b) $f(A)$ c) $f(\cancel{W})$ $\square$ d) $f(\cancel{W+V})$ $\square + \triangle$

$= 2(2)^2 - 1$
$= 7$

$= 2A^2 - 1$

$= 2\square^2 - 1$

$= 2(\square + \triangle)^2 - 1$

Example 8.3.3

From your text: Pg 552 #6ae

Given the functions $f(x)$ and $g(x)$ determine functional equations for $f(g(x))$ and $g(f(x))$ and determine their domains.

a) $\underbrace{f(x) = 3x}$ and $\underbrace{g(x) = \sqrt{x-4}}$

$f(g(x))$ sub $g(x)$ into $f(x)$

$= f(\sqrt{x-4})$

$= 3\sqrt{x-4}$

$D_{f \circ g} = [4, \infty)$

$= g(3x)$

$= \sqrt{3x - 4}$

$D_{g \circ f} = [\frac{4}{3}, \infty)$

e) $f(x) = 10^x$ and $g(x) = \log(x)$

$D(-\infty, \infty)$

$R(0, \infty)$

D: $x > 0$

$R(-\infty, \infty)$

only input >0, so this sets domain

$= f(\log(x))$

$= \underbrace{10^{\log(x)}}_{\text{Inverses}}$

$= x$

$D_{f \circ g} = (0, \infty)$

$= g(10^x)$

$= \log(10^x)$

$= x$

$D_g = (-\infty, \infty)$

$R_g = (0, \infty)$

negative domain undefined as is 0 on input values. will only get + y values.

Now, since $D_f = R_g$

$D_{g \circ f} = (-\infty, \infty)$

Example 8.3.4

From your text: Pg. 553 #7a

Given $h(x)$ find two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$.

a) $h(x) = \sqrt{x^2 + 6}$

1 Let $g(x) = x^2$

$f(x) = \sqrt{x+6}$

2 Let $g(x) = x^2 + 6$

$f(x) = \sqrt{x}$

3 Let $g(x) = x^2 + 3$

$f(x) = \sqrt{x+3}$

Success Criteria:

- I can combine functions by inserting one into the other
  - $(f \circ g)(x) = f(g(x))$
- I can determine the domain and range of these combined functions
  - For domain, it is the set of values, *x*, in the domain of *g* for which *g(x)* is in the domain of *f*.