Chapter 1

Functions: Characteristics and Properties

GOALS

You will be able to

- Review and consolidate your knowledge of the properties and characteristics of functions and their inverses
- Review and consolidate your knowledge of graphing functions using transformations
- Investigate the characteristics of piecewise functions

? What type of function can be used to model the height of a golf ball during its flight, and what information about the relationship between height and time can be found using this function?

1 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
2	R-3
3	R-8, R-12

SKILLS AND CONCEPTS You Need

1. Evaluate $f(x) = x^2 + 3x - 4$ for each of the following values.

a) $f(2)$ **b)** $f(-1)$ **c)** $f\left(\frac{1}{4}\right)$ **d)** $f(a + 1)$

2. Factor each of the following expressions.

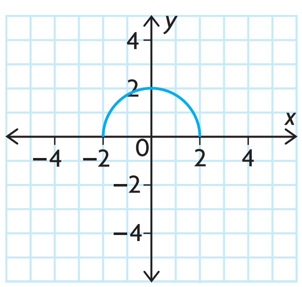
a) $x^2 + 2xy + y^2$ **c)** $(x + y)^2 - 64$
b) $5x^2 - 16x + 3$ **d)** $ax + bx - ay - by$

3. State the **transformations** that are applied to each **parent function**, resulting in the given transformed function. Sketch the graphs of the parent function and transformed function.

a) $f(x) = x^2, y = f(x - 3) + 2$ **c)** $g(x) = \sin x, y = -2g(0.5x)$
b) $f(x) = 2^x, y = f(x - 1) + 2$ **d)** $g(x) = \sqrt{x}, y = -2g(2x)$

4. State the **domain** and **range** of each function.

a)



b) $f(x) = x^2 - 6x - 10$

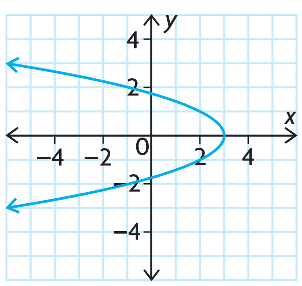
c) $y = \frac{1}{x}$

d) $y = 3 \sin x$

e) $g(x) = 10^x$

5. Which of the following represent functions? Explain.

a)



b) $y = 2(x - 1)^2 + 3$

c) $y = \pm\sqrt{x - 4}$

d) $y = 2^x - 4$

e) $y = \cos(2(x - 30°) + 1)$

6. Consider the **relation** $y = x^3$.

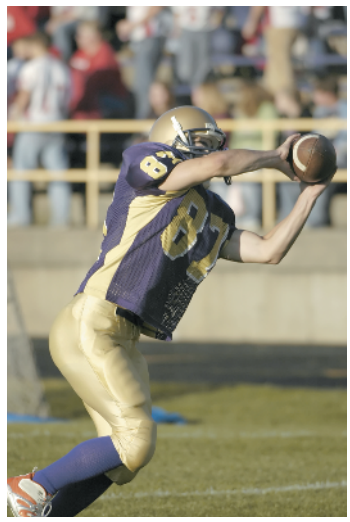
a) If $(2, n)$ is a point on its graph, determine the value of n.
b) If $(m, 20)$ is a point on its graph, determine m correct to two decimal places.

7. A function can be described or defined in many ways. List these different ways, and explain how each can be used to determine whether a relation is a function.

APPLYING What You Know

Modelling the Height of a Football

During a football game, a football is thrown by a quarterback who is 2 m tall. The football travels through the air for 4 s before it is caught by the wide receiver.



? What function can be used to model the height of the football above the ground over time?

A. Explain why the variables time, t, in seconds and height, $h(t)$, in metres are good choices to model this situation.

B. What is $h(0)$? What does it mean in the context of this situation?

C. What happens at $t = 2$ s?

D. What happens at $t = 4$ s?

E. Explain why each of the following functions is *not* a good model for this situation. Support your claim with reasons and a well-labelled sketch.

i) $h(t) = -5t(t - 4)$
ii) $h(t) = -5(t - 4)^2 + 2$
iii) $h(t) = 5t^2 + 4t - 3$

F. Determine a model that can be used to represent the height of the football, given this additional information:

- The ball reached a maximum height of 22 m.
- The wide receiver who caught the ball is also 2 m tall.

G. Use your model from part F to graph the height of the football over the duration of its flight.

1.1 Functions

YOU WILL NEED

- graph paper
- graphing calculator (optional)

GOAL

Represent and describe functions and their characteristics.

LEARN ABOUT the Math

Jonathan and Tina are building an outdoor skating rink. They have enough materials to make a rectangular rink with an area of about 1800 m^2, and they do not want to purchase any additional materials. They know, from past experience, that a good rink must be approximately 30 m longer than it is wide.

? What dimensions should they use to make their rink?

EXAMPLE 1 Representing a situation using a mathematical model

Determine the dimensions that Jonathan and Tina should use to make their rink.

Solution A: Using an algebraic model

Let x represent the length. Let y represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

We know the area must be 1800 m^2, so if we let the width be the **independent variable**, we can write an expression for the length.

The width, in terms of x, is $\frac{1800}{x}$.

Let $f(x)$ represent the difference between the length and the width.

$$f(x) = x - \frac{1800}{x},$$

where $f(x) = 30$.

$$x - \frac{1800}{x} = 30$$

Using **function notation**, write an equation for the difference in length and width. The relation is a **function** because each input produces a unique output. In this case the difference or value of the function must be 30.

$$x(x) - x\left(\frac{1800}{x}\right) = x(30)$$

To solve the equation, multiply all the terms in the equation by the lowest common denominator, *x*, to eliminate any rational expressions.

$$x^2 - 1800 = 30x$$
$$x^2 - 30x - 1800 = 0$$
$$(x - 60)(x + 30) = 0$$

This results in a quadratic equation. Rearrange the equation so that it is in the form $ax^2 + bx + c = 0$. Factor the left side.

$$x - 60 = 0 \text{ or } x + 30 = 0$$
$$x = 60 \text{ or } x = -30$$

Solve for each factor. $x = -30$ is outside the domain of the function, since length cannot be negative. This is an inadmissible solution.

The length is 60 m.

$$y = \frac{1800}{60} = 30$$

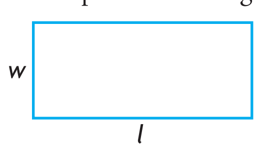
Calculate the width.

The width is 30 m.

The dimensions that are 30 m apart and will produce an area of 1800 m^2 are 60 m × 30 m.

Solution B: Using a numerical model

Let *l* represent the length. Let *w* represent the width.



Length is the independent variable.
Its domain is $0 < l < 1800$.
Width is the dependent variable.

$$A = lw$$
$$1800 = lw$$
$$\frac{1800}{l} = w$$

Write an equation for the width in terms of length for a fixed area of 1800 m^2.

Guess 1: $l = 200$

$$w = \frac{1800}{200} = 9$$

Check: $l - w = 200 - 9 \neq 30$

Guess 2: $l = 100$

$$w = \frac{1800}{100} = 18$$

Check: $l - w = 100 - 18 \neq 30$

Use different values for the length to calculate possible widths. Check to see if the difference between the length and width is 30.

Area (m²)	Length (m)	Width (m)	Length – Width
1800	100	18	82
1800	90	20	70
1800	80	22.5	57.5
1800	70	25.71	44.29
1800	60	30	30
1800	50	36	14
1800	40	45	−5
1800	30	60	−30
1800	20	90	−70

Create a table of values to investigate the difference between the length and the width for a variety of lengths.

The dimensions that are 30 m apart and produce an area of 1800 m^2 are 60 m × 30 m.

A function can also be represented with a graph. A graph provides a visual display of how the variables in the function are related.

Solution C: Using a graphical model

Let x represent the length. Let y represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

Using length (x) as the independent variable, write an expression for width (y).

The width, in terms of x, is $\frac{1800}{x}$.

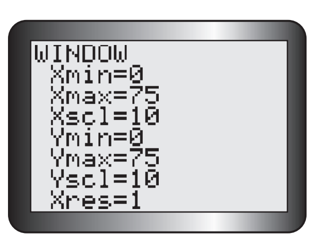
Let $f(x)$ represent the difference between the dimensions.

$$f(x) = x - \frac{1800}{x}$$

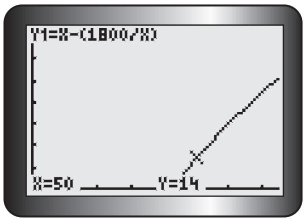
Determine the appropriate window settings to graph $f(x)$ on a graphing calculator.

The value for x (length of rink) will be positive but surely less than 75 m, so we use Xmin = 0 and Xmax = 75.

We use the same settings for the range of f(x), for simplicity.



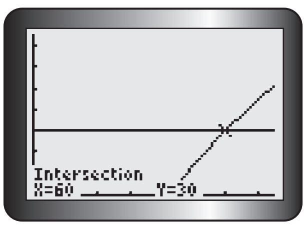
Graph the difference function.

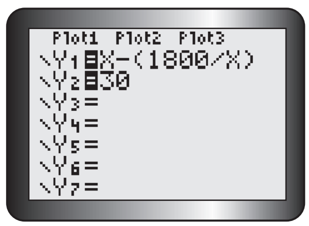


Use the TRACE feature on the graph to investigate points with the ordered pairs (length, length − width) on $f(x)$.

A length of 50 m gives a 14 m difference between the length and the width.

Determine the length that exceeds the width by 30 m.





To determine the length that is 30 m longer than the width, graph $g(x) = 30$ in Y_2 and locate the point of intersection for $g(x)$ and $f(x)$.

The dimensions that are 30 m apart and produce an area of 1800 m^2 are 60 m × 30 m.

Tech Support

For help using the graphing calculator to find points of intersection, see Technical Appendix, T-12.

Reflecting

A. Would the function change if width was used as the independent variable instead of length? Explain.

B. Is it necessary to restrict the domain and range in this problem? Explain.

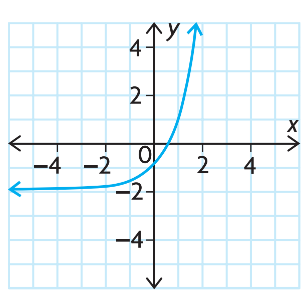
C. Why was it useful to think of the relationship between the length and the width as a function to solve this problem?

APPLY the Math

EXAMPLE 2 Using reasoning to decide whether a relation is a function

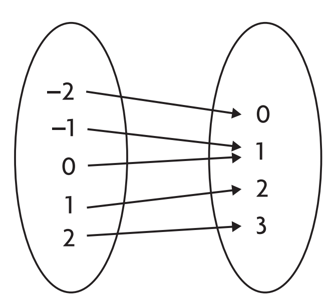
Decide whether each of the following relations is a function. State the domain and range.

a)



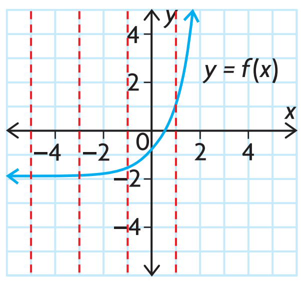
b) $y = \dfrac{1}{x^2}$

c)



Solution

a)



Apply the **vertical line test**. Any vertical line drawn on the graph of a function passes through, at most, a single point. This indicates that each number in the domain corresponds to only one number in the range, which is the condition for the relation to be a function.

The graph represents an **exponential function**.

$D = \{x \in \mathbf{R}\}$

$R = \{y \in \mathbf{R} \mid y > -2\}$

Since the graph of this function has no breaks, or vertical **asymptotes**, and continues indefinitely in both the positive and negative direction, its domain consists of all the **real numbers**.

The function has a **horizontal asymptote** defined by the equation $y = -2$. All its values lie above this horizontal line.

b)

x	−3	−2	−1	0	1	2	3
***f*(*x*)**	$\frac{1}{9}$	$\frac{1}{4}$	1	undefined	1	$\frac{1}{4}$	$\frac{1}{9}$

Create a table of values.

The table indicates that each number in the domain corresponds to only one number in the range.

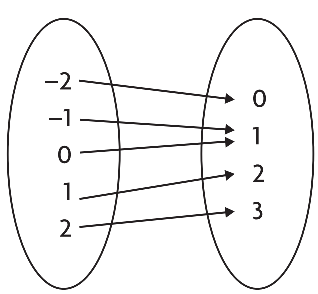
$f(x) = \frac{1}{x^2}$ is a function.

$D = \{x \in \mathbf{R} \mid x \neq 0\}$

$R = \{y \in \mathbf{R} \mid y > 0\}$

$f(x) = \frac{1}{x^2}$ has a **vertical asymptote** defined by $x = 0$. Its domain consists of all the real numbers, except 0. It has a horizontal asymptote defined by the equation $y = 0$. All its values are positive, since x is squared, so they lie above this horizontal line.

c)



The mapping diagram indicates that each number in the domain corresponds to only one number in the range.

A function can have converging arrows but cannot have diverging arrows in a mapping diagram.

This is a function.

$D = \{-2, -1, 0, 1, 2\}$

$R = \{0, 1, 2, 3\}$

The first oval represents the elements found in the domain. The second oval represents the elements found in the range.

EXAMPLE 3 Using reasoning to determine the domain and range of a function

Naill rides a Ferris wheel that has a diameter of 6 m. The axle of the Ferris wheel is 4 m above the ground. The Ferris wheel takes 90 s to make one complete rotation, and Naill rides for 10 rotations. What are the domain and range of the function that models Naill's height above the ground, in terms of time, while he rides the Ferris wheel?

Solution

$h(t) = a \sin[k(t - d)] + c$

or

$h(t) = a \cos[k(t - d)] + c$

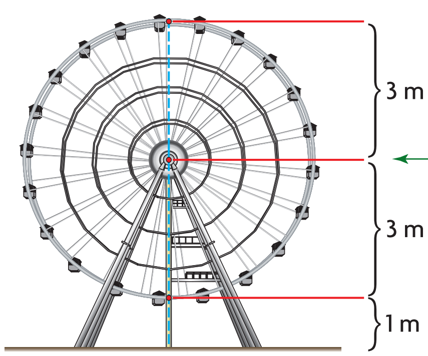
This situation involves circular motion, which can be modelled by a sine or cosine function.

$D = \{t \in \mathbf{R} \mid 0 \le t \le 900\}$

Examine the conditions on the independent variable time to determine the domain. Time cannot be negative, so the lower boundary is 0. The wheel rotates once every 90 s, and Naill rides for 10 complete rotations.

$90 \times 10 = 900$

The upper boundary is 900 s.



$R = \{h(t) \in \mathbf{R} \mid 1 \le h(t) \le 7\}$

Examine the conditions on the dependent variable height to determine the range. The radius of the wheel is 3 m. Since the axle is located 4 m above the ground, the lowest height that Naill can be above the ground is the difference between the height of the axle and the radius of the wheel: $4 - 3 = 1$ m. This is the lower boundary of the range.

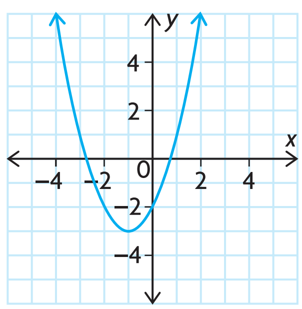
The greatest height he reaches is the sum of the height of the axle and the radius of the wheel: $4 + 3 = 7$ m. This is the upper boundary of the range.

In Summary

Key Ideas

- A function is a relation in which there is a unique output for each input. This means that each value of the independent variable (the domain) must correspond to one, and only one, value of the dependent variable (the range).
- Functions can be represented graphically, numerically, or algebraically.

Graphical Example



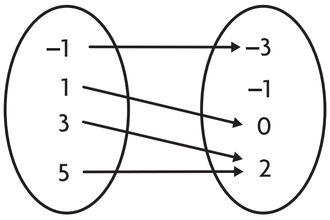
Numerical Examples

Set of ordered pairs:
$\{(1, 3), (3, 5), (-2, 9), (5, 11)\}$

Table of values:

x	y
-2	4
-1	1
0	0
1	1
2	4

Mapping diagram:

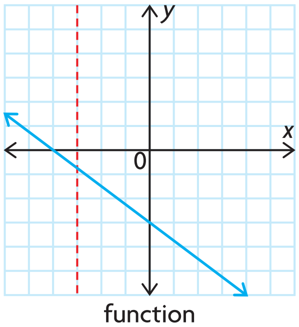


Algebraic Examples

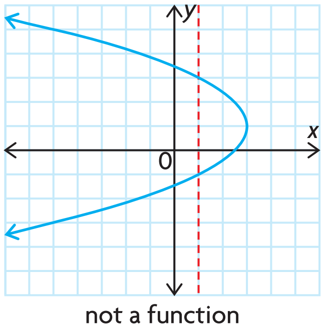
$y = 2 \sin (3x) + 4$
or
$f(x) = 2 \sin (3x) + 4$

Need to Know

- Function notation, $f(x)$, is used to represent the values of the dependent variable in a function, so $y = f(x)$.
- You can use the vertical line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in, at most, one point. This shows that there is only one element in the range for each element in the domain.
- The domain and range of a function depend on the type of function.
- The domain and range of a function that models a particular situation may need to be restricted, based on the situation. For example, negative values may not have meaning when dealing with variables such as time.



function

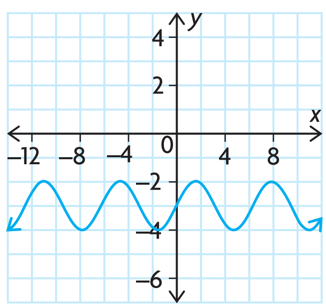


not a function

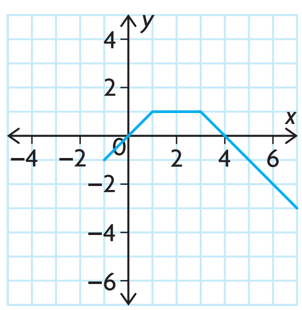
CHECK Your Understanding

1. State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

a)



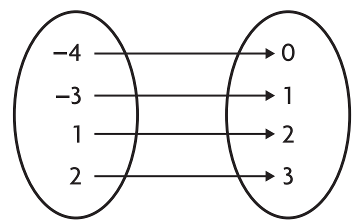
b)



c) $\{(1, 4), (1, 9), (2, 7), (3, -5), (4, 11)\}$

d) $y = 3x - 5$

e)



f) $y = -5x^2$

2. State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

a) $y = -2(x + 1)^2 - 3$

b) $y = \dfrac{1}{x + 3}$

c) $y = 2^{-x}$

d) $y = \cos x + 1$

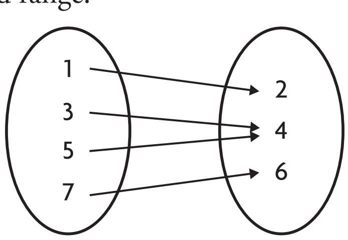
e) $x^2 + y^2 = 9$

f) $y = 2 \sin x$

PRACTISING

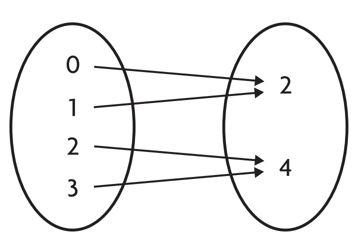
3. Determine whether each relation is a function, and state its domain and range.

a)



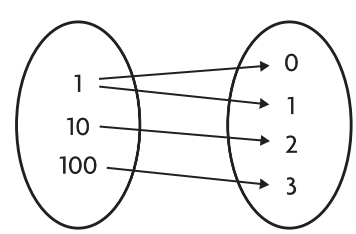
b) $\{(2, 3), (1, 3), (5, 6), (0, -1)\}$

c)



d) $\{(2, 5), (6, 1), (2, 7), (8, 3)\}$

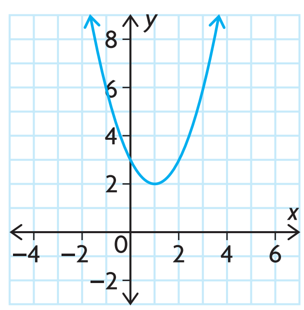
e)



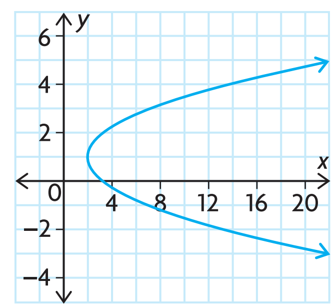
f) $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$

4. Determine whether each relation is a function, and state its domain and range.
K

a)



b)



c) $x^2 = 2y + 1$

d) $x = y^2$

e) $y = \frac{3}{x}$

f) $f(x) = 3x + 1$

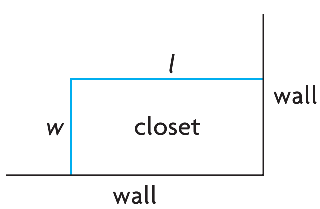
5. Determine the equations that describe the following function rules:

a) The input is 3 less than the output.

b) The output is 5 less than the input multiplied by 2.

c) Subtract 2 from the input and then multiply by 3 to find the output.

d) The sum of the input and output is 5.



6. Martin wants to build an additional closet in a corner of his bedroom. Because the closet will be in a corner, only two new walls need to be built. The total length of the two new walls must be 12 m. Martin wants the length of the closet to be twice as long as the width, as shown in the diagram.

a) Explain why $l = 2w$.

b) Let the function $f(l)$ be the sum of the length and the width. Find the equation for $f(l)$.

c) Graph $y = f(l)$.

d) Find the desired length and width.

7. The following table gives Tina's height above the ground while riding a Ferris wheel, in relation to the time she was riding it.
A

Time (s)	0	20	40	60	80	100	120	140	160	180	200	220	240
Height (m)	5	10	5	0	5	10	5	0	5	10	5	0	5

a) Draw a graph of the relation, using time as the independent variable and height as the dependent variable.

b) What is the domain?

c) What is the range?

d) Is this relation a function? Justify your answer.

e) Another student sketched a graph, but used height as the independent variable. What does this graph look like?

f) Is the relation in part e) a function? Justify your answer.

8. Consider what happens to a relation when the coordinates of all its ordered pairs are switched.

a) Give an example of a function that is still a function when its coordinates are switched.

b) Give an example of a function that is no longer a function when its coordinates are switched.

c) Give an example of a relation that is not a function, but becomes a function when its coordinates are switched.

9. Explain why a relation that fails the vertical line test is not a function.

10. Consider the relation between x and y that consists of all points (x, y) such that the distance from (x, y) to the origin is 5.

a) Is (4, 3) in the relation? Explain.

b) Is (1, 5) in the relation? Explain.

c) Is the relation a function? Explain.

11. The table below lists all the ordered pairs that belong to the function $g(x)$.

x	0	1	2	3	4	5
$g(x)$	3	4	7	12	19	28

a) Determine an equation for $g(x)$.

b) Does $g(3) - g(2) = g(3 - 2)$? Explain.

12. T The factors of 4 are 1, 2, and 4. The sum of the factors is $1 + 2 + 4 = 7$. The sum of the factors is called the sigma function. Therefore, $f(4) = 7$.

a) Find $f(6)$, $f(7)$, and $f(8)$.

b) Is $f(15) = f(3) \times f(5)$?

c) Is $f(12) = f(3) \times f(4)$?

d) Are there others that will work?

13. C Make a concept map to show what you have learned about functions. Put "FUNCTION" in the centre of your concept map, and include the following words:

algebraic model	graphical model	numerical model
dependent variable	independent variable	range
domain	mapping model	vertical line test
function notation		

Communication *Tip*

A concept map is a type of web diagram used for exploring knowledge and gathering and sharing information. A concept map consists of cells that contain a concept, item, or question and links. The links are labelled and denote direction with an arrow symbol. The labelled links explain the relationship between the cells. The arrow describes the direction of the relationship and reads like a sentence.

Extending

14. Consider the relations $x^2 + y^2 = 25$ and $y = \sqrt{25 - x^2}$. Draw the graphs of these relations, and determine whether each relation is a function. State the domain and range of each relation.

15. You already know that y is a function of x if and only if the graph passes the vertical line test. When is x a function of y? Explain.

1.2 Exploring Absolute Value

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Discover the properties of the absolute value function.

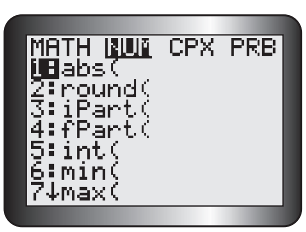
EXPLORE the Math

An average person's blood pressure is dependent on their age and gender. For example, the average systolic blood pressure, P_n, for a 17-year-old girl is about 127 mm Hg. (The symbol mm Hg stands for millimetres of mercury, which is a unit of measure for blood pressure.) The average systolic blood pressure for a 17-year-old boy is about 134 mm Hg.

When doctors measure blood pressure, they compare the blood pressure to the average blood pressure for people in the same age and gender group. This comparison, P_d, is calculated using the formula $P_d = |P - P_n|$, where P is the blood pressure reading and P_n is the average reading for people in the same age and gender group.

Tech | *Support*

To use the **absolute value** command on a graphing calculator, press MATH and scroll right to NUM. Then press ENTER.



? How can the blood pressure readings of a group of people be compared?

A. Jim is a 17-year-old boy whose most recent blood pressure reading was 142 mm Hg. Calculate P_d for Jim.

B. Joe is a 17-year-old boy whose most recent blood pressure reading was 126 mm Hg. Calculate P_d for Joe.

C. Compare the values of $P - P_n$ and $|P - P_n|$ that were used to determine P_d for each boy. What do you notice?

D. Complete the following table by calculating the values of P_d for the given blood pressure readings for 17-year-old boys.

Blood Pressure Reading, *P*	95	100	105	110	115	120	125	130	135	140	145	150	155	160
P_d														

E. Draw a **scatter plot** of P_d as a function of blood pressure, P.

F. Describe these characteristics of your graph:

i) domain
ii) range
iii) zeros
iv) existence of any asymptotes
v) shape of the graph
vi) intervals of the domain in which the values of the function P_d are increasing and decreasing.
vii) behaviour of the values of the function P_d as P becomes larger and smaller

Reflecting

G. Why might you predict the range of your graph to be greater than or equal to zero?

H. What other function with domain greater than P_n could you have used to plot the right side of your graph? Why does this make sense?

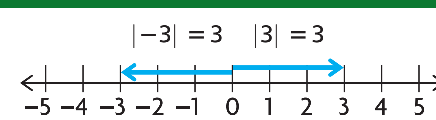
I. What other function with domain less than P_n could you have used to plot the left side of your graph? Why does this make sense?

J. How will the graph of $y = |x|$ compare with the graph of $P_d = |P - P_n|$, if P_d is the y-coordinate and P is the x-coordinate? Use the characteristics you listed in part F to make your comparison.

In Summary

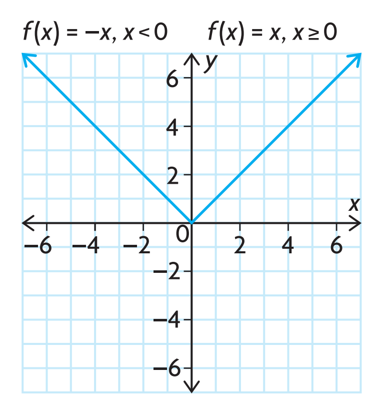
Key Idea

- $f(x) = |x|$ is the absolute value function. On a number line, this function describes the distance, $f(x)$, of any number x from the origin.



Need to Know

- For the function $f(x) = |x|$,
  - there is one zero located at the origin
  - the graph is comprised of two linear functions and is defined as follows:
    $$f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$
  - the graph is symmetric about the y-axis
  - as x approaches large positive values, y approaches large positive values
  - as x approaches large negative values, y approaches large positive values
  - the absolute value function has domain $\{x \in \mathbf{R}\}$ and range $\{y \in \mathbf{R} | y \geq 0\}$
  - every input in an absolute value returns an output that is non-negative



FURTHER Your Understanding

1. Arrange these values in order, from least to greatest:
$|-5|, |20|, |-15|, |12|, |-25|$

2. Evaluate.

a) $|-22|$ **c)** $|-5 - 13|$ **e)** $\frac{|-8|}{-4}$

b) $-|-35|$ **d)** $|4 - 7| + |-10 + 2|$ **f)** $\frac{|-22|}{|-11|} + \frac{-16}{|-4|}$

3. Express using absolute value notation.

a) $x < -3$ or $x > 3$ **c)** $x \leq -1$ or $x \geq 1$

b) $-8 \leq x \leq 8$ **d)** $x \neq \pm 5$

Communication *Tip*

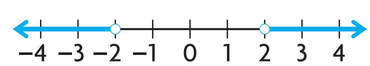
To show that a number is not included in the solution set, use an open dot at this value. A solid dot shows that this value is included in the solution set.

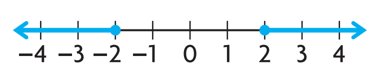
4. Graph on a number line.

a) $|x| < 8$ **b)** $|x| \geq 16$ **c)** $|x| \leq -4$ **d)** $|x| > -7$

5. Rewrite using absolute value notation.

a) 

b) 

c) 

d) 

6. Graph $f(x) = |x - 8|$ and $g(x) = |-x + 8|$.

a) What do you notice?

b) How could you have predicted this?

7. Graph the following functions.

a) $f(x) = |x - 2|$

b) $f(x) = |x| + 2$

c) $f(x) = |x + 2|$

d) $f(x) = |x| - 2$

8. Compare the graphs you drew in question 7. How could you use transformations to describe the graph of $f(x) = |x + 3| - 4$?

9. Predict what the graph of $f(x) = |2x + 1|$ will look like. Verify your prediction using graphing technology.

10. Predict what the graph of $f(x) = 3 - |2x - 5|$ will look like. Verify your prediction using graphing technology.

1.3 Properties of Graphs of Functions

GOAL

Compare and contrast the properties of various types of functions.

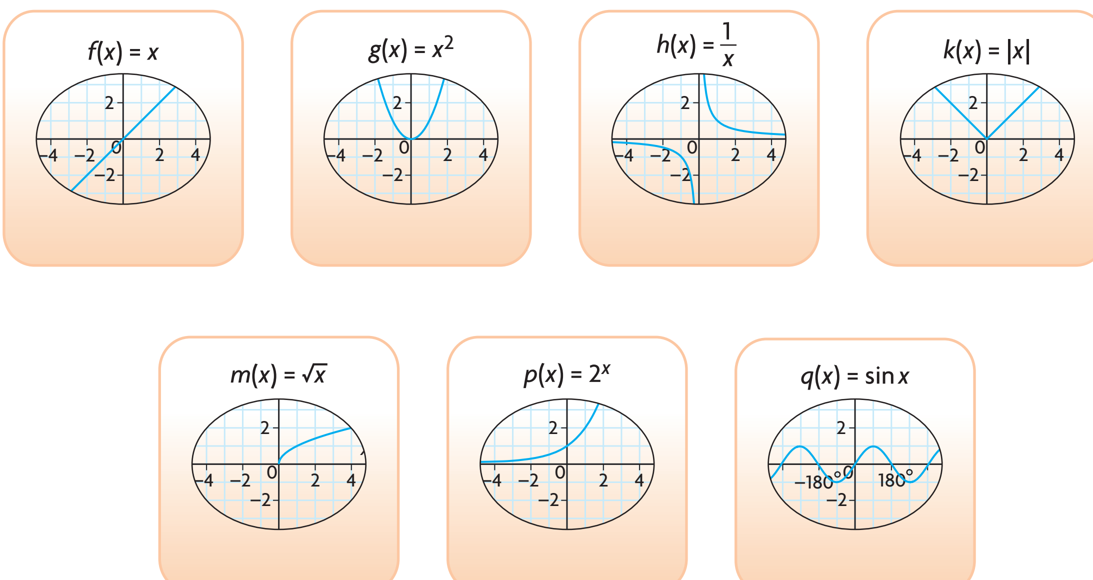
YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

Two students created a game that they called “Which function am I?” In this game, players turn over cards that are placed face down and match the characteristics and properties with the correct functions. The winner is the player who has the most pairs at the end of the game.

The students have studied the following parent functions:



? Which criteria could the students use to differentiate between these different types of functions?

A. Graph each of these parent functions on a graphing calculator, and sketch its graph. State the domain and range of each function, and determine its zeros and y-intercepts.

B. Determine the intervals of increase and the intervals of decrease for each of the parent functions.

interval of increase

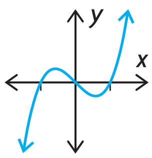
the interval(s) within a function’s domain, where the y-values of the function get larger, moving from left to right

interval of decrease

the interval(s) within a function’s domain, where the y-values of the function get smaller, moving from left to right

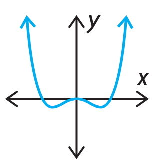
odd function

any function that has rotational symmetry about the origin; algebraically, all odd functions have the property $f(-x) = -f(x)$



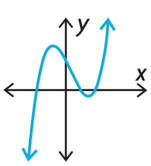
even function

any function that is symmetric about the y-axis; algebraically, all even functions have the property $f(-x) = f(x)$



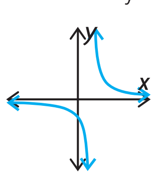
continuous function

any function that does not contain any holes or breaks over its entire domain



discontinuity

a break in the graph of a function is called a point of discontinuity



C. State whether each parent function is an odd function, an even function, or neither.

D. Do any of the functions have vertical or horizontal asymptotes? If so, what are the equations of these asymptotes?

E. Which graphs are continuous? Which have discontinuities?

F. Complete the following statements to describe the end behaviour of each parent function.

a) As x increases to large positive values, y ...

b) As x decreases to large negative values, y ...

Communication *Tip*

It is often convenient to use the symbol for infinity, ∞, and the following notation to write the end behaviour of a function:

- For "As x increases to large positive values, y ...," write "As $x \rightarrow \infty$, $y \rightarrow$ ..."
- For "As x decreases to large negative values, y ...," write "As $x \rightarrow -\infty$, $y \rightarrow$ ..."

G. Summarize your findings.

Reflecting

H. Which of the parent functions can be distinguished by their domain? Which can be distinguished by their range? Which can be distinguished by their zeros?

I. An increasing function is one in which the function's values increase from left to right over its entire domain. A decreasing function is one in which the function's values decrease from left to right over its entire domain. Which of the parent functions are increasing functions? Which are decreasing functions?

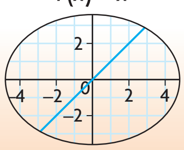
J. Which properties of each function would make the function easy to identify from a description of it?

APPLY the Math

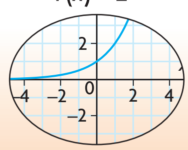
EXAMPLE 1 Connecting the graph of a function with its characteristics

Match each parent function card with a characteristic of its graph. Each card may only be used for one parent function.

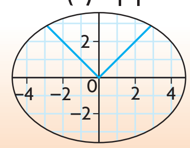
$f(x) = x$



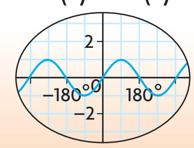
$f(x) = 2^x$



$f(x) = |x|$



$f(x) = \sin(x)$



Range: $\{y \in \mathbb{R} \mid y \geq 0\}$

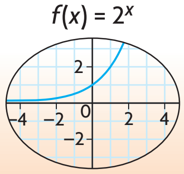
Domain: $\{x \in \mathbb{R}\}$

Infinite Number of Zeros

As $x \rightarrow -\infty$, $y \rightarrow 0$.

Solution

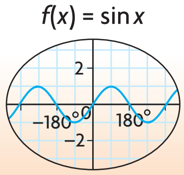
$f(x) = 2^x$



As $x \rightarrow -\infty$, $y \rightarrow 0$.

This property describes the end behaviour: as x becomes negatively large, y approaches zero. The function must have a horizontal asymptote defined by $y = 0$. The function must be $y = 2^x$.

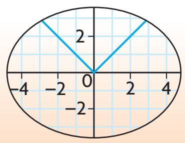
$f(x) = \sin x$



Infinite Number of Zeros

The sine function is periodic and continues infinitely, intersecting the x-axis an infinite number of times.

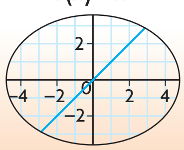
$f(x) = |x|$



Range:
$\{y \in R \mid y \geq 0\}$

The range $\{y \in \mathbf{R} \mid y \geq 0\}$ indicates that all y-values of the function must be non-negative. This is true for both $f(x) = 2^x$ and $f(x) = |x|$. However, $f(x) = 2^x$ has the x-axis as its horizontal asymptote, so $y \neq 0$. Choose $f(x) = |x|$.

$f(x) = x$



Domain:
$\{x \in R\}$

The remaining function, $f(x) = x$, matches the property $\{x \in \mathbf{R}\}$.

If you are given some characteristics of a function, you may be able to determine the equation of the function.

EXAMPLE 2 Using reasoning to determine the equation of a parent function

State which of the parent functions in this lesson have the following characteristics:

a) Domain $= \{x \in \mathbf{R}\}$

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

Solution

a) Domain $= \{x \in \mathbf{R}\}$

$f(x) = x$

$g(x) = x^2$

~~$h(x) = \frac{1}{x}$~~ (Domain $= \{x \in \mathbf{R} \mid x \neq 0\}$)

$k(x) = |x|$

~~$m(x) = \sqrt{x}$~~ (Domain $= \{x \in \mathbf{R} \mid x \geq 0\}$)

$p(x) = 2^x$

$q(x) = \sin x$

There are five parent functions that match this characteristic and two that do not.

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

~~$f(x) = x$~~ (Range $= \{y \in \mathbf{R}\}$)

~~$g(x) = x^2$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

~~$k(x) = |x|$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

~~$p(x) = 2^x$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

~~$q(x) = \sin x$~~

Of these five functions, only the sine function has the range $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$.

Visualizing what the graph of a function looks like can help you remember some of the characteristics of the function.

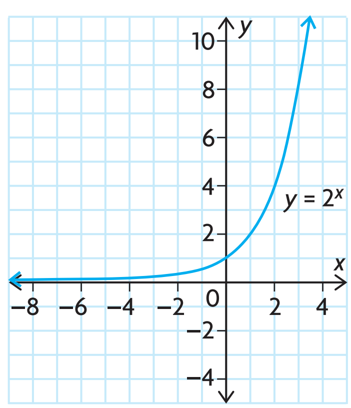
EXAMPLE 3 Connecting the characteristics of a function with its equation

Which of the following are characteristics of the parent function $p(x) = 2^x$? Justify your reasoning.

a) The graph is decreasing for all values in the domain of $p(x)$.
b) The graph is continuous for all values in the domain of $p(x)$.
c) The function $p(x)$ is an even function.
d) The function $p(x)$ has no zeros.

Solution

$p(x) = 2^x$



The function $p(x)$ is an exponential function with a base that is greater than 1.

This type of function is increasing for all values in its domain.

a) This function is increasing for all values in the domain of $p(x)$.

b) The graph is continuous for all values in the domain of $p(x)$.

This function has no breaks.

c) The function $p(x)$ is not an even function.

This type of function is not symmetric about the y-axis. $f(-x) = 2^{-x}$. This substitution does not result in $f(x)$.

d) The function $p(x)$ has no zeros.

As x approaches negative infinity, the graph gets arbitrarily close to the x-axis but does not intersect it.

Only b) and d) are characteristics of $p(x)$.

EXAMPLE 4 Connecting the characteristics of a function with its equation and its graph

Determine a possible transformed parent function that has the following characteristics, and sketch the function:

- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y \geq -2\}$
- decreasing on the interval $(-\infty, 0)$
- increasing on the interval $(0, \infty)$

Communication *Tip*

The interval $(-\infty, 0)$ is described using interval notation and is equivalent to $x < 0$ in set notation. The use of round brackets in interval notation indicates that the endpoint is not included in the interval. The use of square brackets in interval notation indicates that the endpoint is included in the interval. For example, $[-3, 5)$ is equivalent to $-3 \leq x < 5$.

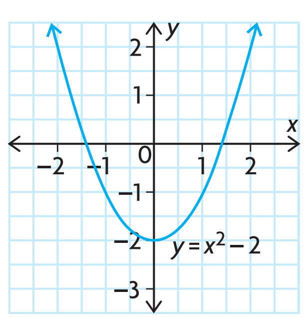
Solution

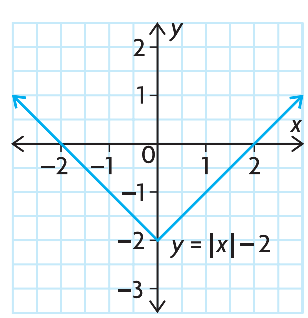
~~$f(x) = x$~~
$g(x) = x^2$
$k(x) = |x|$
~~$p(x) = 2^x$~~
~~$q(x) = \sin x$~~

List the functions that have domain $\{x \in \mathbf{R}\}$. Eliminate the functions that cannot have the range $\{y \in \mathbf{R} \mid y \geq -2\}$. Each of the remaining functions can be translated down two units to have this range.

Function	Intervals of Increase	Intervals of Decrease
$g(x) = x^2$	$(0, \infty)$	$(-\infty, 0)$
$k(x) = \lvert x \rvert$	$(0, \infty)$	$(-\infty, 0)$

State the intervals of increase and decrease for the two remaining functions. Check to see if these intervals match the given conditions. There are two possible parent functions that have the given characteristics.





Sketch the graph of each parent function shifted 2 units down.

In Summary

Key Idea

Functions can be categorized based on their graphical characteristics:

- domain and range
- x-intercepts and y-intercepts
- continuity and discontinuity
- intervals of increase and decrease
- symmetry (even/odd)
- end behaviour

Need to Know

- Given a set of graphical characteristics, the type of function that has these characteristics can be determined by eliminating those that do not have these characteristics.
- Some characteristics are more helpful than others when determining the type of function.

CHECK Your Understanding

1. Which graphical characteristic is the least helpful for differentiating among the parent functions? Why?

2. Which graphical characteristic is the most helpful for differentiating among the parent functions? Why?

3. One of the seven parent functions examined in this lesson is transformed to yield a graph with these characteristics:

- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y > 2\}$
- As $x \to -\infty$, $y \to 2$.

What is the equation of the transformed function?

PRACTISING

4. **K** For each pair of functions, give a characteristic that the two functions have in common and a characteristic that distinguishes between them.

a) $f(x) = \frac{1}{x}$ and $g(x) = x$

b) $f(x) = \sin x$ and $g(x) = x$

c) $f(x) = x$ and $g(x) = x^2$

d) $f(x) = 2^x$ and $g(x) = |x|$

5. For each function, determine $f(-x)$ and $-f(-x)$ and compare it with $f(x)$. Use this to decide whether each function is even, odd, or neither.

a) $f(x) = x^2 - 4$

b) $f(x) = \sin x + x$

c) $f(x) = \frac{1}{x} - x$

d) $f(x) = 2x^3 + x$

e) $f(x) = 2x^2 - x$

f) $f(x) = |2x + 3|$

6. **A** Determine a possible parent function that could serve as a model for each of the following situations, and explain your choice.

a) The number of marks away from the class average that a student's test score is

b) The height of a person above the ground during several rotations of a Ferris wheel

c) The population of Earth throughout time

d) The amount of total money saved if you put aside exactly one dollar every day

7. Identify a parent function whose graph has the given characteristics.

a) The domain is not all real numbers, and $f(0) = 0$.

b) The graph has an infinite number of zeros.

c) The graph is even and has no sharp corners.

d) As x gets negatively large, so does y. As x gets positively large, so does y.

8. Each of the following situations involves a parent function whose graph has been translated. Draw a possible graph that fits the situation.

a) The domain is $\{x \in \mathbf{R}\}$, the interval of increase is $(-\infty, \infty)$, and the range is $\{f(x) \in \mathbf{R} \mid f(x) > -3\}$.

b) The range is $\{g(x) \in \mathbf{R} \mid 2 \leq g(x) \leq 4\}$.

c) The domain is $\{x \in \mathbf{R} \mid x \neq 5\}$, and the range is $\{h(x) \in \mathbf{R} \mid h(x) \neq -3\}$.

9. Sketch a possible graph of a function that has the following characteristics:

- $f(0) = -1.5$
- $f(1) = 2$
- There is a vertical asymptote at $x = -1$.
- As x gets positively large, y gets positively large.
- As x gets negatively large, y approaches zero.

10. **T** **a)** $f(x)$ is a quadratic function. The graph of $f(x)$ decreases on the interval $(-\infty, -2)$ and increases on the interval $(2, \infty)$. It has a y-intercept at $(0, 4)$. What is a possible equation for $f(x)$?

b) Is there only one quadratic function, $f(x)$, that has the characteristics given in part a)?

c) If $f(x)$ is an absolute value function that has the characteristics given in part a), is there only one such function? Explain.

11. $f(x) = x^2$ and $g(x) = |x|$ are similar functions. How might you describe the difference between the two graphs to a classmate, so that your classmate can tell them apart?

12. Copy and complete the following table. In your table, highlight the graphical characteristics that are unique to each function and could be used to distinguish it easily from other parent functions.

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \lvert x \rvert$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain							
Range							
Intervals of Increase							
Intervals of Decrease							
Location of Discontinuities and Asymptotes							
Zeros							
y-Intercepts							
Symmetry							
End Behaviours							

13. C Linear, quadratic, reciprocal, absolute value, square root, exponential, and sine functions are examples of different types of functions, with different properties and characteristics. Why do you think it is useful to name these different types of functions?

Extending

14. Consider the parent function $f(x) = x^3$. Graph $f(x)$, and compare and contrast this function with the parent functions you have learned about in this lesson.

15. Explain why it is not necessary to have $h(x) = \cos(x)$ defined as a parent function.

16. Suppose that $g(x) = |x|$ is translated around the coordinate plane. How many zeros can its graph have? Discuss all possibilities, and give an example of each.

Mid-Chapter Review

FREQUENTLY ASKED Questions

Study | ***Aid***

- See Lesson 1.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: What is a function, and which of its representations is the best for solving problems and making predictions?

A: A function is a relation between two variables, in which each input has a unique output. Functions can be represented using words, graphs, numbers, and algebra.

<table>
<tr><th>Word Example</th><th>Graphical Example</th><th colspan="2">Numerical Example</th><th>Algebraic Example</th></tr>
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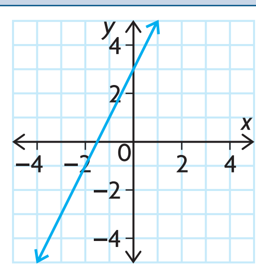
<td>One number is three more than twice another number.</td>
<td>
</td>
<td>Table of values:
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>−4</td><td>−5</td></tr>
<tr><td>−3</td><td>−3</td></tr>
<tr><td>−2</td><td>−1</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>3</td></tr>
<tr><td>1</td><td>5</td></tr>
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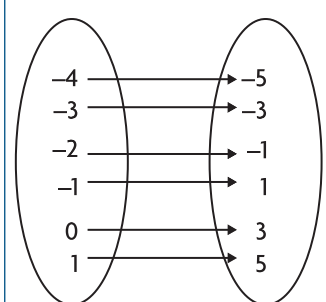

</td>
<td>$f(x) = 2x + 3$</td>
</tr>
</table>

The algebraic model is the most useful and most accurate. If you know the value of one variable, you can substitute this value into the function to create an equation, which can then be solved using an appropriate strategy. This leads to an accurate answer. Both numerical and graphical models are limited in their use because they represent the function for only small intervals of the domain and range. When using a graphical model, it may be necessary to interpolate or extrapolate. This can lead to approximate answers.

Study | ***Aid***

- See Lesson 1.2.
- Try Mid-Chapter Review Questions 4 and 5.

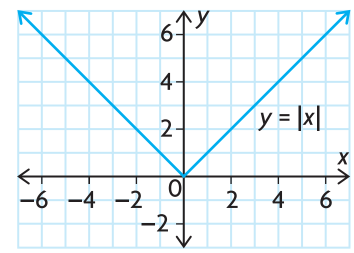
Q: What is the absolute value function, and what are the characteristics of its graph?

A: The absolute value function is $f(x) = |x|$. On a number line, $|x|$ is the distance of any value, x, from the origin. The absolute value function consists of two linear pieces, each defined by a different equation:

$$f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

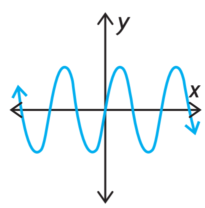
This function has the following characteristics:

- x-intercept: $x = 0$
- y-intercept: $y = 0$
- domain: $D = \{x \in \mathbf{R}\}$; range: $\mathbf{R} = \{y \in \mathbf{R} \mid y \geq 0\}$
- interval of decrease: $(-\infty, 0)$; interval of increase: $(0, \infty)$
- end behaviour: As $x \to \infty, y \to \infty$; as $x \to -\infty, y \to \infty$.

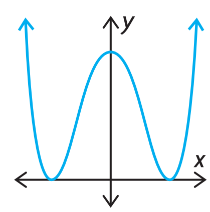


Q: What is the difference between an odd function and an even function, and how are the parent functions differentiated by this characteristic?

A: The graph of an odd function has rotational symmetry about the origin.



The graph of an even function is symmetric about the y-axis.



To test algebraically whether a function is odd or even, substitute $-x$ for x and simplify:

- If $f(-x) = -f(x)$, then the function is odd.
- If $f(-x) = f(x)$, then the function is even.

Odd Parent Functions: $f(x) = x, f(x) = \frac{1}{x}, f(x) = \sin x$

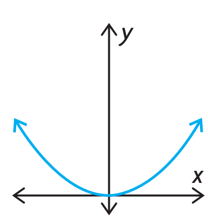
Even Parent Functions: $f(x) = x^2, f(x) = |x|, f(x) = \cos x$

Study Aid

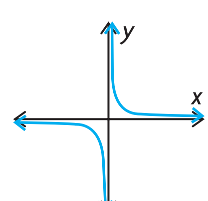
- See Lesson 1.3, Examples 3 and 4.
- Try Mid-Chapter Review Questions 6, 7, and 8.

Q: What is a discontinuity, and what is a continuous function?

A: A discontinuity is a break in the graph of a function. A function is continuous if it has no discontinuities; that is, no holes or breaks in its graph over its entire domain.



a continuous function



The function $y = \frac{1}{x}$ has a discontinuity at $x = 0$.

Study Aid

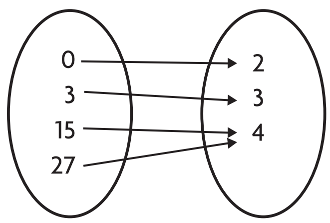
- See Lesson 1.3.
- Try Mid-Chapter Review Question 9.

PRACTICE Questions

Lesson 1.1

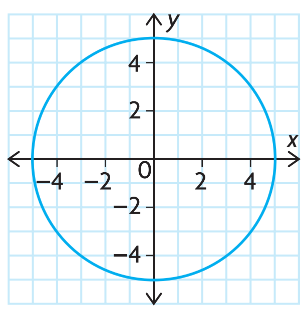
1. Determine whether each relation is a function, and state its domain and range.

a)



b) $y = 2x + 3$

c)



d) $\{(2, 7), (1, 3), (2, 6), (10, -1)\}$

2. The height of a bungee jumper above the ground is modelled by the following data.

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Height (m)	50	40	30	20	10	20	30	40	45	35	25

a) Is the relationship between height and time a function? Explain.
b) What is the domain?
c) What is the range?

3. Determine the domain and range for each of the following and state whether it is a function:

a) $f(x) = 3x + 1$ **c)** $y = \sqrt{5 - x}$
b) $x^2 + y^2 = 9$ **d)** $x^2 - y = 2$

Lesson 1.2

4. Arrange the following values in order, from least to greatest:
$|-3|, -|3|, |5|, |-4|, |0|$

5. Sketch the graph of each function.

a) $f(x) = |x| + 3$ **c)** $f(x) = |-2x|$
b) $f(x) = |x| - 2$ **d)** $f(x) = |0.5x|$

Lesson 1.3

6. Determine a parent function that matches each set of characteristics.

a) The graph is neither even nor odd, and as $x \to \infty, y \to \infty$.
b) $(-\infty, 0)$ and $(0, \infty)$ are both intervals of decrease.
c) The domain is $[0, \infty)$.

7. Determine algebraically if each function is even, odd, or neither.

a) $f(x) = |2x|$ **c)** $f(x) = x + 4$
b) $f(x) = (-x)^2$ **d)** $f(x) = 4x^5 + 3x^3 - 1$

8. Each set of characteristics describes a parent function that has been shifted. Draw a possible graph, and state whether the graph is continuous.

a) There is a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 3$.
b) The range is $\{f(x) \in \mathbf{R} | -3 \le f(x) \le -1\}$.
c) The interval of increase is $(-\infty, \infty)$, and there is a horizontal asymptote at $y = -10$.

9. Sketch a graph that has the following characteristics:

- The function is odd.
- The function is continuous.
- The function has zeros at $x = -3$, 0, and 3.
- The function is increasing on the intervals $x \in (-\infty, -2)$ or $x \in (2, \infty)$.
- The function is decreasing on the interval $x \in (-2, 2)$.

1.4 Sketching Graphs of Functions

GOAL

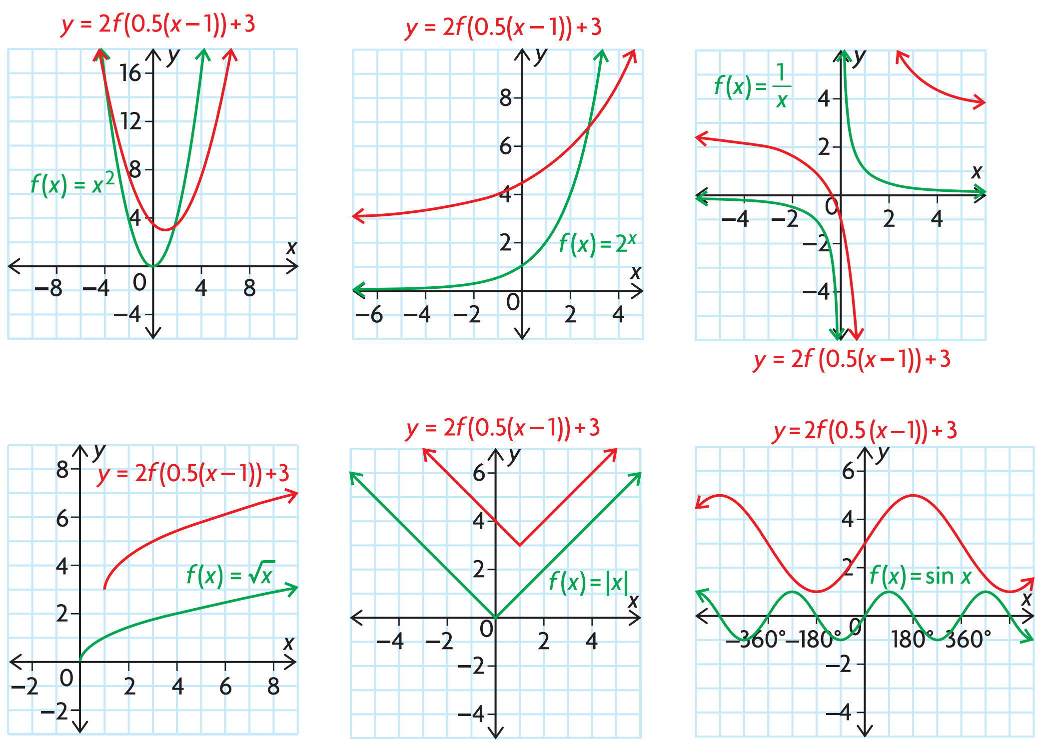
Apply transformations to parent functions, and use the most efficient methods to sketch the graphs of the functions.

YOU WILL NEED

- graph paper
- graphing calculator

INVESTIGATE the Math

The same transformations have been applied to six different parent functions, as shown below.



? How do the transformations defined by $y = 2f(0.5(x - 1)) + 3$ affect the characteristics of each parent function?

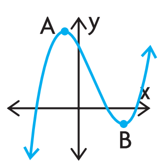
A. Identify the parent function for each graph.

B. Copy and complete the following table for each parent function.

Parent Fuction	$y = x^2$	$y = \frac{1}{x}$	$y = \lvert x \rvert$	$y = 2^x$	$y = \sqrt{x}$	$y = \sin x$
Domain						
Range						
Intervals of Increase						
Intervals of Decrease						
Turning Points						

turning point

a point on a curve where the function changes from increasing to decreasing, or vice versa; for example, A and B are turning points on the following curve



C. Identify the transformations (in the correct order) that were performed on each parent function to arrive at the transformed function.

D. State the transformation(s) that affected each of the following characteristics for each of the parent functions in the table above.

i) domain
ii) range
iii) intervals of increase/decrease
iv) turning points
v) the equation(s) of any vertical asymptotes
vi) the equation(s) of any horizontal asymptotes

E. What transformations to the graph of $y = f(x)$ result in the graph of $y = -\frac{1}{2}f(x + 2) - 1$?

Reflecting

F. For which parent functions are the domain, range, intervals of increase/decrease, and turning points affected when their graphs are transformed?

G. Describe the most efficient order that can be used to graph a transformed function when performing multiple transformations.

H. The most general equation of a transformed function is $y = af(k(x - d)) + c$, where a, k, c, and d are real numbers. Describe the transformations that would be performed on the parent function $y = f(x)$ in terms of the parameters a, k, c, and d.

APPLY the Math

EXAMPLE 1 Connecting transformations to the equation of a function

State the function that would result from vertically compressing $y = f(x)$ by a factor of $\frac{1}{2}$ and then translating the graph 5 units to the right.

Solution

$y = \frac{1}{2}f(x)$ ← This is the function that has a vertical compression by a factor of $\frac{1}{2}$.

$y = \frac{1}{2}f(x - 5)$ ← This is the function has also has a translation 5 units to the right.

EXAMPLE 2 Connecting transformations to the characteristics of a function

Use transformations to help you describe the characteristics of the transformed function $y = 3\sqrt{x} - 2$.

Solution

In the general function $y = af(k(x - d)) + c$, the parameters k and d affect the x-coordinates of each point on the parent function, and the parameters a and c affect the y-coordinates. Each point (x, y) on the parent function is mapped onto $\left(\frac{x}{k} + d, ay + c\right)$ on the transformed function.

← The parameters k and a are related to stretches/compressions and reflections, while the parameters d and c are related to translations. Since division and multiplication must be performed before addition, all stretches/compression and reflections must be applied before any translations, due to the order of operations.

The equation $y = 3\sqrt{x} - 2$ indicates that two transformations have been applied to the parent function $y = \sqrt{x}$:

← In this equation, $a = 3$ and $c = -2$.

1. a vertical stretch by a factor of 3
2. a vertical translation 2 units down

$(x, y) \rightarrow (x, 3y)$

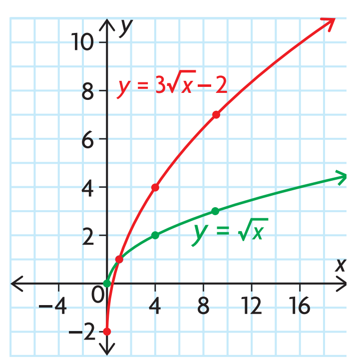
Parent Function $y = \sqrt{x}$	Stretched Function $y = 3\sqrt{x}$
(0, 0)	(0, 3(0)) = (0, 0)
(1, 1)	(1, 3(1)) = (1, 3)
(4, 2)	(4, 3(2)) = (4, 6)
(9, 3)	(9, 3(3)) = (9, 9)

Vertically stretching the graph by a factor of 3 occurs when all the *y*-coordinates on the graph of the parent function are multiplied by 3.

$(x, 3y) \rightarrow (x, 3y - 2)$

Stretched Function $y = 3\sqrt{x}$	Final Transformed Function $y = 3\sqrt{x} - 2$
(0, 0)	(0, 0 − 2) = (0, −2)
(1, 3)	(1, 3 − 2) = (1, 1)
(4, 6)	(4, 6 − 2) = (4, 4)
(9, 9)	(9, 9 − 2) = (9, 7)

Translating the graph 2 units down occurs when 2 is subtracted from all the *y*-coordinates on the graph of the stretched function.



Plot the key points of $y = \sqrt{x}$ and the new points of the transformed function.

Since the domain of both the parent function and transformed function is the same, the interval of increase is also the same: $[0, \infty)$. The difference occurs in the range. The *y*-values of the transformed function increase faster than the *y*-values of the parent function.

These two transformations act on the *y* values only; there is no change to the *x* values. The domain is unchanged; it is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range changes from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \geq -2\}$.

EXAMPLE 3 Reasoning about the characteristics of a transformed function

Graph the function $f(x) = \cos(x)$ and the transformed function $y = 2f(3x)$, where $0° \le x \le 360°$. State the impact of the transformations on the domain, range, intervals of increase/decrease, and turning points of the transformed function.

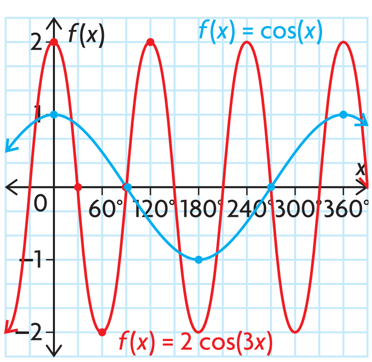
Solution

$(x, y) \rightarrow \left(\frac{1}{3}x, 2y\right)$

Parent Function $y = \cos(x)$	Final Transformed Function $y = 2\cos(3x)$
$(0°, 1)$	$\left(\frac{1}{3}(0°), 2(1)\right) = (0°, 2)$
$(90°, 0)$	$\left(\frac{1}{3}(90°), 2(0)\right) = (30°, 0)$
$(180°, -1)$	$\left(\frac{1}{3}(180°), 2(-1)\right) = (60°, -2)$
$(270°, 0)$	$\left(\frac{1}{3}(270°), 2(0)\right) = (90°, 0)$
$(360°, 1)$	$\left(\frac{1}{3}(360°), 2(1)\right) = (120°, 2)$

Apply a horizontal compression by a factor of $\frac{1}{3}$ and a vertical stretch by a factor of 2.

On the graph of $f(x) = \cos(x)$, multiply the x-coordinates by $\frac{1}{3}$ and the y-coordinates by 2.



Plot the key points of the parent function and the transformed points.

Within the specified domain,

- the transformed function decreases on the intervals $(0°, 60°)$, $(120°, 180°)$, and $(240°, 300°)$ and increases on the intervals $(60°, 120°)$, $(180°, 240°)$, and $(300°, 360°)$
- the transformed function has the following turning points: $(60°, -2)$, $(120°, 2)$, $(180°, -2)$, $(240°, 2)$, and $(300°, -2)$

The domain consists of all real numbers; this is not changed by the horizontal compression and translation.
Domain $= \{x \in \mathbf{R}\}$.
The vertical stretch has changed the range from $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$ to $\{y \in \mathbf{R} \mid -2 \le y \le 2\}$.

EXAMPLE 4 Reasoning about the order of transformations

Describe the order in which you would apply the transformations defined by $y = -2f(3(x + 1)) - 4$ to $f(x) = \sqrt{x}$. Then state the impact of the transformations on the domain, range, intervals of increase/decrease, and end behaviours of the transformed function.

Solution

$(x, y) \rightarrow \left(\frac{1}{3}x, -2y\right)$

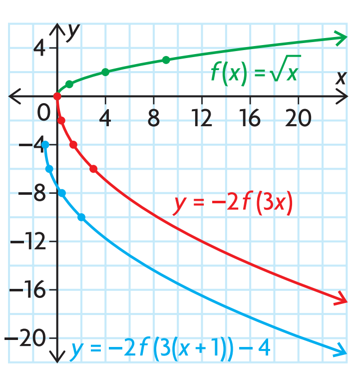
Parent Function $y = \sqrt{x}$	Stretched/Compressed Function $y = -2\sqrt{3x}$
(0, 0)	$\left(\frac{1}{3}(0), -2(0)\right) = (0, 0)$
(1, 1)	$\left(\frac{1}{3}(1), -2(1)\right) = \left(\frac{1}{3}, -2\right)$
(4, 2)	$\left(\frac{1}{3}(4), -2(2)\right) = \left(\frac{4}{3}, -4\right)$
(9, 3)	$\left(\frac{1}{3}(9), -2(3)\right) = (3, -6)$

Since multiplication must be done before addition, apply a horizontal compression by a factor of $\frac{1}{3}$, a vertical stretch by a factor of 2, and a reflection in the x-axis. To do this, multiply the x-coordinates of points on the parent function by $\frac{1}{3}$ and the y-coordinates by -2.

$\left(\frac{1}{3}x, -2y\right) \rightarrow \left(\frac{1}{3}x - 1, -2y - 4\right)$

Stretched/Compressed Function $y = -2\sqrt{3x}$	Final Transformed Function $y = -2\sqrt{3(x + 1)} - 4$
(0, 0)	$(0 - 1, 0 - 4) = (-1, -4)$
$\left(\frac{1}{3}, -2\right)$	$\left(\frac{1}{3} - 1, -2 - 4\right) = \left(-\frac{2}{3}, -6\right)$
$\left(\frac{4}{3}, -4\right)$	$\left(\frac{4}{3} - 1, -4 - 4\right) = \left(\frac{1}{3}, -8\right)$
(3, −6)	$(3 - 1, -6 - 4) = (2, -10)$

Apply all translations next. Translate the graph of $f(x) = -2f(3x)$ 1 unit to the left and 4 units down. To do this, subtract 1 from the x-coordinates and 4 from the y-coordinates of points on the previous function.



The transformed function is now a decreasing function on the interval $[-1, \infty)$.
The transformed function has the following end behaviours:

As $x \rightarrow -1$, $y \rightarrow -4$ and as $x \rightarrow \infty$, $y \rightarrow -\infty$.

Plot the points of the final transformed function. The horizontal translation changed the domain from $\{x \in \mathbf{R} \mid x \geq 0\}$ to $\{x \in \mathbf{R} \mid x \geq -1\}$.

The reflection in the x-axis and the vertical translation changed the range from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \leq -4\}$.

In Summary

Key Ideas

- Transformations on a function $y = af(k(x - d)) + c$ must be performed in a particular order: horizontal and vertical stretches/compressions (including any reflections) must be performed before translations. All points on the graph of the parent function $y = f(x)$ are changed as follows: $(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$
- When using transformations to graph, you can apply a and k together, and then c and d together, to get the desired graph in the fewest number of steps.

Need to Know

- The value of a determines whether there is a vertical stretch or compression, or a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - When $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines whether there is a horizontal stretch or compression, or a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines whether there is a horizontal translation:
  - For $d > 0$, the graph is translated to the right.
  - For $d < 0$, the graph is translated to the left.
- The value of c determines whether there is a vertical translation:
  - For $c > 0$, the graph is translated up.
  - For $c < 0$, the graph is translated down.

CHECK Your Understanding

1. State the transformations defined by each equation in the order they would be applied to $y = f(x)$.

a) $y = f(x) - 1$

b) $y = f(2(x - 1))$

c) $y = -f(x - 3) + 2$

d) $y = -2f(4x)$

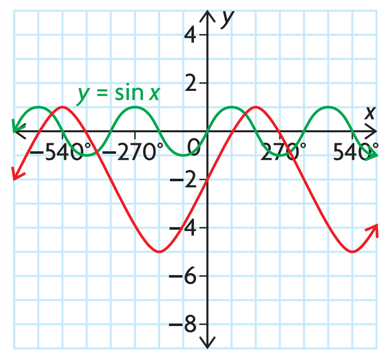
e) $y = -f(-(x + 2)) - 3$

f) $y = \frac{1}{2}f\left(\frac{1}{4}(x - 5)\right) + 6$

2. Identify the appropriate values for a, k, c, and d in $y = af(k(x - d)) + c$ to describe each set of transformations below.

a) horizontal stretch by a factor of 2, vertical translation 3 units up, reflection in the x-axis

b)



3. The point $(2, 3)$ is on the graph of $y = f(x)$. Determine the corresponding coordinates of this point on the graph of $y = -2(f(2(x + 5)) - 4$.

PRACTISING

4. The ordered pairs $(2, 3)$, $(4, 7)$, $(-2, 5)$, and $(-4, 6)$ belong to a function f. List the ordered pairs that belong to each of the following:

a) $y = 2f(x)$

b) $y = f(x - 3)$

c) $y = f(x)+2$

d) $y = f(x + 1) - 3$

e) $y = f(-x)$

f) $y = f(2x) - 1$

5. **K** For each of the following equations, state the parent function and the transformation that was applied. Graph the transformed function.

a) $y = (x + 1)^2$

b) $y = 2|x|$

c) $y = \sin (3x) + 1$

d) $y = \frac{1}{x} + 3$

e) $y = 2^{0.5x}$

f) $y = \sqrt{2(x - 6)}$

6. State the domain and range of each function in question 5.

7. a) Graph the parent function $y = 2^x$ and the transformed function defined by $y = -2f(3(x - 1)) + 4$.

b) State the impact of the transformations on the domain and range, intervals of increase/decrease, and end behaviours.

c) State the equation of the transformed function.

8. The graph of $y = \sqrt{x}$ is stretched vertically by a factor of 3, reflected in the x-axis, and shifted 5 units to the right. Determine the equation that results from these transformations, and graph it.

9. The point $(1, 8)$ is on the graph of $y = f(x)$. Find the corresponding coordinates of this point on each of the following graphs.

a) $y = 3f(x - 2)$
b) $y = f(2(x + 1)) - 4$
c) $y = -2f(-x) - 7$
d) $y = -f(4(x + 1))$
e) $y = -f(-x)$
f) $y = 0.5f(0.5(x + 3)) + 3$

10. **A** Given $f(x) = \sqrt{x}$, find the domain and range for each of the following:

a) $g(x) = f(x - 2)$
b) $h(x) = 2f(x - 1) + 4$
c) $k(x) = f(-x) + 1$
d) $j(x) = 3f(2(x - 5)) - 3$

11. Greg thinks that the graphs of $y = 5x^2 - 3$ and $y = 5(x^2 - 3)$ are the same. Explain why he is incorrect.

12. Given $f(x) = x^3 - 3x^2$, $g(x) = f(x - 1)$, and $h(x) = -f(x)$, graph each function and compare $g(x)$ and $h(x)$ with $f(x)$.

13. **T** Consider the parent function $y = x^2$.

a) Describe the transformation that produced the equation $y = 4x^2$.
b) Describe the transformation that produced the equation $y = (2x)^2$.
c) Show algebraically that the two transformations produce the same equation and graph.

14. **C** Use a flow chart to show the sequence and types of transformations required to transform the graph of $y = f(x)$ into the graph of $y = af(k(x - d)) + c$.

Extending

15. The point $(3, 6)$ is on the graph of $y = 2f(x + 1) - 4$. Find the original point on the graph of $y = f(x)$.

16. **a)** Describe the transformations that produce $y = f(3(x + 2))$.
b) The graph of $y = f(3x + 6)$ is produced by shifting 6 units to the left and then compressing the graph by a factor of $\frac{1}{3}$. Why does this produce the same result as the transformations you described in part a)?
c) Using $f(x) = x^2$ as the parent function, graph the transformations described in parts a) and b) to show that they result in the same transformed function.

1.5 Inverse Relations

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Determine the equation of an inverse relation and the conditions for an inverse relation to be a function.

LEARN ABOUT the Math

The owners of a candy company are creating a spherical container to hold their small chocolates. They are trying to decide what size to make the sphere and how much volume the sphere will hold, based on its radius.

The volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

? How can you use this relationship to find the radius of any sphere for a given volume?

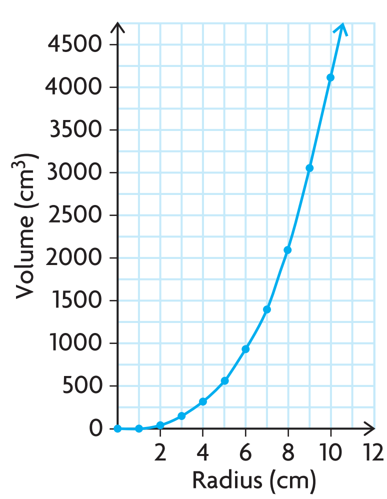
EXAMPLE 1 Representing the inverse using a table of values and a graph

Use a table of values and a graphical model to represent the relationship between the radius of a sphere and any given volume.

Solution

$$V = \frac{4}{3}\pi r^3$$

Radius (cm)	Volume (cm^3)
0.0	0.0
1.0	4.2
2.0	33.5
3.0	113.1
4.0	268.1
5.0	523.6
6.0	904.8
7.0	1436.8
8.0	2144.7
9.0	3053.6
10.0	4188.8

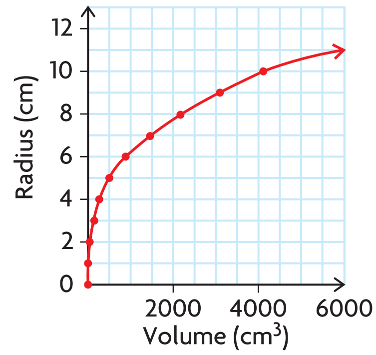


Radius is the independent variable, and volume is the dependent variable.

Create a table of values, and calculate the volume for a specific radius.

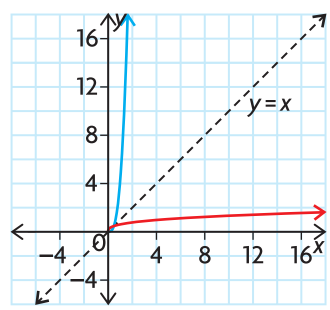
Draw a scatter plot of volume in terms of radius. Draw a smooth curve through the points since the function is continuous.

Volume (cm³)	Radius (cm)
0.0	0
4.2	1
33.5	2
113.1	3
268.1	4
523.6	5
904.8	6
1436.8	7
2144.7	8
3053.6	9
4188.8	10



To graph radius in terms of volume, switch the variables in the table, making radius the dependent variable and volume the independent variable.

The red curve shows volume as the independent variable and radius as the dependent variable.



If we ignore units and plot both relations on the same graph, the red curve is a reflection of the blue curve in the line $y = x$. This is reasonable, given that the x-values and y-values were switched on the graph. The red curve is the inverse relation, and it is also a function.

The inverse was found by switching the independent and dependent variables in the table of values. The independent and dependent variables can also be switched in the equation of the relation to determine the equation of the inverse relation.

EXAMPLE 2 Representing the inverse using an equation

Recall that the volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

Determine the equation of the inverse.

Solution

$$V = \frac{4}{3}\pi r^3$$

To express V in terms of r, rearrange the formula using inverse operations.

$$3 \times V = 3 \times \left(\frac{4}{3}\pi r^3\right)$$

Multiply both sides by 3 to eliminate the fraction.

$$3V = 4\pi r^3$$

$$\frac{3V}{4\pi} = \frac{4\pi r^3}{4\pi}$$

Divide both sides by 4π (the coefficient of r^3) to isolate r^3.
Take the cube root of both sides to isolate r.

$$\sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{r^3}$$

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

The radius is now expressed as a function of volume and can be determined for different values of V.

Reflecting

A. Compare the domain and range of this function and its inverse.

B. Will an **inverse of a function** always be a function? Explain.

C. Why is it reasonable to switch the V and the r in Example 2 to determine the inverse relation?

APPLY the Math

EXAMPLE 3 Using an algebraic strategy to determine the inverse relation

Given $f(x) = x^2$.

a) Find the inverse relation.
b) Compare the domain and range of the function and its inverse.
c) Determine if the inverse relation is also a function.

Solution

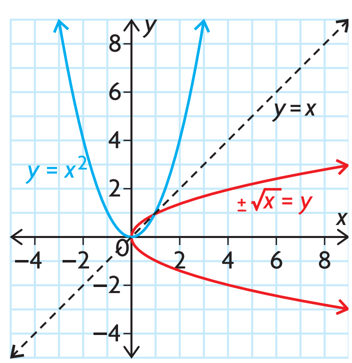
a) $y = x^2$ — Rewrite the function using x and y.

$x = y^2$ — Interchange x and y in the relation.

$\pm\sqrt{x} = \sqrt{y^2}$ — Solve for y by taking the square root of both sides.

$\pm\sqrt{x} = y$

b)



The graph of the inverse relation is a reflection of the original relation in the line $y = x$.

Only non-negative values of x work in the square root function. The square root of a negative number is undefined. Since $\pm$ in the inverse indicates that the output, y, will include both positive and negative values, the range will include all the real numbers.

The domain of $y = x^2$ is $\{x \in \mathbf{R}\}$. The range is $\{x \in \mathbf{R} \mid y \geq 0\}$.

The domain of the inverse relation is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range is $\{y \in \mathbf{R}\}$.

c) The inverse relation is not a function, but it can be split in the middle into the two functions, $y = \sqrt{x}$ and $y = -\sqrt{x}$. — Based on the equation of the inverse relation, each input of x will have two outputs for y, one positive and one negative. The only exception is $x = 0$.

Communication *Tip*

The domain of the square root function is $\{x \in \mathbf{R} \mid x \geq 0\}$; we say the values of x are non-negative. The range of the exponential function $y = 2^x$ is $\{y \in \mathbf{R} \mid y > 0\}$; we say the values of y are positive. The distinction is because zero is neither negative nor positive.

The inverse relation is useful to solve problems, particularly when you are given a value of the dependent variable and need to determine the value of the corresponding independent variable.

EXAMPLE 4 Selecting a strategy that involves the inverse relation to solve a problem

Archaeologists use models for the relationship between height and footprint length to determine the height of a person based on the lengths of the bones they discover. The relationship between height, $h(x)$, in centimetres and footprint length, x, in centimetres is given by $h(x) = 1.1x + 143.6$. Use this relationship to predict the footprint length for a person who is 170 cm tall.

Solution

$h(x) = 1.1x + 143.6$

Let $y = h(x)$.

$y = 1.1x + 143.6$

To predict the footprint length, rewrite the relationship with footprint length as the dependent variable and $h(x)$ as the independent variable.

$x = 1.1y + 143.6$

Interchange x and y.

$x - 143.6 = 1.1y$

$\dfrac{x - 143.6}{1.1} = y = h^{-1}(x)$

Solve for y.

$h^{-1}(170) = \dfrac{170 - 143.6}{1.1}$

$= 24$ cm

Evaluate $h^{-1}(170)$.

A person who is 170 cm tall may have a footprint length of 24 cm.

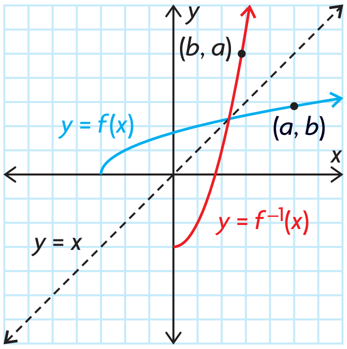
Communication *Tip*

When an inverse relation is also a function, the notation $f^{-1}(x)$ can be used to define the inverse function.

In Summary

Key Ideas

- The inverse function of $f(x)$ is denoted by $f^{-1}(x)$. Function notation can only be used when the inverse is a function.
- The graph of the inverse function is a reflection in the line $y = x$.



(continued)

Need to Know

- Not all inverse relations are functions. The domain and/or range of the original function may need to be restricted to ensure that the inverse of a function is also a function.
- To find the inverse algebraically, write the function equation using y instead of $f(x)$. Interchange x and y. Solve for y.
- If (a, b) represents a point on the graph of $f(x)$, then (b, a) represents a point on the graph of the corresponding f^{-1}.
- Given a table of values or a graph of a function, the independent and dependent variables can be interchanged to get a table of values or a graph of the inverse relation.
- The domain of a function is the range of its inverse. The range of a function is the domain of its inverse.

CHECK Your Understanding

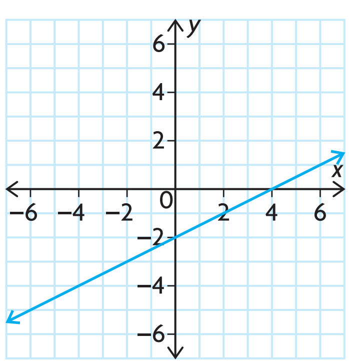
1. Each of the following ordered pairs is a point on a function. State the corresponding point on the inverse relation.

a) $(2, 5)$
b) $(-5, -6)$
c) $(4, -8)$
d) $f(1) = 2$
e) $g(-3) = 0$
f) $h(0) = 7$

2. Given the domain and range of a function, state the domain and range of the inverse relation.

a) $D = \{x \in \mathbf{R}\}, R = \{y \in \mathbf{R}\}$
b) $D = \{x \in \mathbf{R} \mid x \geq 2\}, R = \{y \in \mathbf{R}\}$
c) $D = \{x \in \mathbf{R} \mid x \geq -5\}, R = \{y \in \mathbf{R} \mid y < 2\}$
d) $D = \{x \in \mathbf{R} \mid x < -2\}, R = \{y \in \mathbf{R} \mid -5 < y < 10\}$

3. Match the inverse relations to their corresponding functions.

A


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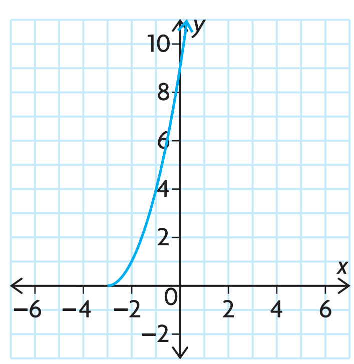

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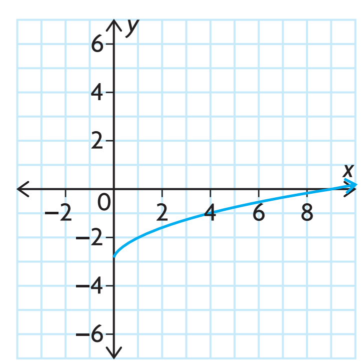

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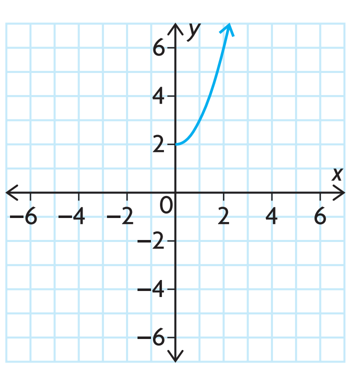

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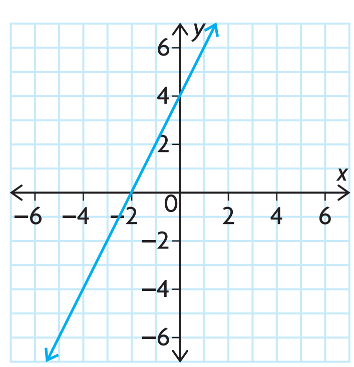

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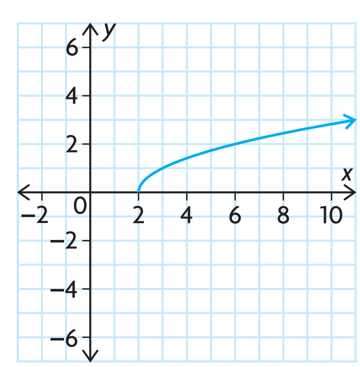


PRACTISING

4. Consider the function $f(x) = 2x^3 + 1$.

K

a) Find the ordered pair $(4, f(4))$ on the function.

b) Find the ordered pair on the inverse relation that corresponds to the ordered pair from part a).

c) Find the domain and range of f.

d) Find the domain and the range of the inverse relation of f.

e) Is the inverse relation a function? Explain.

5. Repeat question 4 for the function $g(x) = x^4 - 8$.

6. Graph each function and its inverse relation on the same set of axes. Determine whether the inverse relation is a function.

a) $f(x) = x^2 + 1$

b) $g(x) = \sin x$, where $-360° \leq x \leq 360°$

c) $h(x) = -x$

d) $m(x) = |x| + 1$

7. **a)** The equation $F = \frac{9}{5}C + 32$ can be used to convert a known Celsius temperature, C, to the equivalent Fahrenheit temperature, F. Find the inverse of this relation, and describe what it can be used for.

A

b) Use the equation given in part a) to convert 20 °C to its equivalent Fahrenheit temperature. Use the inverse relation to convert this Fahrenheit temperature back to its equivalent Celsius temperature.

8. **a)** The formula $A = \pi r^2$ is convenient for calculating the area of a circle when the radius is known. Find the inverse of the relation, and describe what it can be used for.

b) Use the equation given in part a) to calculate the area of a circle with a radius of 5 cm. (Express the area as an exact value in terms of π.) Use the inverse relation to calculate the radius of the circle with the area you calculated.

9. If $f(x) = kx^3 - 1$ and $f^{-1}(15) = 2$, find k.

T

10. Given the function $h(x) = 2x + 7$, find

a) $h(3)$

b) $h(9)$

c) $\dfrac{h(9) - h(3)}{9 - 3}$

d) $h^{-1}(3)$

e) $h^{-1}(9)$

f) $\dfrac{h^{-1}(9) - h^{-1}(3)}{9 - 3}$

11. Suppose that the variable a represents a particular student and $f(a)$ represents the student's overall average in all their subjects. Is the inverse relation of f a function? Explain.

12. Determine the inverse of each function.

a) $f(x) = 3x + 4$
b) $h(x) = -x$
c) $g(x) = x^3 - 1$
d) $m(x) = -2(x + 5)$

13. A function g is defined by $g(x) = 4(x - 3)^2 + 1$.

a) Determine an equation for the inverse of $g(x)$.
b) Solve for y in the equation for the inverse of $g(x)$.
c) Graph $g(x)$ and its inverse using graphing technology.
d) At what points do the graphs of $g(x)$ and its inverse intersect?
e) State **restrictions** on the domain or range of g so that its inverse is a function.
f) Suppose that the domain of $g(x)$ is $\{x \in \mathbf{R} \mid 2 \le x \le 5\}$. Is the inverse a function? Justify your answer.

14. A student writes, "The inverse of $y = -\sqrt{x + 2}$ is $y = x^2 - 2$." Explain why this statement is not true.

15. Do you have to restrict either the domain or the range of the function $y = \sqrt{x + 2}$ to make its inverse a function? Explain.

16. C John and Katie are discussing inverse relationships. John says, "A function is a rule, and the inverse is the rule performed in reverse order with opposite operations. For example, suppose that you cube a number, divide by 4, and add 2. The inverse is found by subtracting 2, multiplying by 4, and taking the cube root." Is John correct? Justify your answer algebraically, numerically, and graphically.

Extending

17. $f(x) = x$ is an interesting function because it is its own inverse. Can you find three more functions that have the same property? Can you convince yourself that there are an infinite number of functions that satisfy this property?

18. The inverse relation of a function is also a function if the original function passes the horizontal line test (in other words, if any horizontal line hits the function in at most one location). Explain why this is true.

1.6 Piecewise Functions

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Understand, interpret, and graph situations that are described by piecewise functions.

LEARN ABOUT the Math

A city parking lot uses the following rules to calculate parking fees:

- A flat rate of \$5.00 for any amount of time up to and including the first hour
- A flat rate of \$12.50 for any amount of time over 1 h and up to and including 2 h
- A flat rate of \$13 plus \$3 per hour for each hour after 2 h

? How can you describe the function for parking fees in terms of the number of hours parked?

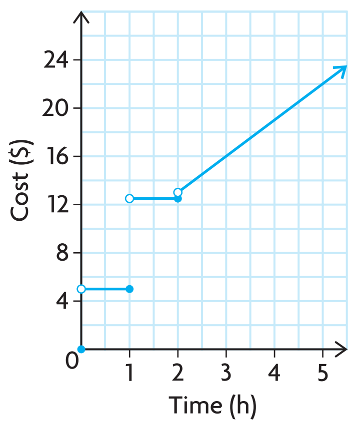
EXAMPLE 1 Representing the problem using a graphical model

Use a graphical model to represent the function for parking fees.

Solution

Time (h)	Parking Fee (\$)
0	0
0.25	5.00
0.50	5.00
1.00	5.00
1.25	12.50
1.50	12.50
2.00	12.50
2.50	14.50
3.00	16.00
4.00	19.00

Create a table of values.



The domain of this piecewise function is $x \geq 0$.

The function is linear over the domain, but it is discontinuous at $x = 0, 1,$ and 2.

Plot the points in the table of values. Use a solid dot to include a value in an interval. Use an open dot to exclude a value from an interval.

There is a solid dot at (0, 0) and an open dot at (0, 5) because the parking fee at 0 h is $0.00.

There is a closed dot at (1, 5) and an open dot at (1, 12.50) because the parking fee at 1 h is $5.00.

There is a closed dot at (2, 12.50) and an open dot at (2, 13) because the parking fee at 2 h is $12.50.

The last part of the graph continues in a straight line since the rate of change is constant after 2 h.

piecewise function
a function defined by using two or more rules on two or more intervals; as a result, the graph is made up of two or more pieces of similar or different functions

Each part of a piecewise function can be described using a specific equation for the interval of the domain.

EXAMPLE 2 Representing the problem using an algebraic model

Use an algebraic model to represent the function for parking fees.

Solution

$y_1 = 0$ if $x = 0$

$y_2 = 5$ if $0 < x \leq 1$

$y_3 = 12.50$ if $1 < x \leq 2$

$y_4 = 3x + 13$ if $x > 2$

Write the relation for each rule.

$$f(x) = \begin{cases} 0, \text{ if } x = 0 \\ 5, \text{ if } 0 < x \leq 1 \\ 12.50, \text{ if } 1 < x \leq 2 \\ 3x + 13, \text{ if } x > 2 \end{cases}$$

Combine the relations into a piecewise function.

The domain of the function is $x \geq 0$.

The function is discontinuous at $x = 0$, 1, and 2 because there is a break in the function at each of these points.

Reflecting

A. How do you sketch the graph of a piecewise function?

B. How do you create the algebraic representation of a piecewise function?

C. How do you determine from a graph or from the algebraic representation of a piecewise function if there are any discontinuities?

APPLY the Math

EXAMPLE 3 Representing a piecewise function using a graph

Graph the following piecewise function.

$$f(x) = \begin{cases} x^2, \text{ if } x < 2 \\ 2x + 3, \text{ if } x \geq 2 \end{cases}$$

Solution

Create a table of values.

$f(x) = x^2$

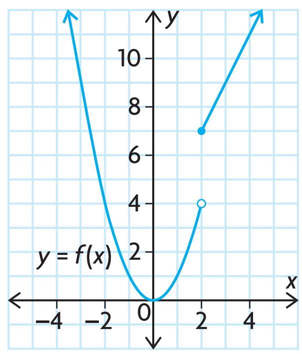
x	*f(x)*
−2	4
−1	1
0	0
1	1
2	4

$f(x) = 2x + 3$

x	*f(x)*
2	7
3	9
4	11
5	13
6	15

From the equations given, the graph consists of part of a parabola that opens up and a line that rises from left to right.

Both tables include $x = 2$ since this is where the description of the function changes.



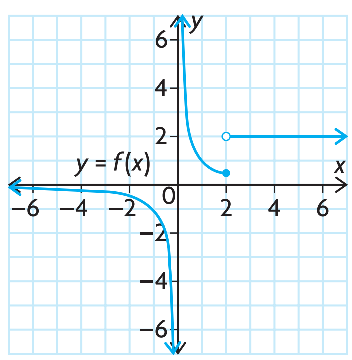
Plot the points, and draw the graph.

A solid dot is placed at (2, 7) since $x = 2$ is included with $f(x) = 2x + 3$. An open dot is placed at (2, 4) since $x = 2$ is excluded from $f(x) = x^2$.

$f(x)$ is discontinuous at $x = 2$.

EXAMPLE 4 Representing a piecewise function using an algebraic model

Determine the algebraic representation of the following piecewise function.



Solution

$$f(x) = \begin{cases} \frac{1}{x}, \text{ if } x \leq 2 \\ 2, \text{ if } x > 2 \end{cases}$$

The graph is made up of two pieces. One piece is part of the reciprocal function defined by $y = \frac{1}{x}$ when $x \leq 2$. The other piece is a horizontal line defined by $y = 2$ when $x > 2$. The solid dot indicates that point $\left(2, \frac{1}{2}\right)$ belongs with the reciprocal function.

EXAMPLE 5 Reasoning about the continuity of a piecewise function

Is this function continuous at the points where it is pieced together? Explain.

$$g(x) = \begin{cases} x + 1, \text{ if } x \leq 0 \\ 2x + 1, \text{ if } 0 < x < 3 \\ 4 - x^2, \text{ if } x \geq 3 \end{cases}$$

Solution

The function is continuous at the points where it is pieced together if the functions being joined have the same y-values at these points.

The graph is made up of three pieces. One piece is part of an increasing line defined by $y = x + 1$ when $x \leq 0$. The second piece is an increasing line defined by $y = 2x + 1$ when $0 < x < 3$. The third piece is part of a parabola that opens down, defined by $y = 4 - x^2$ when $x \geq 3$.

Calculate the values of the function at $x = 0$ using the relevant equations:

$y = x + 1$ $\quad$ $y = 2x + 1$

$y = 0 + 1$ $\quad$ $y = 2(0) + 1$

$y = 1$ $\quad$ $y = 1$

The two y-values are the same, so the two linear pieces join each other at $x = 0$.

Calculate the values of the function at $x = 3$ using the relevant equations:

$y = 2x + 1$ $\quad$ $y = 4 - x^2$

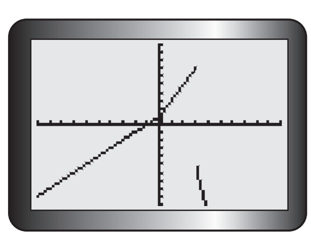
$y = 2(3) + 1$ $\quad$ $y = 4 - 3^2$

$y = 7$ $\quad$ $y = -5$

The two y-values are different, so the second linear piece does not join with the parabola at $x = 3$.

The function is discontinuous, since there is a break in the graph at $x = 3$.

Verify by graphing.



Tech Support

For help using a graphing calculator to graph a piecewise function, see Technical Appendix, T-16.

In Summary

Key Ideas

- Some functions are represented by two or more "pieces." These functions are called piecewise functions.
- Each piece of a piecewise function is defined for a specific interval in the domain of the function.

Need to Know

- To graph a piecewise function, graph each piece of the function over the given interval.
- A piecewise function can be either continuous or not. If all the pieces of the function join together at the endpoints of the given intervals, then the function is continuous. Otherwise, it is discontinuous at these values of the domain.

CHECK *Your Understanding*

1. Graph each piecewise function.

a) $f(x) = \begin{cases} 2, \text{ if } x < 1 \\ 3x, \text{ if } x \geq 1 \end{cases}$

b) $f(x) = \begin{cases} -2x, \text{ if } x < 0 \\ x + 4, \text{ if } x \geq 0 \end{cases}$

c) $f(x) = \begin{cases} |x|, \text{ if } x \leq -2 \\ -x^2, \text{ if } x > -2 \end{cases}$

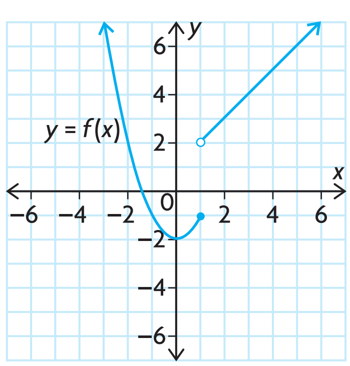
d) $f(x) = \begin{cases} |x + 2|, \text{ if } x \leq -1 \\ -x^2 + 2, \text{ if } x > -1 \end{cases}$

e) $f(x) = \begin{cases} \sqrt{x}, \text{ if } x < 4 \\ 2^x, \text{ if } x \geq 4 \end{cases}$

f) $f(x) = \begin{cases} \dfrac{1}{x}, \text{ if } x < 1 \\ -x, \text{ if } x \geq 1 \end{cases}$

2. State whether each function in question 1 is continuous or not. If not, state where it is discontinuous.

3. Write the algebraic representation of each piecewise function, using function notation.

a)


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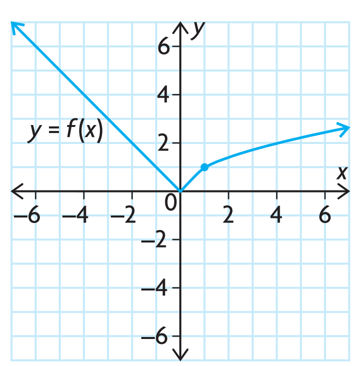


4. State the domain of each piecewise function in question 3, and comment on the continuity of the function.

PRACTISING

5. **K** Graph the following piecewise functions. Determine whether each function is continuous or not, and state the domain and range of the function.

a) $f(x) = \begin{cases} 2, \text{ if } x < -1 \\ 3, \text{ if } x \geq -1 \end{cases}$

b) $f(x) = \begin{cases} -x, \text{ if } x \leq 0 \\ x, \text{ if } x > 0 \end{cases}$

c) $f(x) = \begin{cases} x^2 + 1, \text{ if } x < 2 \\ 2x + 1, \text{ if } x \geq 2 \end{cases}$

d) $f(x) = \begin{cases} 1, \text{ if } x < -1 \\ x + 2, \text{ if } -1 \leq x \leq 3 \\ 5, \text{ if } x > 3 \end{cases}$

6. **A** Graham's long-distance telephone plan includes the first 500 min per month in the \$15.00 monthly charge. For each minute after 500 min, Graham is charged \$0.02. Write a function that describes Graham's total long-distance charge in terms of the number of long distance minutes he uses in a month.

7. Many income tax systems are calculated using a tiered method. Under a certain tax law, the first \$100 000 of earnings are subject to a 35% tax; earnings greater than \$100 000 and up to \$500 000 are subject to a 45% tax. Any earnings greater than \$500 000 are taxed at 55%. Write a piecewise function that models this situation.

8. **T** Find the value of k that makes the following function continuous. Graph the function.

$$f(x) = \begin{cases} x^2 - k, \text{ if } x < -1 \\ 2x - 1, \text{ if } x \geq -1 \end{cases}$$

9. The fish population, in thousands, in a lake at any time, x, in years is modelled by the following function:

$$f(x) = \begin{cases} 2^x, \text{ if } 0 \leq x \leq 6 \\ 4x + 8, \text{ if } x > 6 \end{cases}$$

This function describes a sudden change in the population at time $x = 6$, due to a chemical spill.

a) Sketch the graph of the piecewise function.

b) Describe the continuity of the function.

c) How many fish were killed by the chemical spill?

d) At what time did the population recover to the level it was before the chemical spill?

e) Describe other events relating to fish populations in a lake that might result in piecewise functions.

10. Create a flow chart that describes how to plot a piecewise function with two pieces. In your flow chart, include how to determine where the function is continuous.

11. C An absolute value function can be written as a piecewise function that involves two linear functions. Write the function $f(x) = |x + 3|$ as a piecewise function, and graph your piecewise function to check it.

12. The demand for a new CD is described by

$$D(p) = \begin{cases} \frac{1}{p^2}, \text{ if } 0 < p \leq 15 \\ 0, \text{ if } p > 15 \end{cases}$$

where D is the demand for the CD at price p, in dollars. Determine where the demand function is discontinuous and continuous.

Extending

13. Consider a function, $f(x)$, that takes an element of its domain and rounds it down to the nearest 10. Thus, $f(15.6) = 10$, while $f(21.7) = 20$ and $f(30) = 30$. Draw the graph, and write the piecewise function. You may limit the domain to $x \in [0, 50)$. Why do you think graphs like this one are often referred to as *step functions*?

14. Explain why there is no value of k that will make the following function continuous.

$$f(x) = \begin{cases} 5x, \text{ if } x < -1 \\ x + k, \text{ if } -1 \leq x \leq 3 \\ 2x^2, \text{ if } x > 3 \end{cases}$$

15. The *greatest integer function* is a step function that is written as $f(x) = [x]$, where $f(x)$ is the greatest integer less than or equal to x. In other words, the greatest integer function rounds any number down to the nearest integer. For example, the greatest integer less than or equal to the number [5.3] is 5, while the greatest integer less than or equal to the number $[-5.3]$ is -6. Sketch the graph of $f(x) = [x]$.

16. **a)** Create your own piecewise function using three different transformed parent functions.
b) Graph the function you created in part a).
c) Is the function you created continuous or not? Explain.
d) If the function you created is not continuous, change the interval or adjust the transformations used as required to change it to a continuous function.

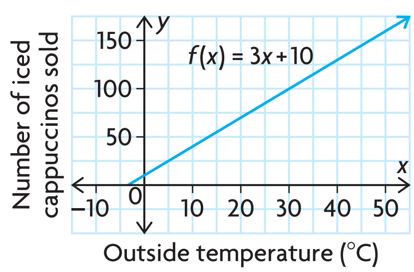
1.7 Exploring Operations with Functions

GOAL

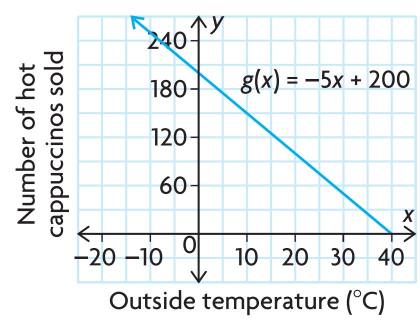
Explore the properties of the sum, difference, and product of two functions.

A popular coffee house sells iced cappuccino for \$4 and hot cappuccino for \$3. The manager would like to predict the relationship between the outside temperature and the total daily revenue from each type of cappuccino sold. The manager discovers that every 1 °C increase in temperature leads to an increase in the sales of cold drinks by three cups per day and to a decrease in the sales of hot drinks by five cups per day.

The function $f(x) = 3x + 10$ can be used to model the number of iced cappuccinos sold.



The function $g(x) = -5x + 200$ can be used to model the number of hot cappuccinos sold.



In both functions, x represents the daily average outside temperature. In the first function, $f(x)$ represents the daily average number of iced cappuccinos sold. In the second function, $g(x)$ represents the daily average number of hot cappuccinos sold.

How does the outside temperature affect the daily revenue from cappuccinos sold?

A. Make a table of values for each function, with the temperature in intervals of 5°, from 0° to 40°.

B. What does $h(x) = f(x) + g(x)$ represent?

C. Simplify $h(x) = (3x + 10) + (-5x + 200)$.

D. Make a table of values for the function in part C, with the temperature in intervals of 5°, from 0° to 40°. How do the values compare with the values in each table you made in part A? How do the domains of $f(x)$, $g(x)$, and $h(x)$ compare?

E. What does $h(x) = f(x) - g(x)$ represent?

F. Simplify $h(x) = (3x + 10) - (-5x + 200)$.

G. Make a table of values for the function in part F, with the temperature in intervals of 5°, from 0° to 40°. How do the values compare with the values in each table you made in part A? How do the domains of $f(x)$, $g(x)$, and $h(x)$ compare?

H. What does $R(x) = 4f(x) + 3g(x)$ represent?

I. Simplify $R(x) = 4(3x + 10) + 3(-5x + 200)$.

J. Make a table of values for the function in part I, with the temperature in intervals of 5°, from 0° to 40°. How do the values compare with the values in each table you made in part A? How do the domains of $f(x)$, $g(x)$, and $R(x)$ compare?

K. How does temperature affect the daily revenue from cappuccinos sold?

Reflecting

L. Explain how the sum function, $h(x)$, would be different if
a) both $f(x)$ and $g(x)$ were increasing functions
b) both $f(x)$ and $g(x)$ were decreasing functions

M. What does the function $k(x) = g(x) - f(x)$ represent? Is its graph identical to the graph of $h(x) = f(x) - g(x)$? Explain.

N. Determine the function $h(x) = f(x) \times g(x)$. Does this function have any meaning in the context of the daily revenue from cappuccinos sold? Explain how the table of values for this function is related to the tables of values you made in part A.

O. If you are given the graphs of two functions, explain how you could create a graph that represents
a) the sum of the two functions
b) the difference between the two functions
c) the product of the two functions

In Summary

Key Idea

- If two functions have domains that overlap, they can be added, subtracted, or multiplied to create a new function on that shared domain.

Need to Know

- Two functions can be added, subtracted, or multiplied graphically by adding, subtracting, or multiplying the values of the dependent variable for identical values of the independent variable.
- Two functions can be added, subtracted, or multiplied algebraically by adding, subtracting, or multiplying the expressions for the dependent variable and then simplifying.
- The properties of each original function have an impact on the properties of the new function.

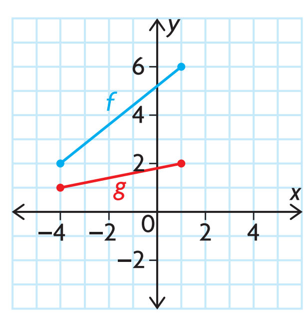
FURTHER *Your Understanding*

1. Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$. Determine:

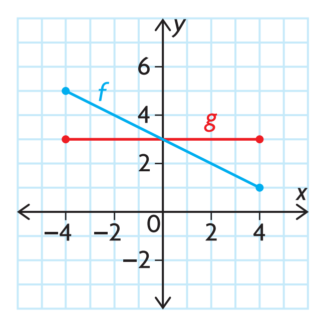
a) $f + g$ **b)** $f - g$ **c)** $g - f$ **d)** fg

2. Use the graphs of f and g to sketch the graphs of $f + g$.

a)

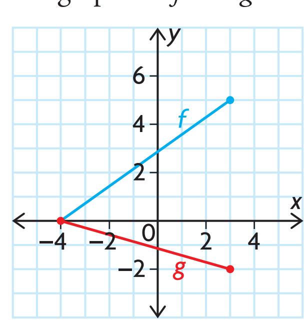


b)

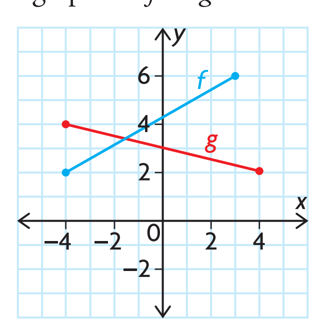


3. Use the graphs of f and g to sketch the graphs of $f - g$.

a)

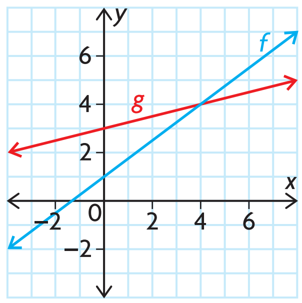


b)

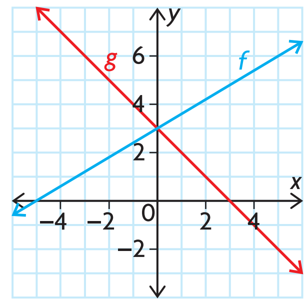


4. Use the graphs of f and g to sketch the graphs of fg.

a)



b)



5. Determine the equation of each new function, and then sketch its graph.

a) $h(x) = f(x) + g(x)$, where $f(x) = x^2$ and $g(x) = -x^2$

b) $p(x) = m(x) - n(x)$, where $m(x) = x^2$ and $n(x) = -7x + 12$

c) $r(x) = s(x) + t(x)$, where $s(x) = |x|$ and $t(x) = 2^x$

d) $a(x) = b(x) \times c(x)$, where $b(x) = x$ and $c(x) = x^2$

6. a) Using the graphs you sketched in question 5, compare and contrast the relationship between the properties of the original functions and the properties of the new function.

b) Which properties of the original functions determined the properties of the new function?

7. Let $f(x) = x + 3$ and $g(x) = -x^2 + 5, x \in \mathbf{R}$.

a) Sketch each graph on the same set of axes.

b) Make a table of values for $-3 \leq x \leq 3$, and determine the corresponding values of $h(x) = f(x) \times g(x)$.

c) Use the table to sketch $h(x)$ on the same axes. Describe the shape of the graph.

d) Determine the algebraic model for $h(x)$. What is its degree?

e) What is the domain of $h(x)$? How does this domain compare with the domains of $f(x)$ and $g(x)$?

8. Let $f(x) = x^2 + 2$ and $g(x) = x^2 - 2, x \in \mathbf{R}$.

a) Sketch each graph on the same set of axes.

b) Make a table of values for $-3 \leq x \leq 3$, and determine the corresponding values of $h(x) = f(x) \times g(x)$.

c) Use the table to sketch $h(x)$ on the same axes. Describe the shape of the graph.

d) Determine the algebraic model for $h(x)$. What is its degree?

e) What is the domain of $h(x)$?

1 Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 1.4, Examples 2, 3, and 4.
- Try Chapter Review Questions 7, 8, and 9.

Q: In what order are transformations performed on a function?

Function → Perform all stretches/compressions and reflections. → Perform all translations.

A: All stretches/compressions (vertical and horizontal) and reflections can be applied at the same time by multiplying the x- and y-coordinates on the parent function by the appropriate factors. Both vertical and horizontal translations can then be applied by adding or subtracting the relevant numbers to the x- and y-coordinates of the points.

Study *Aid*

- See Lesson 1.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 10 to 13.

Q: How do you find the inverse relation of a function?

A: You can find the inverse relation of a function numerically, graphically, or algebraically.

To find the inverse relation of a function numerically, using a table of values, switch the values for the independent and dependent variables.

$f(x)$	f^{-1}
(x, y)	(y, x)

To find the inverse relation graphically, reflect the graph of the function in the line $y = x$. This is accomplished by switching the x- and y-coordinates in each ordered pair.

To find the algebraic representation of the inverse relation, interchange the positions of the x- and y-variables in the function and solve for y.

Q: Is an inverse of a function always a function?

A: No; if an element in the domain of the original function corresponds to more than one number in the range, then the inverse relation is not a function.

Study *Aid*

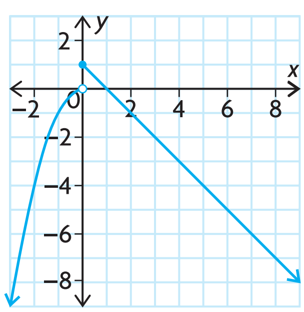
- See Lesson 1.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 10 to 13.

Q: What is a piecewise function?

A: A piecewise function is a function that has two or more function rules for different parts of its domain.

For example, the function defined by $f(x) = \begin{cases} -x^2, \text{ if } x < 0 \\ -x + 1, \text{ if } x \geq 0 \end{cases}$

consists of two pieces. The first equation defines half of a parabola that opens down when $x < 0$. The second equation defines a decreasing line with a y-intercept of 1 when $x \geq 0$. The graph confirms this.



Study *Aid*

- See Lesson 1.6, Examples 1, 2, 3, and 4.
- Try Chapter Review Questions 14 to 17.

Q: If you are given the graphs or equations of two functions, how can you create a new function?

A: You can create a new function by adding, subtracting, or multiplying the two given functions.

This can be done graphically by adding, subtracting, or multiplying the y-coordinates in each pair of ordered pairs that have identical x-coordinates.

This can be done algebraically by adding, subtracting, or multiplying the expressions for the dependent variable and then simplifying.

Study *Aid*

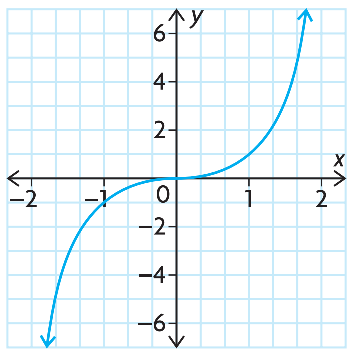
- See Lesson 1.7.
- Try Chapter Review Questions 18 to 21.

PRACTICE Questions

Lesson 1.1

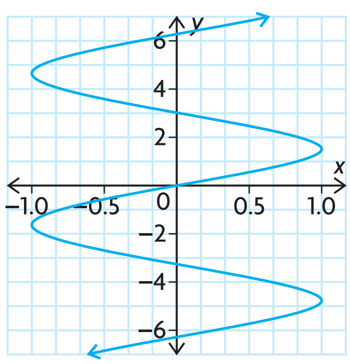
1. Determine whether each relation is a function, and state its domain and range.

a)



b) $3x^2 + 2y = 6$

c)



d) $x = 2^y$

2. A cell phone company charges a monthly fee of \$30, plus \$0.02 per minute of call time.

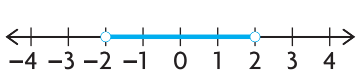
a) Write the monthly cost function, $C(t)$, where t is the amount of time in minutes of call time during a month.

b) Find the domain and range of C.

Lesson 1.2

3. Graph $f(x) = 2|x + 3| - 1$, and state the domain and range.

4. Describe this interval using absolute value notation.



Lesson 1.3

5. For each pair of functions, give a characteristic that the two functions have in common and a characteristic that distinguishes them.

a) $f(x) = x^2$ and $g(x) = \sin x$

b) $f(x) = \frac{1}{x}$ and $g(x) = x$

c) $f(x) = |x|$ and $g(x) = x^2$

d) $f(x) = 2^x$ and $g(x) = x$

6. Identify the intervals of increase/decrease, the symmetry, and the domain and range of each function.

a) $f(x) = 3x$

b) $f(x) = x^2 + 2$

c) $f(x) = 2^x - 1$

Lesson 1.4

7. For each of the following equations, state the parent function and the transformations that were applied. Graph the transformed function.

a) $y = |x + 1|$

b) $y = -0.25\sqrt{3(x + 7)}$

c) $y = -2\sin(3x) + 1,\ 0 \le x \le 360°$

d) $y = 2^{-2x} - 3$

8. The graph of $y = x^2$ is horizontally stretched by a factor of 2, reflected in the x-axis, and shifted 3 units down. Find the equation that results from the transformation, and graph it.

9. (2, 1) is a point on the graph of $y = f(x)$. Find the corresponding point on the graph of each of the following functions.

a) $y = -f(-x) + 2$

b) $y = f(-2(x + 9)) - 7$

c) $y = f(x - 2) + 2$

d) $y = 0.3f(5(x - 3))$

e) $y = 1 - f(1 - x)$

f) $y = -f(2(x - 8))$

Lesson 1.5

10. For each point on a function, state the corresponding point on the inverse relation.

a) $(1, 2)$
b) $(-1, -9)$
c) $(0, 7)$
d) $f(5) = 7$
e) $g(0) = -3$
f) $h(1) = 10$

11. Given the domain and range of a function, state the domain and range of the inverse relation.

a) $D = \{x \in \mathbf{R}\}, R = \{y \in \mathbf{R} \mid -2 < y < 2\}$
b) $D = \{x \in \mathbf{R} \mid x \geq 7\}, R = \{y \in \mathbf{R} \mid y < 12\}$

12. Graph each function and its inverse relation on the same set of axes. Determine whether the inverse relation is a function.

a) $f(x) = x^2 - 4$
b) $g(x) = 2^x$

13. Find the inverse of each function.

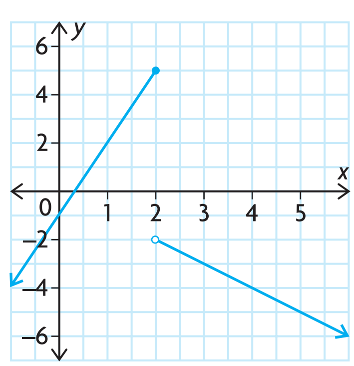
a) $f(x) = 2x + 1$
b) $g(x) = x^3$

Lesson 1.6

14. Graph the following function. Determine whether it is discontinuous and, if so, where. State the domain and the range of the function.

$$f(x) = \begin{cases} 2x, \text{ if } x < 1 \\ x + 1, \text{ if } x \geq 1 \end{cases}$$

15. Write the algebraic representation for the following piecewise function, using function notation.



16. If $f(x) = \begin{cases} x^2 + 1, \text{ if } x < 1 \\ 3x, \text{ if } x \geq 1 \end{cases}$

is $f(x)$ continuous at $x = 1$? Explain.

17. A telephone company charges \$30 a month and gives the customer 200 free call minutes. After the 200 min, the company charges \$0.03 a minute.

a) Write the function using function notation.
b) Find the cost for talking 350 min in a month.
c) Find the cost for talking 180 min in a month.

Lesson 1.7

18. Given $f = \{(0, 6), (1, 3), (4, 7), (5, 8)\}$ and $g = \{(-1, 2), (1, 4), (2, 3), (4, 8), (8, 9)\}$, determine the following.

a) $f(x) + g(x)$
b) $f(x) - g(x)$
c) $[f(x)][g(x)]$

19. Given $f(x) = 2x^2 - 2x, -2 \leq x \leq 3$ and $g(x) = -4x, -3 \leq x \leq 5$, graph the following.

a) f
b) g
c) $f + g$
d) $f - g$
e) fg

20. $f(x) = x^2 + 2x$ and $g(x) = x + 1$. Match the answer with the operation.

Answer:	Operation:
a) $x^3 + 3x^2 + 2x$	**A** $f(x) + g(x)$
b) $-x^2 - x + 1$	**B** $f(x) - g(x)$
c) $x^2 + 3x + 1$	**C** $g(x) - f(x)$
d) $x^2 + x - 1$	**D** $f(x) \times g(x)$

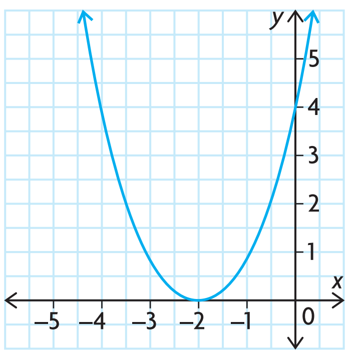
21. $f(x) = x^3 + 2x^2$ and $g(x) = -x + 6$,

a) Complete the table.

x	−3	−2	−1	0	1	2
$f(x)$						
$g(x)$						
$(f + g)(x)$						

b) Use the table to graph $f(x)$ and $g(x)$ on the same axes.
c) Graph $(f + g)(x)$ on the same axes as part b).
d) State the equation of $(f + g)(x)$.
e) Verify the equation of $(f + g)(x)$ using two of the ordered pairs in the table.

1 Chapter Self-Test



1. Consider the graph of the relation shown.
   a) Is the relation a function? Explain.
   b) State the domain and range.

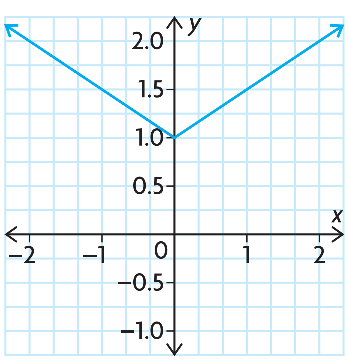
2. Given the following information about a function:
   - $D = \{x \in \mathbf{R}\}$
   - $R = \{y \in \mathbf{R} \mid y \geq -2\}$
   - decreasing on the interval $(-\infty, 0)$
   - increasing on the interval $(0, \infty)$

   a) What is a possible parent function?
   b) Draw a possible graph of the function.
   c) Describe the transformation that was performed.

3. Show algebraically that the function $f(x) = |3x| + x^2$ is an even function.

4. Both $f(x) = x^2$ and $g(x) = 2^x$ have a domain of all real numbers. List as many characteristics as you can to distinguish the two functions.

5. Describe the transformations that must be applied to $y = x^2$ to obtain the function $f(x) = -(x + 3)^2 - 5$, then graph the function.



6. Given the graph shown, describe the transformations that were performed to get this function. Write the algebraic representation, using function notation.

7. (3, 5) is a point on the graph of $y = f(x)$. Find the corresponding point on the graph of each of the following relations.
   a) $y = 3f(-x + 1) + 2$ b) $y = f^{-1}(x)$

8. Find the inverse of $f(x) = -2(x + 1)$.

9. A certain tax policy states that the first \$50 000 of income is taxed at 5% and any income above \$50 000 is taxed at 12%.
   a) Calculate the tax on \$125 000.
   b) Write a function that models the tax policy.

10. a) Sketch the graph of $f(x)$ where $f(x) = \begin{cases} 2^x + 1, \text{ if } x < 0 \\ \sqrt{x} + 3, \text{ if } x \geq 0 \end{cases}$
    b) Is $f(x)$ continuous over its entire domain? Explain.
    c) State the intervals of increase and decrease.
    d) State the domain and range of this function.

1 Chapter Task

Modelling with Functions

In 1950, a team of chemists led by Dr. W. F. Libby developed a method for determining the age of any natural specimen, up to approximately 60 000 years of age. Dr. Libby's method is based on the fact that all living materials contain traces of carbon-14. His method involves measuring the percent of carbon-14 that remains when a specimen is found.

The percent of carbon-14 that remains in a specimen after various numbers of years is shown in the table below.

Years	Carbon-14 Remaining (%)
5 730	50.0
11 460	25.0
17 190	12.5
22 920	6.25
28 650	3.125
34 380	1.5625

? How can you use the function $P(t) = 100(0.5)^{\frac{t}{5730}}$ to model this situation and determine the age of a natural specimen?

A. What percent of carbon is remaining for $t = 0$? What does this mean in the context of Dr. Libby's method?

B. Draw a graph of the function $P(t) = 100(0.5)^{\frac{t}{5730}}$, using the given table of values.

C. What is a reasonable domain for $P(t)$? What is a reasonable range?

D. Determine the approximate age of a specimen, given that $P(t) = 70$.

E. Draw the graph of the inverse function.

F. What information does the inverse function provide?

G. What are the domain and range of the inverse function?

Task | Checklist

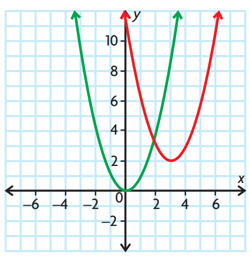
- ✔ Did you show all your steps?
- ✔ Did you draw and label your graphs accurately?
- ✔ Did you determine the age of the specimen that had 70% carbon-14 remaining?
- ✔ Did you explain your thinking clearly?

Answers

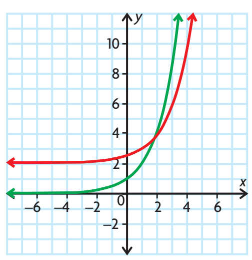
Chapter 1

Getting Started, p. 2

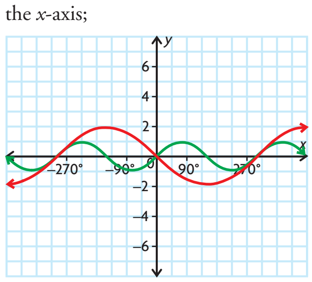
1. a) 6 c) $-\frac{51}{16}$
   b) -6 d) $a^2 + 5a$
2. a) $(x + y)(x + y)$
   b) $(5x - 1)(x - 3)$
   c) $(x + y + 8)(x + y - 8)$
   d) $(a + b)(x - y)$
3. a) horizontal translation 3 units to the right, vertical translation 2 units up;



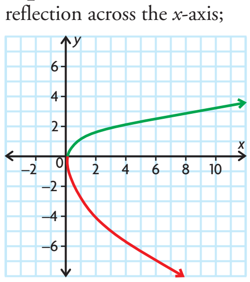
   b) horizontal translation 1 unit to the right, vertical translation 2 units up;



   c) horizontal stretch by a factor of 2, vertical stretch by a factor of 2, reflection across the x-axis;



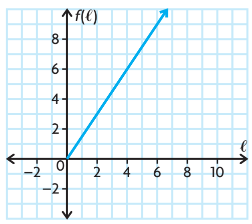
   d) horizontal compression by a factor of $\frac{1}{2}$, vertical stretch by a factor of 2, reflection across the x-axis;

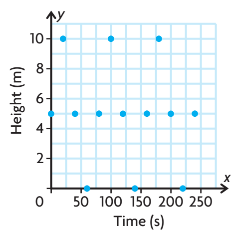


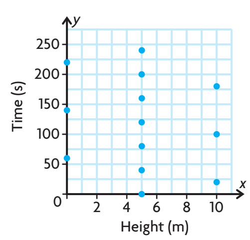
4. a) $D = \{x \in \mathbf{R} \mid -2 \leq x \leq 2\}$,
   $R = \{y \in \mathbf{R} \mid 0 \leq y \leq 2\}$
   b) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y \geq -19\}$
   c) $D = \{x \in \mathbf{R} \mid x \neq 0\}$,
   $R = \{y \in \mathbf{R} \mid y \neq 0\}$
   d) $D = \{x \in \mathbf{R}\}$,
   $R = \{y \in \mathbf{R} \mid -3 \leq y \geq 3\}$
   e) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y > 0\}$
5. a) This is not a function; it does not pass the vertical line test.
   b) This is a function; for each x-value, there is exactly one corresponding y-value.
   c) This is not a function; for each x-value greater than 0, there are two corresponding y-values.
   d) This is a function; for each x-value, there is exactly one corresponding y-value.
   e) This is a function; for each x-value, there is exactly one corresponding y-value.
6. a) 8
   b) about 2.71
7. If a relation is represented by a set of ordered pairs, a table, or an arrow diagram, one can determine if the relation is a function by checking that each value of the independent variable is paired with no more than one value of the dependent variable. If a relation is represented using a graph or scatter plot, the vertical line test can be used to determine if the relation is a function. A relation may also be represented by a description/rule or by using function notation or an equation. In these cases, one can use reasoning to determine if there is more than one value of the dependent variable paired with any value of the independent variable.

Lesson 1.1, pp. 11–13

1. a) $D = \{x \in \mathbf{R}\}$;
   $R = \{y \in \mathbf{R} \mid -4 \leq y \leq -2\}$; This is a function because it passes the vertical line test.
   b) $D = \{x \in \mathbf{R} \mid -1 \leq x \leq 7\}$;
   $R = \{y \in \mathbf{R} \mid -3 \leq y \leq 1\}$; This is a function because it passes the vertical line test.
   c) $D = \{1, 2, 3, 4\}$;
   $R = \{-5, 4, 7, 9, 11\}$; This is not a function because 1 is sent to more than one element in the range.
   d) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$; This is a function because every element in the domain produces exactly one element in the range.
   e) $D = \{-4, -3, 1, 2\}$; $R = \{0, 1, 2, 3\}$; This is a function because every element of the domain is sent to exactly one element in the range.
   f) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y \leq 0\}$; This is a function because every element in the domain produces exactly one element in the range.
2. a) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y \leq -3\}$; This is a function because every element in the domain produces exactly one element in the range.
   b) $D = \{x \in \mathbf{R} \mid x \neq -3\}$;
   $R = \{y \in \mathbf{R} \mid y \neq 0\}$; This is a function because every element in the domain produces exactly one element in the range.
   c) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y > 0\}$; This is a function because every element in the domain produces exactly one element in the range.
   d) $D = \{x \in \mathbf{R}\}$;
   $R = \{y \in \mathbf{R} \mid 0 \leq y \leq 2\}$; This is a function because every element in the domain produces exactly one element in the range.
   e) $D = \{x \in \mathbf{R} \mid -3 \leq x \leq 3\}$;
   $R = \{y \in \mathbf{R} \mid -3 \leq y \leq 3\}$; This is not a function because (0, 3) and (0, 3) are both in the relation.
   f) $D = \{x \in \mathbf{R}\}$;
   $R = \{y \in \mathbf{R} \mid -2 \leq y \leq 2\}$; This is a function because every element in the domain produces exactly one element in the range.
3. a) function; $D = \{1, 3, 5, 7\}$;
   $R = \{2, 4, 6\}$
   b) function; $D = \{0, 1, 2, 5\}$;
   $R = \{-1, 3, 6\}$
   c) function; $D = \{0, 1, 2, 3\}$; $R = \{2, 4\}$
   d) not a function; $D = \{2, 6, 8\}$;
   $R = \{1, 3, 5, 7\}$
   e) not a function; $D = \{1, 10, 100\}$;
   $R = \{0, 1, 2, 3\}$
   f) function; $D = \{1, 2, 3, 4\}$;
   $R = \{1, 2, 3, 4\}$
4. a) function; $D = \{x \in \mathbf{R}\}$;
   $R = \{y \in \mathbf{R} \mid y \geq 2\}$.
   b) not a function; $D = \{x \in \mathbf{R} \mid x \geq 2\}$;
   $R = \{y \in \mathbf{R}\}$
   c) function; $D = \{x \in \mathbf{R}\}$;
   $R = \{y \in \mathbf{R} \mid y \geq -0.5\}$
   d) not a function; $D = \{x \in \mathbf{R} \mid x \geq 0\}$;
   $R = \{y \in \mathbf{R}\}$
   e) function; $D = \{x \in \mathbf{R} \mid x \neq 0\}$;
   $R = \{y \in \mathbf{R} \mid y \neq 0\}$
   f) function; $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$
5. a) $y = x + 3$ c) $y = 3(x - 2)$
   b) $y = 2x - 5$ d) $y = -x + 5$

6. a) The length is twice the width.
b) $f(l) = \frac{3}{2}l$
c)

d) length = 8 m; width = 4 m

7. a)


b) D = {0, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220, 240}
c) R = {0, 5, 10}
d) It is a function because it passes the vertical line test.
e)


f) It is not a function because (5, 0) and (5, 40) are both in the relation.

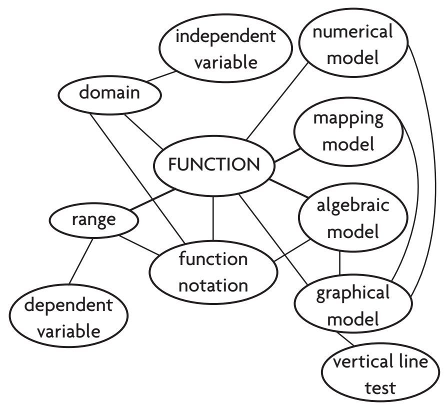
8. a) $\{(1, 2), (3, 4), (5, 6)\}$
b) $\{(1, 2), (3, 2), (5, 6)\}$
c) $\{(2, 1), (2, 3), (5, 6)\}$

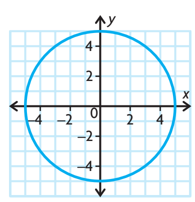
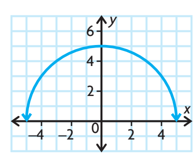
9. If a vertical line passes through a function and hits two points, those two points have identical *x*-coordinates and different *y*-coordinates. This means that one *x*-coordinate is sent to two different elements in the range, violating the definition of *function*.

10. a) Yes, because the distance from (4, 3) to (0, 0) is 5.
b) No, because the distance from (1, 5) to (0, 0) is not 5.
c) No, because (4, 3) and (4, −3) are both in the relation.

11. a) $g(x) = x^2 + 3$
b) $g(3) - g(2) = 12 - 7$
$= 5$
$g(3 - 2) = g(1)$
$= 4$
So, $g(3) - g(2) \neq g(3 - 2)$.

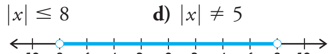
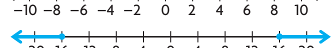
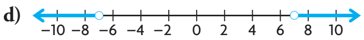
12. a) $f(6) = 12; f(7) = 8; f(8) = 15$
b) Yes, $f(15) = f(3) \times f(5)$
c) Yes, $f(12) = f(3) \times f(4)$
d) Yes, there are others that will work. $f(a) \times f(b) = f(a \times b)$ whenever a and b have no common factors other than 1.

13. Answers may vary. For example:


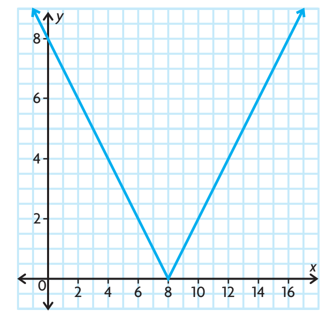
14.


The first is not a function because it fails the vertical line test:
$D = \{x \in \mathbf{R} \mid -5 \leq x \leq 5\}$;
$R = \{y \in \mathbf{R} \mid -5 \leq y \leq 5\}$.
The second is a function because it passes the vertical line test:
$D = \{x \in \mathbf{R} \mid -5 \leq x \leq 5\}$;
$R = \{y \in \mathbf{R} \mid 0 \leq y \leq 5\}$.

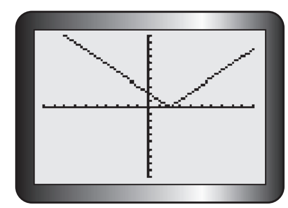
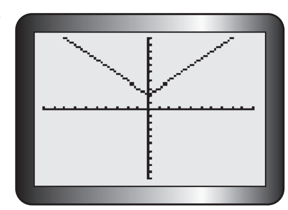
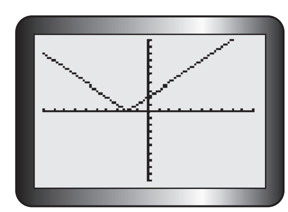
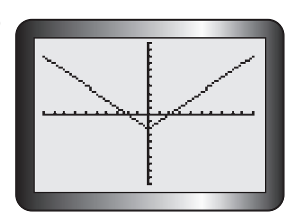
15. x is a function of y if the graph passes the horizontal line test. This occurs when any horizontal line hits the graph at most once.

Lesson 1.2, p. 16

1. $|-5|, |12|, |-15|, |20|, |-25|$
2. a) 22 c) 18 e) −2
b) −35 d) 11 f) −2
3. a) $|x| > 3$ c) $|x| \geq 1$
b) $|x| \leq 8$ d) $|x| \neq 5$
4. a)

b)

c) The absolute value of a number is always greater than or equal to 0. There are no solutions to this inequality.
d)


5. a) $|x| \leq 3$ c) $|x| \geq 2$
b) $|x| > 2$ d) $|x| < 4$

6.

a) The graphs are the same.
b) Answers may vary. For example, $x - 8 = -(-x + 8)$, so they are negatives of each other and have the same absolute value.

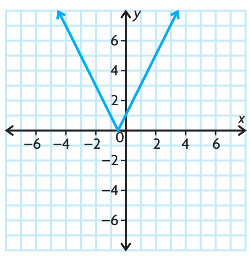
7. a)

b)

c)

d)


8. When the number you are adding or subtracting is inside the absolute value signs, it moves the function to the left (when adding) or to the right (when subtracting) of the origin. When the number you are

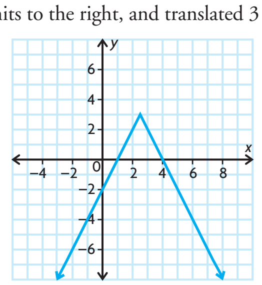
Answers

adding or subtracting is outside the absolute value signs, it moves the function down (when subtracting) or up (when adding) from the origin.
The graph of the function will be the absolute value function moved to the left 3 units and up 4 units from the origin.

9. This is the graph of $g(x) = |x|$ horizontally compressed by a factor of $\frac{1}{2}$ and translated $\frac{1}{2}$ unit to the left.



10. This is the graph of $g(x) = |x|$ horizontally compressed by a factor of $\frac{1}{2}$, reflected over the x-axis, translated $2\frac{1}{2}$ units to the right, and translated 3 units up.



Lesson 1.3, pp. 23–25

1. Answers may vary. For example, domain because most of the parent functions have all real numbers as a domain.

2. Answers may vary. For example, the end behaviour because the only two that match are x^2 and $|x|$.

3. Given the horizontal asymptote, the function must be derived from 2^x. But the asymptote is at $y = 2$, so it must have been translated up two. Therefore, the function is $f(x) = 2^x + 2$.

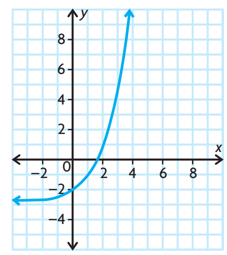
4. **a)** Both functions are odd, but their domains are different.
b) Both functions have a domain of all real numbers, but sin (x) has more zeros.
c) Both functions have a domain of all real numbers, but different end behaviour.
d) Both functions have a domain of all real numbers, but different end behaviour.

5. **a)** even **d)** odd
b) odd **e)** neither even nor odd
c) odd **f)** neither even nor odd

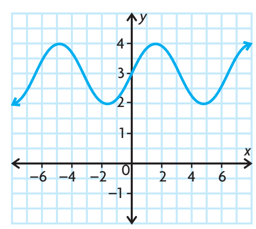
6. **a)** $|x|$, because it is a measure of distance from a number
b) sin (x), because the heights are periodic
c) 2^x, because population tends to increase exponentially
d) x, because there is \$1 on the first day, \$2 on the second, \$3 on the third, etc.

7. **a)** $f(x) = \sqrt{x}$ **c)** $f(x) = x^2$
b) $f(x) = \sin x$ **d)** $f(x) = x$

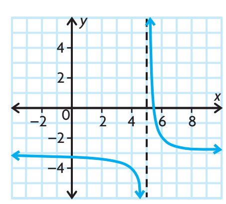
8. **a)** $f(x) = 2^x - 3$



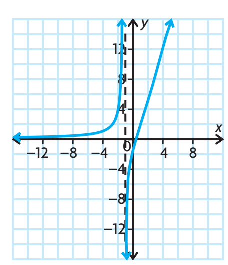
b) $g(x) = \sin x + 3$



c) $h(x) = \frac{1}{x-5} - 3 = \frac{16-3x}{x-5}$



9.



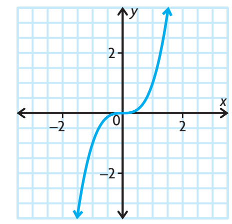
10. **a)** $f(x) = (x-2)^2$
b) There is not only one function. $f(x) = \frac{3}{4}(x-2)^2 + 1$ works as well.
c) There is more than one function that satisfies the property.
$f(x) = |x-2| + 2$ and
$f(x) = 2|x-2|$ both work.

11. x^2 is a smooth curve, while $|x|$ has a sharp, pointed corner at (0, 0).

12. See next page.

13. It is important to name parent functions in order to classify a wide range of functions according to similar behaviour and characteristics.

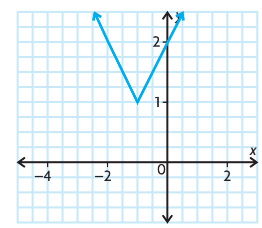
14.



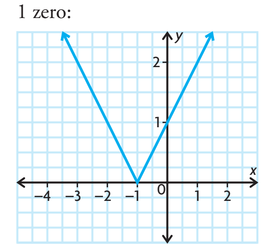
D = $\{x \in \mathbf{R}\}$, R = $\{f(x) \in \mathbf{R}\}$; interval of increase = $(-\infty, \infty)$, no interval of decrease, no discontinuities, x- and y-intercept at (0, 0), odd, $x \to \infty$, $y \to \infty$, and $x \to -\infty$, $y \to -\infty$. It is very similar to $f(x) = x$. It does not, however, have a constant slope.

15. No, cos x is a horizontal translation of sin x.

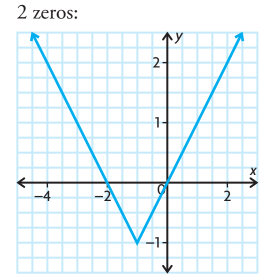
16. The graph can have 0, 1, or 2 zeros.
0 zeros:



1 zero:



2 zeros:

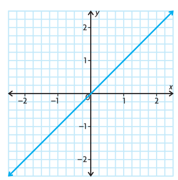
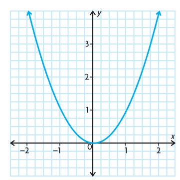
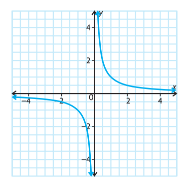
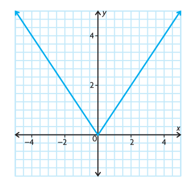
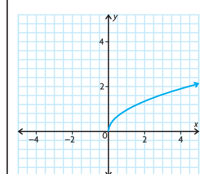
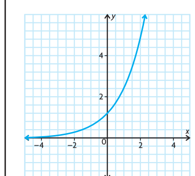
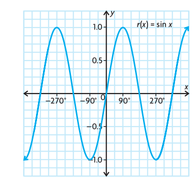


Mid-Chapter Review, p. 28

1. **a)** function; D = {0, 3, 15, 27}, R = {2, 3, 4}
b) function; D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R}\}$
c) not a function;
D = $\{x \in \mathbf{R} \mid -5 \le x \le 5\}$,
R = $\{y \in \mathbf{R} \mid -5 \le y \le 5\}$
d) not a function; D = {1, 2, 10}, R = {−1, 3, 6, 7}

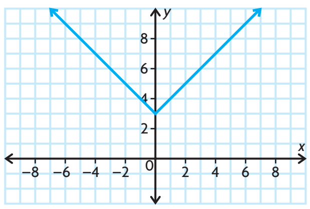
2. **a)** Yes. Every element in the domain gets sent to exactly one element in the range.
b) D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
c) R = {10, 20, 25, 30, 35, 40, 45, 50}

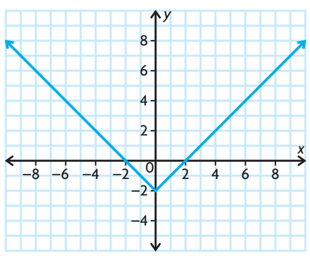
12.

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \lvert x \rvert$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \neq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \geq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$
Range	$\{f(x) \in \mathbf{R}\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \neq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) > 0\}$	$\{f(x) \in \mathbf{R} \mid -1 \leq f(x) \leq 1\}$
Intervals of Increase	$(-\infty, \infty)$	$(0, \infty)$	None	$(0, \infty)$	$(0, \infty)$	$(-\infty, \infty)$	$[90(4k + 1), 90(4k + 3)]$ $K \in \mathbf{Z}$
Intervals of Decrease	None	$(-\infty, 0)$	$(-\infty, 0)$ $(0, \infty)$	$(-\infty, 0)$	None	None	$[90(4k + 3), 90(4k + 1)]$ $K \in \mathbf{Z}$
Location of Discontinuities and Asymptotes	None	None	$y = 0$ $x = 0$	None	None	$y = 0$	None
Zeros	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	None	$180k$ $K \in \mathbf{Z}$
y-Intercepts	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	(0, 1)	(0, 0)
Symmetry	Odd	Even	Odd	Even	Neither	Neither	Odd
End Behaviours	$x \to \infty, y \to \infty$ $x \to -\infty, y \to -\infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to 0$ $x \to -\infty, y \to 0$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to \infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to 0$	Oscillating

3. a) D = $\{x \in \mathbf{R}\}$, R = $\{f(x) \in \mathbf{R}\}$; function
b) D = $\{x \in \mathbf{R} \mid -3 \le x \le 3\}$, R = $\{y \in \mathbf{R} \mid -3 \le y \le 3\}$; not a function
c) D = $\{x \in \mathbf{R} \mid x \le 5\}$, R = $\{y \in \mathbf{R} \mid y \ge 0\}$; function
d) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} \mid y \ge -2\}$; function

4. $-|3|, |0|, |-3|, |-4|, |5|$

5. a)


b)


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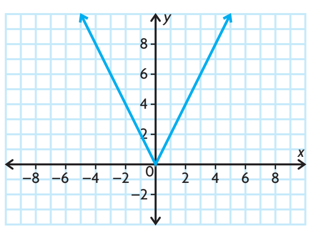
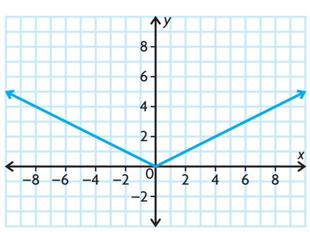
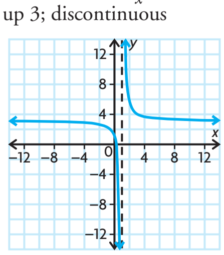
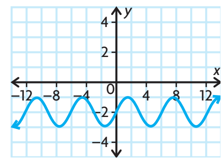


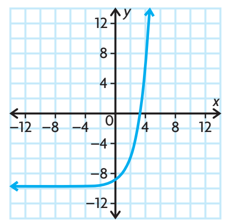
d)


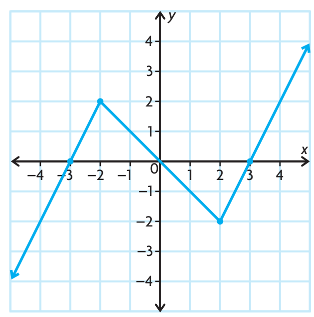
6. a) $f(x) = 2^x$
b) $f(x) = \frac{1}{x}$
c) $f(x) = \sqrt{x}$

7. a) even **c)** neither odd nor even
b) even **d)** neither odd nor even

8. a) This is $f(x) = \frac{1}{x}$ translated right 1 and up 3; discontinuous


b) This if $f(x) = \sin x$ translated down 2; continuous


c) This is $f(x) = 2^x$ translated down 10; continuous


9.


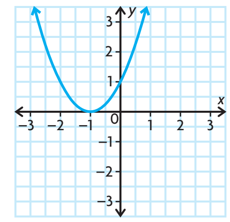
Lesson 1.4, pp. 35–37

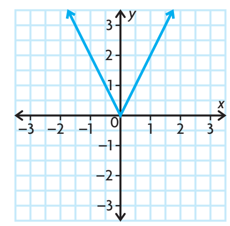
1. a) translation 1 unit down
b) horizontal compression by a factor of $\frac{1}{2}$, translation 1 unit right
c) reflection over the x-axis, translation 2 units up, translation 3 units right
d) reflection over the x-axis, vertical stretch by a factor of 2, horizontal compression by a factor of $\frac{1}{4}$
e) reflection over the x-axis, translation 3 units down, reflection over the y-axis, translation 2 units left
f) vertical compression by a factor of $\frac{1}{2}$, translation 6 units up, horizontal stretch by a factor of 4, translation 5 units right

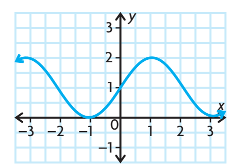
2. a) $a = -1, k = \frac{1}{2}, d = 0, c = 3$
b) $a = 3, k = \frac{1}{2}, d = 0, c = -2$

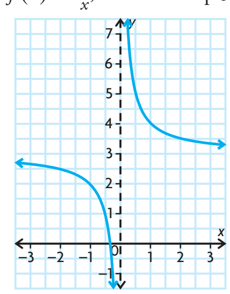
3. (2, 3), (1, 3), (1, 6), (1, −6), (−4, −6), (−4, −10)

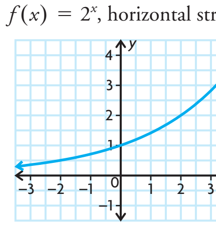
4. a) (2, 6), (4, 14), (−2, 10), (−4, 12)
b) (5, 3), (7, 7), (1, 5), (−1, 6)
c) (2, 5), (4, 9), (−2, 7), (−4, 8)
d) (1, 0), (3, 4), (−3, 2), (−5, 3)
e) (2, 5), (4, 6), (−2, 3), (−4, 7)
f) (1, 2), (2, 6), (−1, 4), (−2, 5)

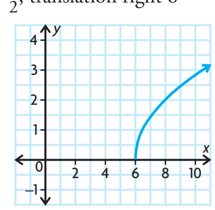
5. a) $f(x) = x^2$, translated left 1


b) $f(x) = |x|$, vertical stretch by 2


c) $f(x) = \sin x$, horizontal compression of $\frac{1}{3}$, translation up 1


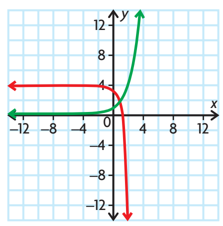
d) $f(x) = \frac{1}{x}$, translation up 3


e) $f(x) = 2^x$, horizontal stretch by 2


f) $f(x) = \sqrt{x}$, horizontal compression by $\frac{1}{2}$, translation right 6


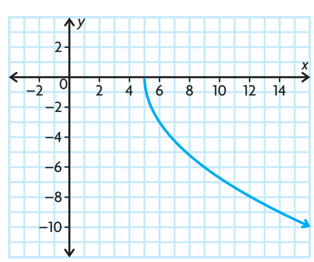
6. **a)** D = $\{x \in \mathbf{R}\}$,
R = $\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$
b) D = $\{x \in \mathbf{R}\}$,
R = $\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$
c) D = $\{x \in \mathbf{R}\}$,
R = $\{f(x) \in \mathbf{R} \mid 0 \leq f(x) \leq 2\}$
d) D = $\{x \in \mathbf{R} \mid x \neq 0\}$,
R = $\{f(x) \in \mathbf{R} \mid f(x) \neq 3\}$
e) D = $\{x \in \mathbf{R}\}$,
R = $\{f(x) \in \mathbf{R} \mid f(x) > 0\}$
f) D = $\{x \in \mathbf{R} \mid x \geq 6\}$,
R = $\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$

7. **a)**



b) The domain remains unchanged at D = $\{x \in \mathbf{R}\}$. The range must now be less than 4: R = $\{f(x) \in \mathbf{R} \mid f(x) < 4\}$. It changes from increasing on $(-\infty, \infty)$ to decreasing on $(-\infty, \infty)$. The end behaviour becomes as $x \to -\infty$, $y \to 4$, and as $x \to \infty$, $y \to -\infty$.
c) $g(x) = -2(2^{3(x-1)} + 4)$

8. $y = -3\sqrt{x - 5}$

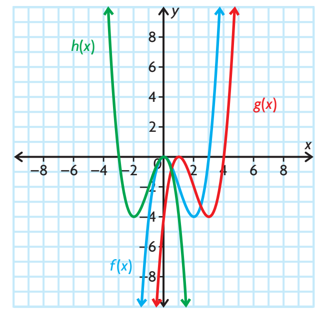


9. **a)** (3, 24) **b)** (−0.5, 4) **c)** (−1, 9) **d)** (−0.75, −8) **e)** (−1, −8) **f)** (−1, 7)

10. **a)** D = $\{x \in \mathbf{R} \mid x \geq 2\}$,
R = $\{g(x) \in \mathbf{R} \mid g(x) \geq 0\}$
b) D = $\{x \in \mathbf{R} \mid x \geq 1\}$,
R = $\{h(x) \in \mathbf{R} \mid h(x) \geq 4\}$
c) D = $\{x \in \mathbf{R} \mid x \leq 0\}$,
R = $\{k(x) \in \mathbf{R} \mid k(x) \geq 1\}$
d) D = $\{x \in \mathbf{R} \mid x \geq 5\}$,
R = $\{j(x) \in \mathbf{R} \mid j(x) \geq -3\}$

11. $y = 5(x^2 - 3)$ is the same as $y = 5x^2 - 15$, not $y = 5x^2 - 3$.

12.



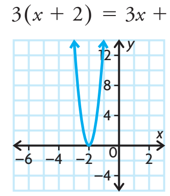
13. **a)** a vertical stretch by a factor of 4
b) a horizontal compression by a factor of $\frac{1}{2}$
c) $(2x)^2 = 2^2x^2 = 4x^2$

14. Answers may vary. For example:

horizontal stretch or compression, based on value of k
↓
vertical stretch or compression, based on value of a
↓
reflection in x-axis if $a < 0$; reflection in y-axis if $k < 0$
↓
horizontal translation based on value of d
↓
vertical translation based on value of c

15. (4, 5)

16. **a)** horizontal compression by a factor of $\frac{1}{3}$, translation 2 units to the left
b) because they are equivalent expressions: $3(x + 2) = 3x + 6$
c)



Lesson 1.5, pp. 43–45

1. **a)** (5, 2) **b)** (−6, −5) **c)** (−8, 4) **d)** (2, 1) **e)** (0, −3) **f)** (7, 0)

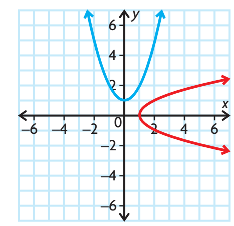
2. **a)** D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R}\}$
b) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} \mid y \geq 2\}$
c) D = $\{x \in \mathbf{R} \mid x < 2\}$,
R = $\{y \in \mathbf{R} \mid y \geq -5\}$
d) D = $\{x \in \mathbf{R} \mid -5 < x < 10\}$,
R = $\{y \in \mathbf{R} \mid y < -2\}$

3. A and D match; B and F match; C and E match

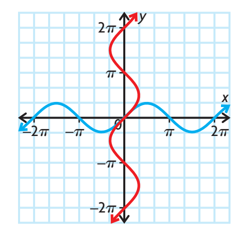
4. **a)** (4, 129)
b) (129, 4)
c) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R}\}$
d) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R}\}$
e) Yes; it passes the vertical line test.

5. **a)** (4, 248)
b) (248, 4)
c) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} \mid y \geq -8\}$
d) D = $\{x \in \mathbf{R} \mid x \geq -8\}$ R = $\{y \in \mathbf{R}\}$
e) No; (248, 4) and (248, −4) are both on the inverse relation.

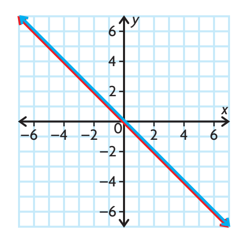
6. **a)** Not a function



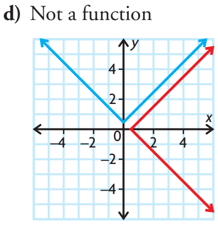
b) Not a function



c) Function



d) Not a function



7. **a)** $C = \frac{5}{9}(F - 32)$; this allows you to convert from Fahrenheit to Celsius.
b) 20 °C = 68 °F

8. **a)** $r = \sqrt{\frac{A}{\pi}}$; this can be used to determine the radius of a circle when its area is known.
b) $A = 25\pi$ cm², $r = 5$ cm

9. $k = 2$

10. **a)** 13 **b)** 25 **c)** 2 **d)** −2 **e)** 1 **f)** $\frac{1}{2}$

11. No; several students could have the same grade point average.

12. **a)** $f^{-1}(x) = \frac{1}{3}(x - 4)$

b) $h^{-1}(x) = -x$

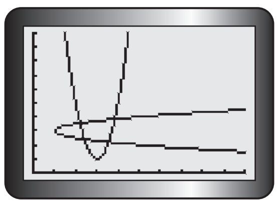
c) $g^{-1}(x) = \sqrt[3]{x + 1}$

d) $m^{-1}(x) = -\frac{x}{2} - 5$

13. **a)** $x = 4(y - 3)^2 + 1$

b) $y = \pm\sqrt{\frac{x - 1}{4}} + 3$

c)



d) (2.20, 3.55), (2.40, 2.40), (3.55, 2.20), (3.84, 3.84)

e) $x \geq 3$ because a negative square root is undefined.

f) $g(2) = 5$, but $g^{-1}(5) = 2$ or 4; the inverse is not a function if this is the domain of g.

14. For $y = -\sqrt{x + 2}$, $D = \{x \in \mathbf{R} \mid x \geq -2\}$ and $R = \{y \in \mathbf{R} \mid y \leq 0\}$. For $y = x^2 - 2$, $D = \{x \in \mathbf{R}\}$ and $R = \{y \in \mathbf{R} \mid y \geq -2\}$. The student would be correct if the domain of $y = x^2 - 2$ is restricted to $D = \{x \in \mathbf{R} \mid x \leq 0\}$.

15. Yes; the inverse of $y = \sqrt{x + 2}$ is $y = x^2 - 2$ so long as the domain of this second function is restricted to $D = \{x \in \mathbf{R} \mid x \geq 0\}$.

16. John is correct.

Algebraic: $y = \frac{x^3}{4} + 2$; $y - 2 = \frac{x^3}{4}$; $4(y - 2) = x^3$; $x = \sqrt[3]{4(y - 2)}$.

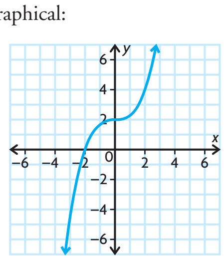
Numeric: Let $x = 4$.

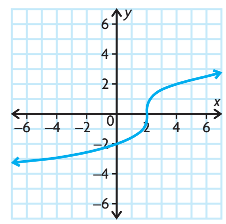
$y = \frac{4^3}{4} + 2 = \frac{64}{4} + 2 = 16 + 2 = 18$;

$x = \sqrt[3]{4(y - 2)} = \sqrt[3]{4(18 - 2)}$

$= \sqrt[3]{4(16)} = \sqrt[3]{64} = 4.$

Graphical:





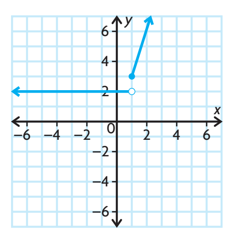
The graphs are reflections over the line $y = x$.

17. $f(x) = k - x$ works for all $k \in \mathbf{R}$.
$y = k - x$
Switch variables and solve for y: $x = k - y$
$y = k - x$
So the function is its own inverse.

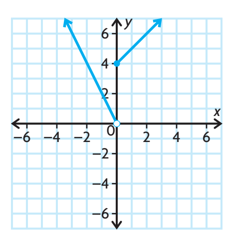
18. If a horizontal line hits the function in two locations, that means there are two points with equal y-values and different x-values. When the function is reflected over the line $y = x$ to find the inverse relation, those two points become points with equal x-values and different y-values, thus violating the definition of a function.

Lesson 1.6, pp. 51–53

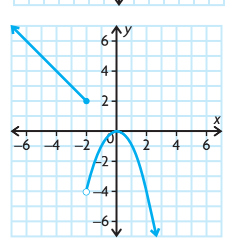
1. **a)**



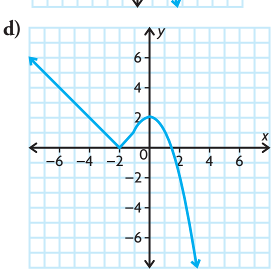
b)



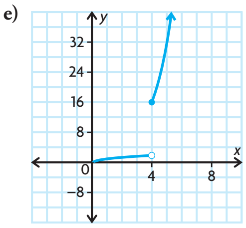
c)



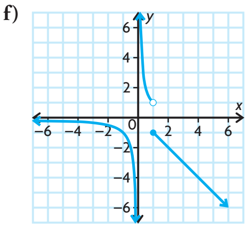
d)



e)



f)



2. **a)** Discontinuous at $x = 1$

b) Discontinuous at $x = 0$

c) Discontinuous at $x = -2$

d) Continuous

e) Discontinuous at $x = 4$

f) Discontinuous at $x = 1$ and $x = 0$

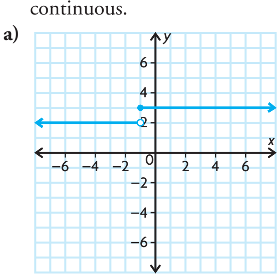
3. **a)** $f(x) = \begin{cases} x^2 - 2, \text{ if } x \leq 1 \\ x + 1, \text{ if } x > 1 \end{cases}$

b) $f(x) = \begin{cases} |x|, \text{ if } x < 1 \\ \sqrt{x}, \text{ if } x \geq 1 \end{cases}$

4. **a)** $D = \{x \in \mathbf{R}\}$; the function is discontinuous at $x = 1$.

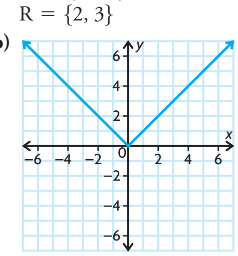
b) $D = \{x \in \mathbf{R}\}$; the function is continuous.

5. **a)**



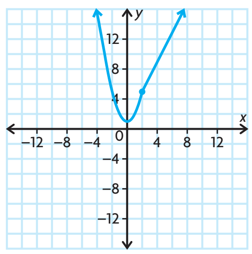
The function is discontinuous at $x = -1$.
$D = \{x \in \mathbf{R}\}$
$R = \{2, 3\}$

b)



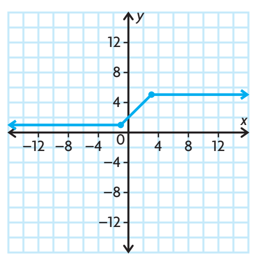
The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$

c)



The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} \mid f(x) \geq 1\}$

d)

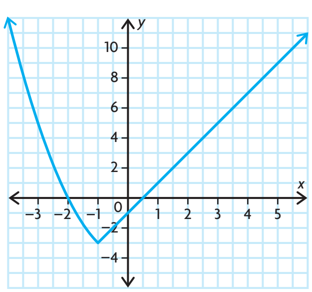


The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} \mid 1 \leq f(x) \leq 5\}$

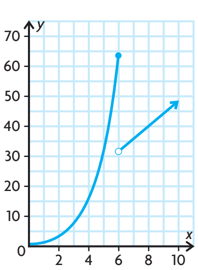
6. $f(x) = \begin{cases} 15, \text{ if } 0 \leq x \leq 500 \\ 15 + 0.02x, \text{ if } x \geq 500 \end{cases}$

7. $f(x) =$
$\begin{cases} 0.35x, \text{ if } 0 \leq x \leq 100\,000 \\ 0.45x - 10\,000, \text{ if } 100\,000 < x \leq 500\,000 \\ 0.55x - 60\,000, \text{ if } x > 500\,000 \end{cases}$

8. $k = 4$



9. a)

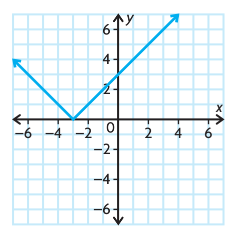


b) The function is discontinuous at $x = 6$.
c) 32 fish
d) $4x + 8 = 64$; $4x = 56$; $x = 14$
e) Answers may vary. For example, three possible events are environmental changes, introduction of a new predator, and increased fishing.

10. Answers may vary. For example:

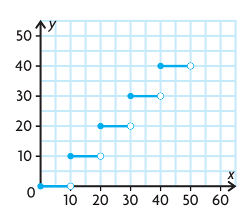
Plot the function for the left interval.
↓
Plot the function for the right interval.
↓
Determine if the plots for the left and right intervals meet at the *x*-value that serves as the common end point for the intervals; if so, the function is continuous at this point.
↓
Determine continuity for the two intervals using standard methods.

11. $f(x) = |x + 3| = \begin{cases} x + 3, \text{ if } x \geq -3 \\ -x - 3, \text{ if } x < -3 \end{cases}$



12. discontinuous at $p = 0$ and $p = 15$; continuous at $0 < p < 15$ and $p > 15$

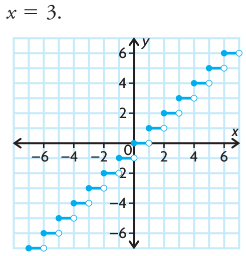
13. $f(x) = \begin{cases} 0, \text{ if } 0 \leq x < 10 \\ 10, \text{ if } 10 \leq x < 20 \\ 20, \text{ if } 20 \leq x < 30 \\ 30, \text{ if } 30 \leq x < 40 \\ 40, \text{ if } 40 \leq x < 50 \end{cases}$



It is often referred to as a step function because the graph looks like steps.

14. To make the first two pieces continuous, $5(-1) = -1 + k$, so $k = -4$. But if $k = -4$, the graph is discontinuous at $x = 3$.

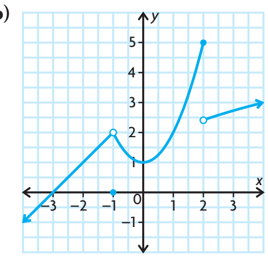
15.



16. Answers may vary. For example:

a) $f(x) = \begin{cases} x + 3, \text{ if } x < -1 \\ x^2 + 1, \text{ if } -1 \leq x \leq 2 \\ \sqrt{x} + 1, \text{ if } x > 2 \end{cases}$

b)



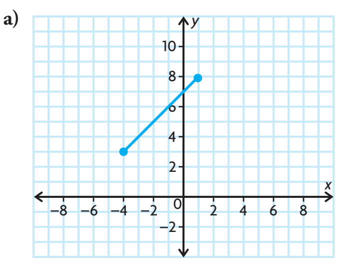
c) The function is not continuous. The last two pieces do not have the same value for $x = 2$.

d) $f(x) = \begin{cases} x + 3, \text{ if } x < -1 \\ x^2 + 1, \text{ if } -1 \leq x \leq 1 \\ \sqrt{x} + 1, \text{ if } x > 1 \end{cases}$

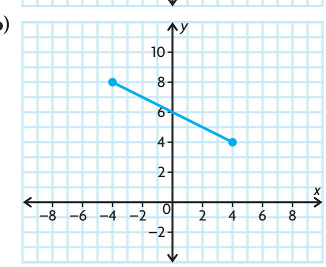
Lesson 1.7, pp. 56–57

1. a) $\{(-4, 6), (-2, 5), (1, 5), (4, 10)\}$
b) $\{(-4, 2), (-2, 3), (1, 1), (4, 2)\}$
c) $\{(-4, -2), (-2, -3), (1, -1), (4, -2)\}$
d) $\{(-4, 8), (-2, 4), (1, 6), (4, 24)\}$

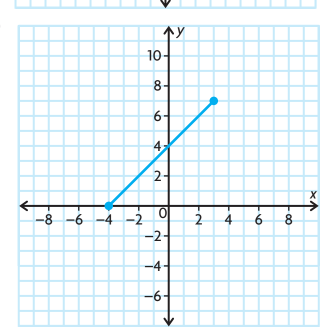
2. a)



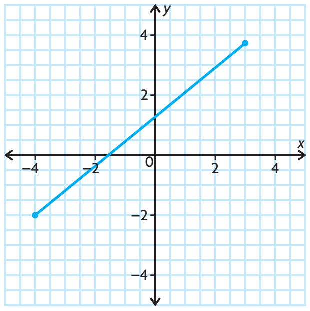
b)



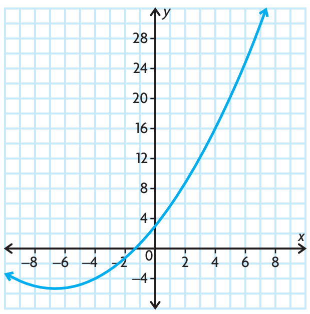
3. a)



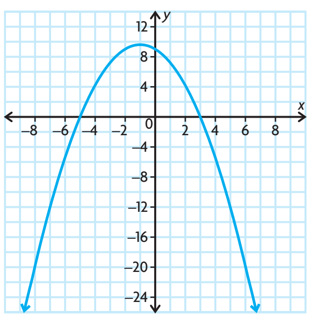
b)



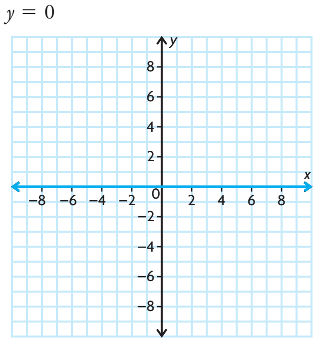
4. a)



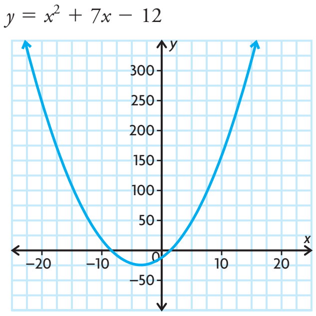
b)



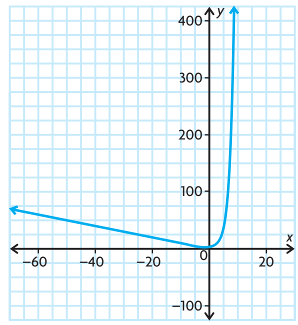
5. a) $y = 0$



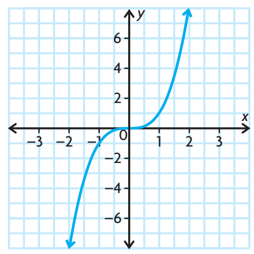
b) $y = x^2 + 7x - 12$



c) $y = |x| + 2^x$

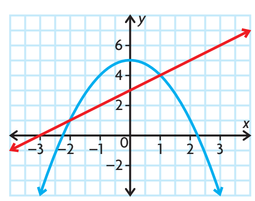


d) $y = x^3$



6. a)–b) Answers may vary. For example, properties of the original graphs such as intercepts and sign at various values of the independent variable figure prominently in the shape of the new function.

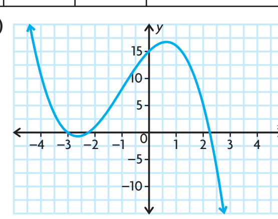
7. a)



b)

x	$f(x)$	$g(x)$	$h(x) = f(x) \times g(x)$
−3	0	−4	0
−2	1	1	1
−1	2	4	8
0	3	5	15
1	4	4	16
2	5	1	5
3	6	−4	−24

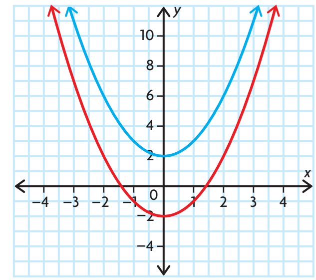
c)



d) $h(x) = (x + 3)(-x^2 + 5)$
$= -x^3 - 3x^2 + 5x + 15$; degree is 3

e) $D = \{x \in \mathbf{R}\}$; this is the same as the domain of both f and g.

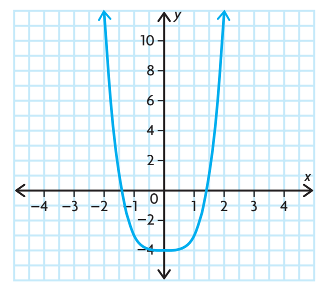
8. a)



b)

x	$f(x)$	$g(x)$	$h(x) = f(x) \times g(x)$
−3	11	7	77
−2	6	2	12
−1	3	−1	−3
0	2	−2	−4
1	3	−1	−3
2	6	2	12
3	11	7	77

c)



d) $h(x) = (x^2 + 2)(x^2 - 2) = x^4 - 4$; degree is 4

e) $D = \{x \in \mathbf{R}\}$

Chapter Review, pp. 60–61

1. a) function; $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$

b) function; $D = \{x \in \mathbf{R}\}$;
$R = \{y \in \mathbf{R} \mid y \leq 3\}$

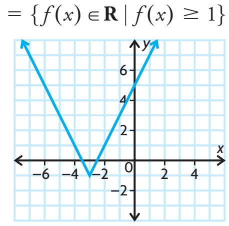
c) not a function;
$D = \{x \in \mathbf{R} \mid -1 \leq x \leq 1\}$;
$R = \{y \in \mathbf{R}\}$

d) function; $D = \{x \in \mathbf{R} \mid x > 0\}$;
$R = \{y \in \mathbf{R}\}$

2. a) $C(t) = 30 + 0.02t$

b) $D = \{t \in \mathbf{R} \mid t \geq 0\}$,
$R = \{C(t) \in \mathbf{R} \mid C(t) \geq 30\}$

3. $D = \{x \in \mathbf{R}\}$,
$R = \{f(x) \in \mathbf{R} \mid f(x) \geq 1\}$

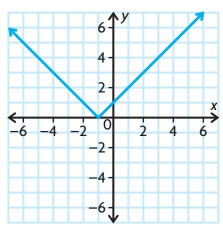


4. $|x| < 2$

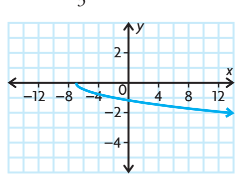
5. **a)** Both functions have a domain of all real numbers, but the ranges differ.
b) Both functions are odd but have different domains.
c) Both functions have the same domain and range, but x^2 is smooth and $|x|$ has a sharp corner at (0, 0).
d) Both functions are increasing on the entire real line, but 2^x has a horizontal asymptote while x does not.

6. **a)** Increasing on $(-\infty, \infty)$; odd; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R}\}$
b) Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$; even; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R} \mid f(x) \geq 2\}$
c) Increasing on $(-\infty, \infty)$; neither even nor odd; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R} \mid f(x) > -1\}$

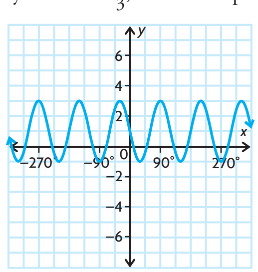
7. **a)** Parent: $y = |x|$; translated left 1



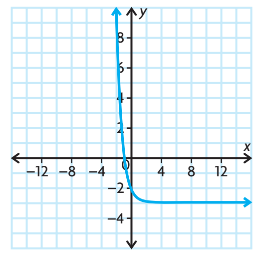
b) Parent: $y = \sqrt{x}$; compressed vertically by a factor of 0.25, reflected across the x-axis, compressed horizontally by a factor of $\frac{1}{3}$, and translated left 7



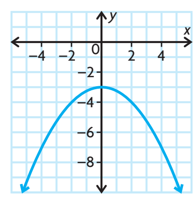
c) Parent: $y = \sin x$; reflected across the x-axis, expanded vertically by a factor of 2, compressed horizontally by a factor of $\frac{1}{3}$, translated up by 1



d) Parent: $y = 2^x$; reflected across the y-axis, compressed horizontally by a factor of $\frac{1}{2}$, and translated down by 3



8. $y = -\left(\frac{1}{2}x\right)^2 - 3$

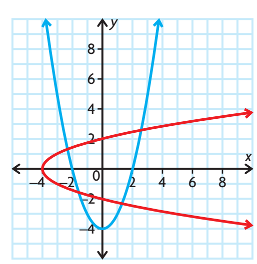


9. **a)** $(-2, 1)$
b) $(-10, -6)$
c) $(4, 3)$
d) $\left(\frac{17}{5}, 0.3\right)$
e) $(-1, 0)$
f) $(9, -1)$

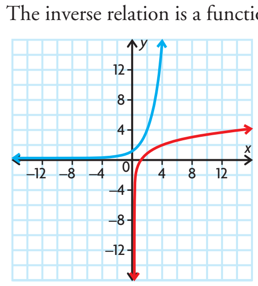
10. **a)** $(2, 1)$
b) $(-9, -1)$
c) $(7, 0)$
d) $(7, 5)$
e) $(-3, 0)$
f) $(10, 1)$

11. **a)** $D = \{x \in \mathbf{R} \mid -2 < x < 2\}$, $R = \{y \in \mathbf{R}\}$
b) $D = \{x \in \mathbf{R} \mid x < 12\}$, $R = \{y \in \mathbf{R} \mid y \geq 7\}$

12. **a)** The inverse relation is not a function.

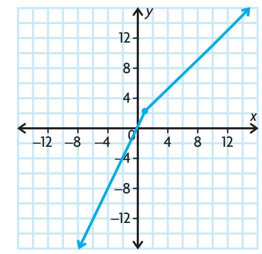


b) The inverse relation is a function.



13. **a)** $f^{-1}(x) = \frac{x - 1}{2}$
b) $g^{-1}(x) = \sqrt[3]{x}$

14.



The function is continuous; $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R}\}$

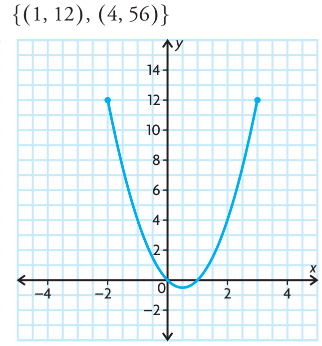
15. $f(x) = \begin{cases} 3x - 1, \text{ if } x \leq 2 \\ -x, \text{ if } x > 2 \end{cases}$; the function is discontinuous at $x = 2$.

16. In order for $f(x)$ to be continuous at $x = 1$, the two pieces must have the same value when $x = 1$.
When $x = 1$, $x^2 + 1 = 2$ and $3x = 3$.
The two pieces are not equal when $x = 1$, so the function is not continuous at $x = 1$.

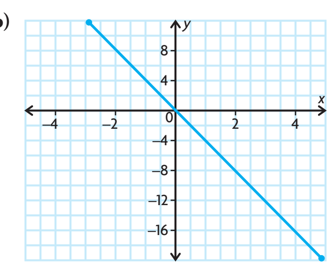
17. **a)** $f(x) = \begin{cases} 30, \text{ if } x \leq 200 \\ 24 + 0.03, \text{ if } x > 200 \end{cases}$
b) \$34.50
c) \$30

18. **a)** $\{(1, 7), (4, 15)\}$
b) $\{(1, -1), (4, -1)\}$
c) $\{(1, 12), (4, 56)\}$

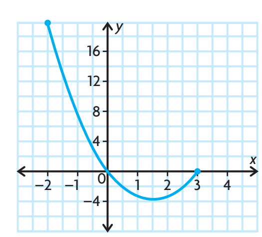
19. **a)**



b)

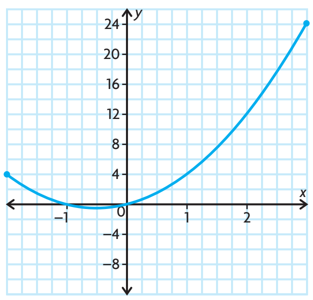


c)

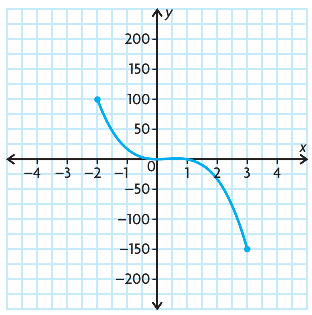


Answers

d)



e)

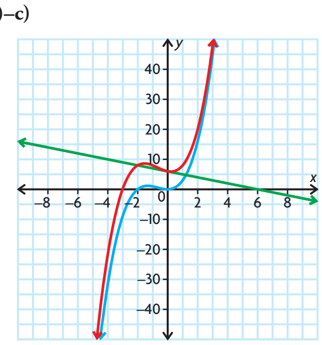


20. **a)** D
b) C
c) A
d) B

21. **a)**

x	−3	−2	−1	0	1	2
***f*(*x*)**	−9	0	1	0	3	16
***g*(*x*)**	9	8	7	6	5	4
(*f* + *g*)(*x*)	0	8	8	6	8	20

b)–c)

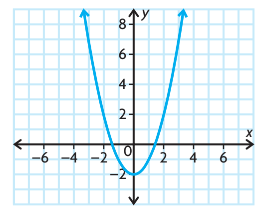


d) $x^3 + 2x^2 - x + 6$
e) Answers may vary. For example, (0, 0) belongs to *f*, (0, 6) belongs to *g* and (0, 6) belongs to $f + g$. Also, (1, 3) belongs to *f*, (1, 5) belongs to *g* and (1, 8) belongs to $f + g$.

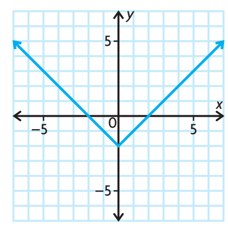
Chapter Self-Test, p. 62

1. **a)** Yes. It passes the vertical line test.
b) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \geq 0\}$

2. **a)** $f(x) = x^2$ or $f(x) = |x|$
b)



or

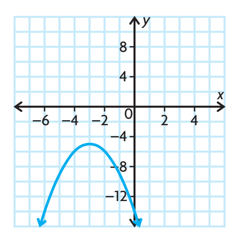


c) The graph was translated 2 units down.

3. $f(-x) = |3(-x)| + (-x)^2$
$= |3x| + x^2 = f(x)$

4. 2^x has a horizontal asymptote while x^2 does not. The range of 2^x is $\{y \in \mathbf{R} \mid y > 0\}$ while the range of x^2 is $\{y \in \mathbf{R} \mid y \geq 0\}$. 2^x is increasing on the whole real line and x^2 has an interval of decrease and an interval of increase.

5. reflection over the *x*-axis, translation down 5 units, translation left 3 units



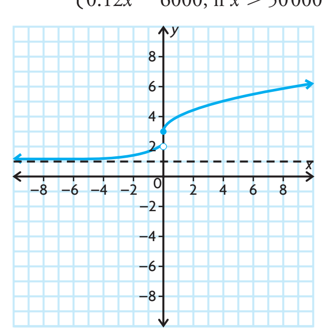
6. horizontal stretch by a factor of 2, translation 1 unit up;
$f(x) =$ if $|\frac{1}{2}x| + 1$

7. **a)** $(-4, 17)$
b) $(5, 3)$

8. $f^{1}(x) = -\frac{x}{2} - 1$

9. **a)** $9000
b) $f(x) = \begin{cases} 0.05, & \text{if } x \leq 50000 \\ 0.12x - 6000, & \text{if } x > 50000 \end{cases}$

10. **a)**



b) $f(x)$ is discontinuous at $x = 0$ because the two pieces do not have the same value when $x = 0$. When $x = 0$, $2^x + 1 = 2$ and $\sqrt{x + 3} = 3$.
c) Intervals of increase: $(-\infty, 0)$, $(0, \infty)$; no intervals of decrease
d) $D = \{x \in \mathbf{R}\}$,
$R = \{y \in \mathbf{R} \mid 0 < y < 2 \text{ or } y \geq 3\}$

Chapter 2

Getting Started, p. 66

1. **a)** $\frac{4}{3}$ **b)** $-\frac{6}{7}$

2. **a)** Each successive first difference is 2 times the previous first difference. The function is exponential.
b) The second differences are all 6. The function is quadratic.

3. **a)** $-\frac{3}{2}, 2$ **c)** $45°, 225°$
b) 0 **d)** $-270°, -90°$

4. **a)** vertical compression by a factor of $\frac{1}{2}$
b) vertical stretch by a factor of 2, horizontal translation 4 units to the right
c) vertical stretch by a factor of 3, reflection across *x*-axis, vertical translation 7 units up
d) vertical stretch by a factor of 5, horizontal translation 3 units to the right, vertical translation 2 units down,

5. **a)** $A = 1000(1.08)^t$
b) $1259.71
c) No, since the interest is compounded each year, each year you earn more interest than the previous year.

6. **a)** 15 m; 1 m
b) 24 s
c) 15 m

7.

Linear relations	**Nonlinear relations**
constant; same as slope of line; positive for lines that slope up from left to right; negative for lines that slope down from left to right; 0 for horizontal lines.	variable; can be positive, negative, or 0 for different parts of the same relation

Rates of Change

Lesson 2.1, pp. 76–78

1. **a)** 19 **c)** 13 **e)** 11.4
b) 15 **d)** 12 **f)** 11.04

2. **a)** **i)** 15 m/s **ii)** −5 m/s