Chapter 3

DERIVATIVES AND THEIR APPLICATIONS

We live in a world that is always in flux. Sir Isaac Newton's name for calculus was "the method of fluxions." He recognized in the seventeenth century, as you probably recognize today, that understanding change is important. Newton was what we might call a "mathematical physicist." He developed his method of fluxions to gain a better understanding of the natural world, including motion and gravity. But change is not limited to the natural world, and, since Newton's time, the use of calculus has spread to include applications in the social sciences. Psychology, business, and economics are just a few of the areas in which calculus continues to be an effective problem-solving tool. As we shall see in this chapter, anywhere functions can be used as models, the derivative is certain to be meaningful and useful.

CHAPTER EXPECTATIONS

In this chapter, you will

- make connections between the concept of motion and the concept of derivatives, **Section 3.1**
- solve problems involving rates of change, **Section 3.1**
- determine second derivatives, **Section 3.1**
- determine the extreme values of a function, **Section 3.2**
- solve problems by applying mathematical models and their derivatives to determine, interpret, and communicate the mathematical results, **Section 3.3**
- solve problems by determining the maximum and minimum values of a mathematical model, **Career Link, Sections 3.3, 3.4**

Review of Prerequisite Skills

In Chapter 2, we developed an understanding of derivatives and differentiation. In this chapter, we will consider a variety of applications of derivatives. The following skills will be helpful:

- graphing polynomial and simple rational functions
- working with circles in standard position
- solving polynomial equations
- finding the equations of tangents and normals
- using the following formulas:
  circle: circumference: $C = 2\pi r$, area: $A = \pi r^2$
  right circular cylinder: surface area: $SA = 2\pi rh + 2\pi r^2$, volume: $V = \pi r^2 h$

Exercise

1. Sketch the graph of each function.

a. $2x + 3y - 6 = 0$

b. $3x - 4y = 12$

c. $y = \sqrt{x}$

d. $y = \sqrt{x - 2}$

e. $y = x^2 - 4$

f. $y = -x^2 + 9$

2. Solve each equation, where $x, t \in \mathbf{R}$.

a. $3(x - 2) + 2(x - 1) - 6 = 0$

b. $\frac{1}{3}(x - 2) + \frac{2}{5}(x + 3) = \frac{x - 5}{2}$

c. $t^2 - 4t + 3 = 0$

d. $2t^2 - 5t - 3 = 0$

e. $\frac{6}{t} + \frac{t}{2} = 4$

f. $x^3 + 2x^2 - 3x = 0$

g. $x^3 - 8x^2 + 16x = 0$

h. $4t^3 + 12t^2 - t - 3 = 0$

i. $4t^4 - 13t^2 + 9 = 0$

3. Solve each inequality, where $x \in \mathbf{R}$.

a. $3x - 2 > 7$

b. $x(x - 3) > 0$

c. $-x^2 + 4x > 0$

4. Determine the area of each figure. Leave your answers in terms of π, where applicable.

a. square: perimeter 20 cm

b. rectangle: length 8 cm, width 6 cm

c. circle: radius 7 cm

d. circle: circumference 12π cm

5. Two measures of each right circular cylinder are given. Calculate the two remaining measures.

	Radius, r	**Height, h**	**Surface Area, $S = 2\pi rh + 2\pi r^2$**	**Volume, $V = \pi r^2 h$**
a.	4 cm	3 cm		
b.	4 cm			96π cm^3
c.		6 cm		216π cm^3
d.	5 cm		120π cm^2	

6. Calculate total surface area and volume for cubes with the following side lengths:

a. 3 cm

b. $\sqrt{5}$ cm

c. $2\sqrt{3}$ cm

d. $2k$ cm

7. Express each set of numbers using interval notation.

a. $\{x \in \mathbf{R} \mid x > 3\}$

b. $\{x \in \mathbf{R} \mid x \le -2\}$

c. $\{x \in \mathbf{R} \mid x < 0\}$

d. $\{x \in \mathbf{R} \mid x \ge -5\}$

e. $\{x \in \mathbf{R} \mid -2 < x \le 8\}$

f. $\{x \in \mathbf{R} \mid -4 < x < 4\}$

8. Express each interval using set notation, where $x \in \mathbf{R}$.

a. $(5, \infty)$

b. $(-\infty, 1]$

c. $(-\infty, \infty)$

d. $[-10, 12]$

e. $(-1, 3)$

f. $[2, 20)$

9. Use graphing technology to graph each function and determine its maximum and/or minimum values.

a. $f(x) = x^2 - 5$

b. $f(x) = -x^2 - 10x$

c. $f(x) = 3x^2 - 30x + 82$

d. $f(x) = |x| - 1$

e. $f(x) = 3\sin x + 2$

f. $f(x) = -2\cos 2x - 5$

CAREER LINK | Investigate

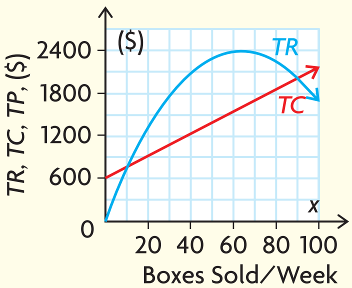
CHAPTER 3: MAXIMIZING PROFITS

We live in a world that demands we determine the best, the worst, the maximum, and the minimum. Through mathematical modelling, calculus can be used to establish optimum operating conditions for processes that seem to have competing variables. For example, minimizing transportation costs for a delivery vehicle would seem to require the driver to travel as fast as possible to reduce hourly wages. Higher rates of speed, however, increase gas consumption. With calculus, an optimal speed can be established to minimize the total cost of driving the delivery vehicle, considering both gas consumption and hourly wages. In this chapter, calculus tools will be used in realistic contexts to solve optimization problems—from business applications (such as minimizing cost) to psychology (such as maximizing learning).

Case Study—Entrepreneurship

In the last 10 years, the Canadian economy has seen a dramatic increase in the number of small businesses. An ability to use graphs to interpret the marginal profit (a calculus concept) will help an entrepreneur make good business decisions.

A person with an old family recipe for gourmet chocolates decides to open her own business. Her weekly total revenue (*TR*) and total cost (*TC*) curves are plotted on the set of axes shown.



DISCUSSION QUESTIONS

Make a rough sketch of the graph in your notes, and answer the following questions:

1. What sales interval would keep the company profitable? What do we call these values?
2. Superimpose the total profit (*TP*) curve over the *TR* and *TC* curves. What would the sales level have to be to obtain maximum profits? Estimate the slopes on the *TR* and *TC* curves at this level of sales. Should they be the same? Why or why not?
3. On a separate set of axes, sketch the marginal profit (the extra profit earned by selling one more box of chocolates), $MP = \frac{dTP}{dx}$. What can you say about the marginal profit as the level of sales progresses from just less than the maximum to the maximum, and then to just above the maximum? Does this make sense? Explain.

Section 3.1—Higher-Order Derivatives, Velocity, and Acceleration

Derivatives arise in the study of motion. The velocity of a car is the rate of change of displacement at a specific point in time. We have already developed the rules of differentiation and learned how to interpret the derivative at a point on a curve. We can now extend the applications of differentiation to higher-order derivatives. This will allow us to discuss the applications of the first and second derivatives to rates of change as an object moves in a straight line, either vertically or horizontally, such as a space shuttle taking off into space or a car moving along a straight section of road.

Higher-Order Derivatives

The function $y = f(x)$ has a first derivative $y = f'(x)$. The **second derivative** of $y = f(x)$ is the derivative of $y = f'(x)$.

The derivative of $f(x) = 10x^4$ with respect to x is $f'(x) = 40x^3$. If we differentiate $f'(x) = 40x^3$, we obtain $f''(x) = 120x^2$. This new function is called the second derivative of $f(x) = 10x^4$, and is denoted $f''(x)$.

For $y = 2x^3 - 5x^2$, the first derivative is $\frac{dy}{dx} = 6x^2 - 10x$ and the second derivative is $\frac{d^2y}{dx^2} = 12x - 10$.

Note the appearance of the superscripts in the second derivative. The reason for this choice of notation is that the second derivative is the derivative of the first derivative. That is, we write $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Other notations that are used to represent first and second derivatives of $y = f(x)$ are $\frac{dy}{dx} = f'(x) = y'$ and $\frac{d^2y}{dx^2} = f''(x) = y''$.

EXAMPLE 1 **Selecting a strategy to determine the second derivative of a rational function**

Determine the second derivative of $f(x) = \frac{x}{1 + x}$ when $x = 1$.

Solution

Write $f(x) = \frac{x}{1 + x}$ as a product, and differentiate.

$$f(x) = x(x + 1)^{-1}$$

$$f'(x) = (1)(x + 1)^{-1} + (x)(-1)(x + 1)^{-2}(1)$$

(Product and power of a function rule)

$$= \frac{1}{x + 1} - \frac{x}{(x + 1)^2}$$

$$= \frac{1(x + 1)}{(x + 1)^2} - \frac{x}{(x + 1)^2}$$ (Simplify)

$$= \frac{x + 1 - x}{(x + 1)^2}$$

$$= \frac{1}{(1 + x)^2}$$ (Rewrite as a power function)

$$= (1 + x)^{-2}$$

Differentiating again to determine the second derivative,

$$f''(x) = -2(1 + x)^{-3}(1)$$ (Power of a function rule)

$$= \frac{-2}{(1 + x)^3}$$

When $x = 1$, $$f''(1) = \frac{-2}{(1 + 1)^3}$$ (Evaluate)

$$= \frac{-2}{8}$$

$$= -\frac{1}{4}$$

Velocity and Acceleration—Motion on a Straight Line

One reason for introducing the derivative is the need to calculate rates of change. Consider the motion of an object along a straight line. Examples are a car moving along a straight section of road, a ball dropped from the top of a building, and a rocket in the early stages of flight.

When studying motion along a line, we assume that the object is moving along a number line, which gives us an origin of reference, as well as positive and negative directions. The position of the object on the line relative to the origin is a function of time, t, and is commonly denoted by $s(t)$.

The rate of change of $s(t)$ with respect to time is the object's **velocity**, $v(t)$, and the rate of change of the velocity with respect to time is its **acceleration**, $a(t)$. The absolute value of the velocity is called **speed**.

Motion on a Straight Line

An object that moves along a straight line with its position determined by a function of time, $s(t)$, has a velocity of $\boldsymbol{v}(t) = \boldsymbol{s}'(t)$ and an acceleration of $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{s}''(t)$ at time t.
In Leibniz notation,

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The speed of the object is $|v(t)|$.

The units of velocity are displacement divided by time. The most common units are m/s.

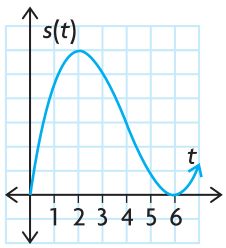
The units of acceleration are displacement divided by $(\text{time})^2$. The most common units are metres per second per second, or metres per second squared, or m/s^2.

Since we are assuming that the motion is along the number line, it follows that when the object is moving to the right at time t, $v(t) > 0$ and when the object is moving to the left at time t, $v(t) < 0$. If $v(t) = 0$, the object is stationary at time t.

The object is accelerating when $a(t)$ and $v(t)$ are both positive or both negative. That is, the product of $a(t)$ and $v(t)$ is positive.

The object is decelerating when $a(t)$ is positive and $v(t)$ is negative, *or* when $a(t)$ is negative and $v(t)$ is positive. This happens when the product of $a(t)$ and $v(t)$ is negative.

EXAMPLE 2 Reasoning about the motion of an object along a straight line



An object is moving along a straight line. Its position, $s(t)$, to the right of a fixed point is given by the graph shown. When is the object moving to the right, when is it moving to the left, and when is it at rest?

Solution

The object is moving to the right whenever $s(t)$ is increasing, or $v(t) > 0$.

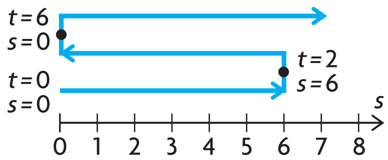
From the graph, $s(t)$ is increasing for $0 < t < 2$ and for $t > 6$.

For $2 < t < 6$, the value of $s(t)$ is decreasing, or $v(t) < 0$, so the object is moving to the left.

At $t = 2$, the direction of motion of the object changes from right to left, $v(t) = 0$, so the object is stationary at $t = 2$.

At $t = 6$, the direction of motion of the object changes from left to right, $v(t) = 0$, so the object is stationary at $t = 6$.

The motion of the object can be illustrated by the following position diagram.



EXAMPLE 3 Connecting motion to displacement, velocity, and acceleration

The position of an object moving on a line is given by $s(t) = 6t^2 - t^3$, $t \geq 0$, where s is in metres and t is in seconds.

a. Determine the velocity and acceleration of the object at $t = 2$.
b. At what time(s) is the object at rest?
c. In which direction is the object moving at $t = 5$?
d. When is the object moving in a positive direction?
e. When does the object return to its initial position?

Solution

a. The velocity at time t is $v(t) = s'(t) = 12t - 3t^2$.

At $t = 2$, $v(2) = 12(2) - 3(2)^2 = 12$.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 12 - 6t$.

At $t = 2$, $a(2) = 12 - 6(2) = 0$.

At $t = 2$, the velocity is 12 m/s and the acceleration is 0 m/s^2.

We note that at $t = 2$, the object is moving at a constant velocity, since the acceleration is 0 m/s^2. The object is neither speeding up nor slowing down.

b. The object is at rest when the velocity is 0—that is, when $v(t) = 0$.

$12t - 3t^2 = 0$

$3t(4 - t) = 0$

$t = 0$ or $t = 4$

The object is at rest at $t = 0$ s and at $t = 4$ s.

c. To determine the direction of motion, we use the velocity at time $t = 5$.

$$v(5) = 12(5) - 3(5)^2 = -15$$

The object is moving in a negative direction at $t = 5$.

d. The object moves in a positive direction when $v(t) > 0$.

$12t - 3t^2 > 0$ (Divide by -3)

$t^2 - 4t < 0$ (Factor)

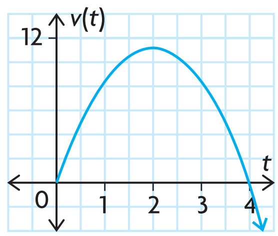
$t(t - 4) < 0$

There are two cases to consider since a product is negative when the first factor is positive and the second is negative, and vice versa.

Case 1	Case 2
$t > 0$ and $t - 4 < 0$	$t < 0$ and $t - 4 > 0$
so $t > 0$ and $t < 4$	so $t < 0$ and $t > 4$
$0 < t < 4$	no solution

Therefore, $0 < t < 4$.

The graph of the velocity function is a parabola opening downward, as shown.



From the graph and the algebraic solution above, we conclude that $v(t) > 0$ for $0 < t < 4$.

The object is moving to the right during the interval $0 < t < 4$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's initial position is at 0.

To find other times when the object is at this point, we solve $s(t) = 0$.

$6t^2 - t^3 = 0$ (Factor)

$t^2(6 - t) = 0$ (Solve)

$t = 0$ or $t = 6$

The object returns to its initial position after 6 s.

EXAMPLE 4 Analyzing motion along a horizontal line

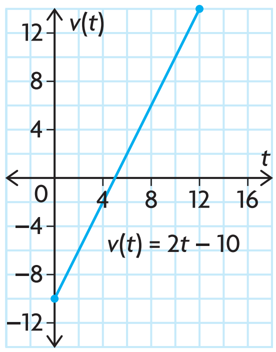
Discuss the motion of an object moving on a horizontal line if its position is given by $s(t) = t^2 - 10t$, $0 \le t \le 12$, where s is in metres and t is in seconds. Include the initial velocity, final velocity, and any acceleration in your discussion.

Solution

The initial position of the object occurs at time $t = 0$. Since $s(0) = 0$, the object starts at the origin.

The velocity at time t is $v(t) = s'(t) = 2t - 10 = 2(t - 5)$.

The object is at rest when $v(t) = 0$.



$$2(t - 5) = 0$$
$$t = 5$$

So the object is at rest after $t = 5$ s.

$v(t) > 0$ for $5 < t \le 12$, therefore the object is moving to the right during this time interval.

$v(t) < 0$ for $0 \le t < 5$, therefore the object is moving to the left during this time interval.

The initial velocity is $v(0) = -10$. So initially, the object is moving 10 m/s to the left.

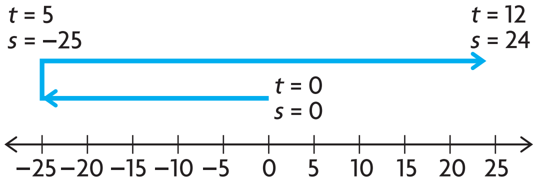
At $t = 12$, $v(12) = 2(12) - 10 = 14$.

So the final velocity is 14 m/s to the right. The velocity graph is shown.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 2$. The acceleration is always 2 m/s^2. This means that the object is constantly increasing its velocity at a rate of 2 metres per second per second.

In conclusion, the object moves to the left for $0 \le t < 5$ and to the right for $5 < t \le 12$. The initial velocity is -10 m/s and the final velocity is 14 m/s.

To draw a diagram of the motion, determine the object's position at $t = 5$ and $t = 12$. (The actual path of the object is back and forth on a line.)



$s(5) = 5^2 - 10(5) = -25$

$s(12) = 12^2 - 10(12) = 24$

EXAMPLE 5 Analyzing motion under gravity near the surface of Earth

A baseball is hit vertically upward. The position function $s(t)$, in metres, of the ball above the ground is $s(t) = -5t^2 + 30t + 1$, where t is in seconds.

a. Determine the maximum height reached by the ball.

b. Determine the velocity of the ball when it is caught 1 m above the ground.

Solution

a. The maximum height occurs when the velocity of the ball is zero—that is, when the slope of the tangent to the graph is zero.

The velocity function is $v(t) = s'(t) = -10t + 30$.

Solving $v(t) = 0$, we obtain $t = 3$. This is the moment when the ball changes direction from up to down.

$$s(3) = -5(3)^2 + 30(3) + 1$$
$$= 46$$

Therefore, the maximum height reached by the ball is 46 m.

b. When the ball is caught, $s(t) = 1$. To find the time at which this occurs, solve

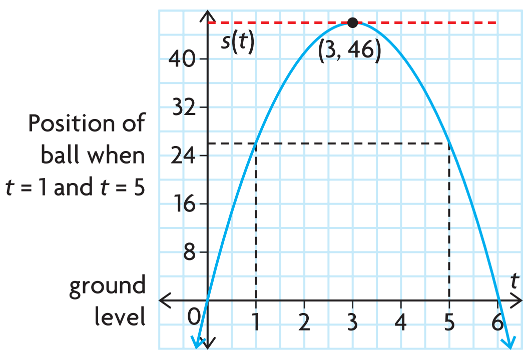
$$1 = -5t^2 + 30t + 1$$
$$0 = -5t(t - 6)$$
$$t = 0 \text{ or } t = 6$$

Since $t = 0$ is the time at which the ball leaves the bat, the time at which the ball is caught is $t = 6$.

The velocity of the ball when it is caught is $v(6) = -10(6) + 30 = -30$ m/s.

This negative value is reasonable, since the ball is falling (moving in a negative direction) when it is caught.

Note, however, that the graph of $s(t)$ does not represent the path of the ball. We think of the ball as moving in a straight line along a vertical s-axis, with the direction of motion reversing when $s = 46$.



To see this, note that the ball is at the same height at time $t = 1$, when $s(1) = 26$, and at time $t = 5$, when $s(5) = 26$.

IN SUMMARY

Key Ideas

- The derivative of the derivative function is called the second derivative.
- If the position of an object, $s(t)$, is a function of time, t, then the first derivative of this function represents the velocity of the object at time t.
  $v(t) = s'(t) = \frac{ds}{dt}$
- Acceleration, $a(t)$, is the instantaneous rate of change of velocity with respect to time. Acceleration is the first derivative of the velocity function and the second derivative of the position function.
  $a(t) = v'(t) = s''(t)$, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Need to Know

- Negative velocity, $v(t) < 0$ or $s'(t) < 0$, indicates that an object is moving in a negative direction (left or down) at time t.
- Positive velocity, $v(t) > 0$ or $s'(t) > 0$, indicates that an object is moving in a positive direction (right or up) at time t.
- Zero velocity, $v(t) = 0$ or $s'(t) = 0$, indicates that an object is stationary and that a possible change in direction may occur at time t.
- Notations for the second derivative are $f''(x)$, $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}[f(x)]$, or y'' of a function $y = f(x)''$.
- Negative acceleration, $a(t) < 0$ or $v'(t) < 0$, indicates that the velocity is decreasing.
- Positive acceleration, $a(t) > 0$ or $v'(t) > 0$, indicates that the velocity is increasing.
- Zero acceleration, $a(t) = 0$ or $v'(t) = 0$, indicates that the velocity is constant and the object is neither accelerating nor decelerating.
- An object is accelerating (speeding up) when its velocity and acceleration have the same signs.
- An object is decelerating (slowing down) when its velocity and acceleration have opposite signs.
- The speed of an object is the magnitude of its velocity at time t.
  speed $= |v(t)| = |s'(t)|$.

Exercise 3.1

PART A

C

1. Explain and discuss the difference in velocity at times $t = 1$ and $t = 5$ for $v(t) = 2t - t^2$.

2. Determine the second derivative of each of the following:

a. $y = x^{10} + 3x^6$

b. $f(x) = \sqrt{x}$

c. $y = (1 - x)^2$

d. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

e. $y = 4x^{\frac{3}{2}} - x^{-2}$

f. $f(x) = \frac{2x}{x + 1}$

g. $y = x^2 + \frac{1}{x^2}$

h. $g(x) = \sqrt{3x - 6}$

i. $y = (2x + 4)^3$

j. $h(x) = \sqrt[3]{x^5}$

K

3. Each of the following position functions describes the motion of an object along a straight line. Find the velocity and acceleration as functions of t, $t \geq 0$.

a. $s(t) = 5t^2 - 3t + 15$

b. $s(t) = 2t^3 + 36t - 10$

c. $s(t) = t - 8 + \frac{6}{t}$

d. $s(t) = (t - 3)^2$

e. $s(t) = \sqrt{t + 1}$

f. $s(t) = \frac{9t}{t + 3}$

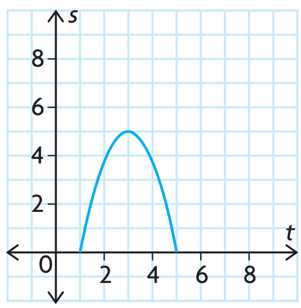
4. Answer the following questions for each position versus time graph below:

i. When is the velocity zero?

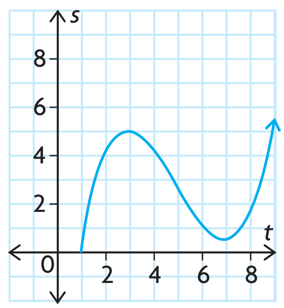
ii. When is the object moving in a positive direction?

iii. When is the object moving in a negative direction?

a.



b.



5. A particle moves along a straight line with the equation of motion $s = \frac{1}{3}t^3 - 2t^2 + 3t, t \geq 0$.
   a. Determine the particle's velocity and acceleration at any time t.
   b. When does the motion of the particle change direction?
   c. When does the particle return to its initial position?

PART B

6. Each function describes the position of an object that moves along a straight line. Determine whether the object is moving in a positive or negative direction at time $t = 1$ and at time $t = 4$.
   a. $s(t) = -\frac{1}{3}t^2 + t + 4$
   b. $s(t) = t(t - 3)^2$
   c. $s(t) = t^3 - 7t^2 + 10t$
7. Starting at $t = 0$, a particle moves along a line so that its position after t seconds is $s(t) = t^2 - 6t + 8$, where s is in metres.
   a. What is its velocity at time t?
   b. When is its velocity zero?
8. When an object is launched vertically from ground level with an initial velocity of 40 m/s, its position after t seconds is $s(t) = 40t - 5t^2$ metres above ground level.
   a. When does the object stop rising?
   b. What is its maximum height?
9. An object moves in a straight line, and its position, s, in metres after t seconds is $s(t) = 8 - 7t + t^2$.
   a. Determine the velocity when $t = 5$
   b. Determine the acceleration when $t = 5$.

A 10. The position function of a moving object is $s(t) = t^{\frac{5}{2}}(7 - t), t \geq 0$, in metres, at time t, in seconds.
   a. Calculate the object's velocity and acceleration at any time t.
   b. After how many seconds does the object stop?
   c. When does the motion of the object change direction?
   d. When is its acceleration positive?
   e. When does the object return to its original position?

11. A ball is thrown upward, and its height, h, in metres above the ground after t seconds is given by $h(t) = -5t^2 + 25t, t \geq 0$.
   a. Calculate the ball's initial velocity.
   b. Calculate its maximum height.
   c. When does the ball strike the ground, and what is its velocity at this time?

12. A dragster races down a 400 m strip in 8 s. Its distance, in metres, from the starting line after t seconds is $s(t) = 6t^2 + 2t$.

    a. Determine the dragster's velocity and acceleration as it crosses the finish line.

    b. How fast was it moving 60 m down the strip?

13. For each of the following position functions, discuss the motion of an object moving on a horizontal line, where s is in metres and t is in seconds. Make a graph similar to that in Example 4, showing the motion for $t \geq 0$. Find the velocity and acceleration, and determine the extreme positions (farthest left and right) for $t \geq 0$.

    a. $s(t) = 10 + 6t - t^2$

    b. $s(t) = t^3 - 12t - 9$

14. If the position function of an object is $s(t) = t^5 - 10t^2$, at what time, t, in seconds, will the acceleration be zero? Is the object moving toward or away from the origin at this instant?

T

15. The position–time relationship for a moving object is given by $s(t) = kt^2 + (6k^2 - 10k)t + 2k$, where k is a non-zero constant.

    a. Show that the acceleration is constant.

    b. Find the time at which the velocity is zero, and determine the position of the object when this occurs.

PART C

16. An elevator is designed to start from a resting position without a jerk. It can do this if the acceleration function is continuous.

    a. Show that the acceleration is continuous at $t = 0$ for the following position function

    $$s(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^3}{t^2 + 1}, \text{ if } t \geq 0 \end{cases}$$

    b. What happens to the velocity and acceleration for very large values of t?

17. An object moves so that its velocity, v, is related to its position, s, according to $v = \sqrt{b^2 + 2gs}$, where b and g are constants. Show that the acceleration of the object is constant.

18. Newton's law of motion for a particle of mass m moving in a straight line says that $F = ma$, where F is the force acting on the particle and a is the acceleration of the particle. In relativistic mechanics, this law is replaced by

    $$F = \frac{m_0 \frac{d}{dt} v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},$$ where m_0 is the mass of the particle measured at rest, v is the velocity of the particle and c is the speed of light. Show that

    $$F = \frac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}.$$

Section 3.2—Maximum and Minimum on an Interval (Extreme Values)

INVESTIGATION The purpose of this investigation is to determine how the derivative can be used to determine the maximum (largest) value or the minimum (smallest) value of a function on a given interval. Together, these are called the **absolute extrema** on an interval.

A. For each of the following functions, determine, by completing the square, the value of x that produces a maximum or minimum function value on the given interval.

   i. $f(x) = -x^2 + 6x - 3, 0 \leq x \leq 5$
   ii. $f(x) = -x^2 - 2x + 11, -3 \leq x \leq 4$
   iii. $f(x) = 4x^2 - 12x + 7, -1 \leq x \leq 4$

B. For each function in part A, determine the value of c such that $f'(c) = 0$.

C. Compare the values obtained in parts A and B for each function. Why does it make sense to say that the pattern you discovered is not merely a coincidence?

D. Using a graphing calculator, graph each of the following functions and determine all the values of x that produce a maximum or minimum value on the given interval.

   i. $f(x) = x^3 - 3x^2 - 8x + 10, -2 \leq x \leq 4$
   ii. $f(x) = x^3 - 12x + 5, -3 \leq x \leq 3$
   iii. $f(x) = 3x^3 - 15x^2 + 9x + 23, 0 \leq x \leq 4$
   iv. $f(x) = -2x^3 + 12x + 7, -2 \leq x \leq 2$
   v. $f(x) = -x^3 - 2x^2 + 15x + 23, -4 \leq x \leq 3$

E. For each function in part D, determine all the values of c such that $f'(c) = 0$.

F. Compare the values obtained in parts D and E for each function. What do you notice?

G. From your comparisons in parts C and F, state a method for using the derivative of a function to determine values of the variable that give maximum or minimum values of the function.

H. Repeat part D for the following functions, using the given intervals.

i. $f(x) = -x^2 + 6x - 3, 4 \leq x \leq 8$

ii. $f(x) = 4x^2 - 12x + 7, 2 \leq x \leq 6$

iii. $f(x) = x^3 - 3x^2 - 9x + 10, -2 \leq x \leq 6$

iv. $f(x) = x^3 - 12x + 5, 0 \leq x \leq 5$

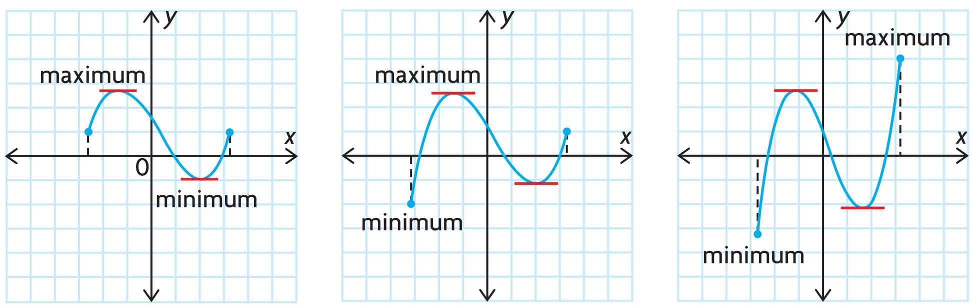
v. $f(x) = x^3 - 5x^2 + 3x + 7, -2 \leq x \leq 5$

I. In parts C and F, you saw that a maximum or minimum can occur at points $(c, f(c))$, where $f'(c) = 0$. From your observations in part H, state other values of the variable that can produce a maximum or minimum in a given interval.

Checkpoint: Check Your Understanding

The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" $(f'(c) = 0)$ or at an endpoint of the interval. The minimum value occurs at a "valley" $(f'(c) = 0)$ or at an endpoint. This is true no matter how many peaks and valleys the graph has in the interval.

In the following three graphs, the derivative equals zero at two points:



Algorithm for Finding Maximum or Minimum (Extreme) Values

If a function $f(x)$ has a derivative at every point in the interval $a \leq x \leq b$, calculate $f(x)$ at

- all points in the interval $a \leq x \leq b$, where $f'(x) = 0$
- the endpoints $x = a$ and $x = b$

The maximum value of $f(x)$ on the interval $a \leq x \leq b$ is the largest of these values, and the minimum value of $f(x)$ on the interval is the smallest of these values.

When using the algorithm above it is important to consider the function $f(x)$ on a finite interval—that is, an interval that includes its endpoints. Otherwise, the function may not attain a maximum or minimum value.

EXAMPLE 1 Selecting a strategy to determine absolute extrema

Find the extreme values of the function $f(x) = -2x^3 + 9x^2 + 4$ on the interval $x \in [-1, 5]$.

Solution

The derivative is $f'(x) = -6x^2 + 18x$.

If we set $f'(x) = 0$, we obtain $-6x(x - 3) = 0$, so $x = 0$ or $x = 3$.

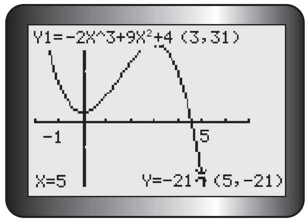
Both values lie in the given interval, $[-1, 5]$.

We can then evaluate $f(x)$ for these values and at the endpoints $x = -1$ and $x = 5$ to obtain

$$f(-1) = 15$$
$$f(0) = 4$$
$$f(3) = 31$$
$$f(5) = -21$$

Therefore, the maximum value of $f(x)$ on the interval $-1 \leq x \leq 5$ is $f(3) = 31$, and the minimum value is $f(5) = -21$.

Graphing the function on this interval verifies our analysis.



EXAMPLE 2 Solving a problem involving absolute extrema

The amount of current, in amperes (A), in an electrical system is given by the function $C(t) = -t^3 + t^2 + 21t$, where t is the time in seconds and $0 \leq t \leq 5$. Determine the times at which the current is at its maximum and minimum, and determine the amount of current in the system at these times.

Solution

The derivative is $\frac{dC}{dt} = -3t^2 + 2t + 21$.

If we set $\frac{dC}{dt} = 0$, we obtain

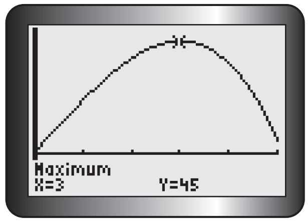
$$-3t^2 + 2t + 21 = 0 \quad \text{(Multiply by } -1\text{)}$$
$$3t^2 - 2t - 21 = 0 \quad \text{(Factor)}$$
$$(3t + 7)(t - 3) = 0 \quad \text{(Solve)}$$

Therefore, $t = -\frac{7}{3}$ or $t = 3$.

Only $t = 3$ is in the given interval, so we evaluate $C(t)$ at $t = 0$, $t = 3$, and $t = 5$ as follows:

$$C(0) = 0$$
$$C(3) = -3^3 + 3^2 + 21(3) = 45$$
$$C(5) = -5^3 + 5^2 + 21(5) = 5$$

The maximum is 45 A at time $t = 3$ s, and the minimum is 0 A at time $t = 0$ s.



Graphing the function on this interval verifies our analysis.

EXAMPLE 3 Selecting a strategy to determine the absolute minimum

The amount of light intensity on a point is given by the function $I(t) = \frac{t^2 + 2t + 16}{t + 2}$, where t is the time in seconds and $t \in [0, 14]$. Determine the time of minimal intensity.

Solution

Note that the function is not defined for $t = -2$. Since this value is not in the given interval, we need not worry about it.

The derivative is

$$I'(t) = \frac{(2t + 2)(t + 2) - (t^2 + 2t + 16)(1)}{(t + 2)^2} \quad \text{(Quotient rule)}$$

$$= \frac{2t^2 + 6t + 4 - t^2 - 2t - 16}{(t + 2)^2} \quad \text{(Expand and simplify)}$$

$$= \frac{t^2 + 4t - 12}{(t + 2)^2}$$

If we set $I'(t) = 0$, we only need to consider when the numerator is 0.

$$t^2 + 4t - 12 = 0 \quad \text{(Factor)}$$
$$(t + 6)(t - 2) = 0 \quad \text{(Solve)}$$
$$t = -6 \text{ or } t = 2$$

Only $t = 2$ is in the given interval, so we evaluate $I(t)$ for $t = 0, 2,$ and 14.

$$I(0) = 8$$
$$I(2) = \frac{4 + 4 + 16}{4} = 6$$
$$I(14) = \frac{14^2 + 2(14) + 16}{16} = 15$$

Note that the calculation can be simplified by rewriting the intensity function as shown.

$$I(t) = \frac{t^2 + 2t}{t + 2} + \frac{16}{t + 2}$$
$$= t + 16(t + 2)^{-1}$$

Then $I'(t) = 1 - 16(t + 2)^{-2}$

$$= 1 - \frac{16}{(t + 2)^2}$$

Setting $I'(t) = 0$ gives

$$1 = \frac{16}{(t + 2)^2}$$
$$t^2 + 4t + 4 = 16$$
$$t^2 + 4t - 12 = 0$$

As before, $t = -6$ or $t = 2$.
The evaluations are also simplified.

$$I(0) = 0 + \frac{16}{2} = 8$$
$$I(2) = 2 + \frac{16}{4} = 6$$
$$I(14) = 14 + \frac{16}{16} = 15$$

Either way, the minimum amount of light intensity occurs at $t = 2$ s on the given time interval.

IN SUMMARY

Key Ideas

- The maximum and minimum values of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" ($f'(c) = 0$) or at an endpoint of the interval, [a, b],
- The minimum value occurs at a "valley" ($f'(c) = 0$) or at an endpoint of the interval, [a, b].

Need to Know

- **Algorithm for Finding Extreme Values:**

  For a function $f(x)$ that has a derivative at every point in an interval [a, b], the maximum or minimum values can be found by using the following procedure:

  1. Determine $f'(x)$. Find all points in the interval $a \leq x \leq b$, where $f'(x) = 0$.
  2. Evaluate $f(x)$ at the endpoints a and b, and at points where $f'(x) = 0$.
  3. Compare all the values found in step 2.
     - The largest of these values is the maximum value of $f(x)$ on the interval $a \leq x \leq b$.
     - The smallest of these values is the minimum value of $f(x)$ on the interval $a \leq x \leq b$.

Exercise 3.2

PART A

C 1. State, with reasons, why the maximum/minimum algorithm can or cannot be used to determine the maximum and minimum values of the following functions:

a. $y = x^3 - 5x^2 + 10, -5 \leq x \leq 5$

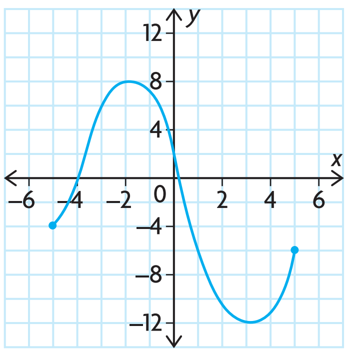
b. $y = \dfrac{3x}{x - 2}, -1 \leq x \leq 3$

c. $y = \dfrac{x}{x^2 - 4}, x \in [0, 5]$

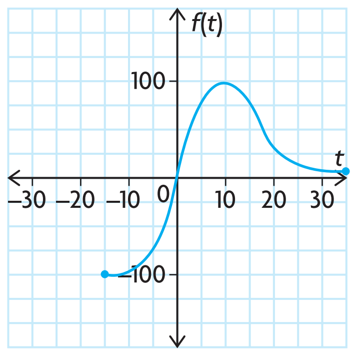
d. $y = \dfrac{x^2 - 1}{x + 3}, x \in [-2, 3]$

2. State the absolute maximum value and the absolute minimum value of each function, if the function is defined on the interval shown.

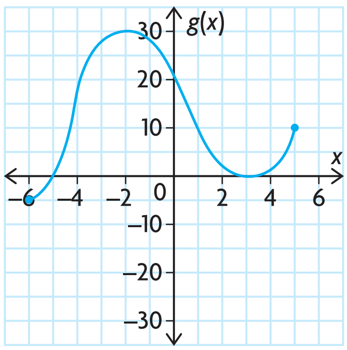
a.



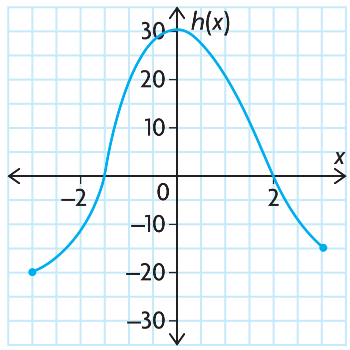
c.



b.



d.



K 3. Determine the absolute extrema of each function on the given interval. Illustrate your results by sketching the graph of each function.

a. $f(x) = x^2 - 4x + 3, 0 \le x \le 3$

b. $f(x) = (x - 2)^2, 0 \le x \le 2$

c. $f(x) = x^3 - 3x^2, -1 \le x \le 3$

d. $f(x) = x^3 - 3x^2, x \in [-2, 1]$

e. $f(x) = 2x^3 - 3x^2 - 12x + 1, x \in [-2, 0]$

f. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, x \in [0, 4]$

PART B

T 4. Using the algorithm for finding maximum or minimum values, determine the absolute extreme values of each function on the given interval.

a. $f(x) = x + \frac{4}{x}, 1 \leq x \leq 10$

b. $f(x) = 4\sqrt{x} - x, x \in [2, 9]$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20, x \in [-3, 4]$

e. $f(x) = \frac{4x}{x^2 + 1}, -2 \leq x \leq 4$

f. $f(x) = \frac{4x}{x^2 + 1}, x \in [2, 4]$

5. a. An object moves in a straight line. Its velocity, in m/s, at time t is $v(t) = \frac{4t^2}{4 + t^3}, t \geq 0$. Determine the maximum and minimum velocities over the time interval $1 \leq t \leq 4$.

b. Repeat part a., if $v(t) = \frac{4t^2}{1 + t^2}, t \geq 0$.

A 6. A swimming pool is treated periodically to control the growth of bacteria. Suppose that t days after a treatment, the number of bacteria per cubic centimetre is $N(t) = 30t^2 - 240t + 500$. Determine the lowest number of bacteria during the first week after the treatment.

7. The fuel efficiency, E, in litres per 100 kilometres, for a car driven at speed v, in km/h, is $E(v) = \frac{1600v}{v^2 + 6400}$.

a. If the speed limit is 100 km/h, determine the legal speed that will maximize the fuel efficiency.

b. Repeat part a., using a speed limit of 50 km/h.

c. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is increasing.

d. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is decreasing.

8. The concentration $C(t)$, in milligrams per cubic centimetre, of a certain medicine in a patient's bloodstream is given by $C(t) = \frac{0.1t}{(t + 3)^2}$, where t is the number of hours after the medicine is taken. Determine the maximum and minimum concentrations between the first and sixth hours after the medicine is taken.

9. Technicians working for the Ministry of Natural Resources found that the amount of a pollutant in a certain river can be represented by $P(t) = 2t + \frac{1}{(162t + 1)}$, $0 \leq t \leq 1$, where t is the time, in years, since a cleanup campaign started. At what time was the pollution at its lowest level?

10. A truck travelling at x km/h, where $30 \leq x \leq 120$, uses gasoline at the rate of $r(x)$ L/100 km, where $r(x) = \frac{1}{4}\left(\frac{4900}{x} + x\right)$. If fuel costs \$1.15/L, what speed will result in the lowest fuel cost for a trip of 200 km? What is the lowest total cost for the trip?

11. The polynomial function $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$, $0 \leq x \leq 75$, models the shape of a roller-coaster track, where f is the vertical displacement of the track and x is the horizontal displacement of the track. Both displacements are in metres. Determine the absolute maximum and minimum heights along this stretch of track.

12. a. Graph the cubic function with an absolute minimum at $(-2, -12)$, a local maximum at $(0, 3)$, a local minimum at $(2, -1)$, and an absolute maximum at $(4, 9)$. *Note*: local maximum and minimum values occur at peaks and valleys of a graph and do not have to be absolute extrema.
    b. What is the domain of this function?
    c. Where is the function increasing? Where is it decreasing?

13. What points on an interval must you consider to determine the absolute maximum or minimum value on the interval? Why?

PART C

T
14. In a certain manufacturing process, when the level of production is x units, the cost of production, in dollars, is $C(x) = 3000 + 9x + 0.05x^2$, $1 \leq x \leq 300$. What level of production, x, will minimize the unit cost, $U(x) = \frac{C(x)}{x}$? Keep in mind that the production level must be an integer.

15. Repeat question 14. If the cost of production is $C(x) = 6000 + 9x + 0.05x^2$, $1 \leq x \leq 300$.

Mid-Chapter Review

1. Determine the second derivative of each of the following functions:
   a. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$
   b. $f(x) = (2x - 5)^3$
   c. $y = \frac{15}{x + 3}$
   d. $g(x) = \sqrt{x^2 + 1}$
2. The displacement of an object in motion is described by $s(t) = t^3 - 21t^2 + 90t$, where the horizontal displacement, s, is measured in metres at t seconds.
   a. Calculate the displacement at 3 s.
   b. Calculate the velocity at 5 s.
   c. Calculate the acceleration at 4 s.
3. A ball is thrown upward. Its motion can be described by $h(t) = -4.9t^2 + 6t + 2$, where the height, h, is measured in metres at t seconds.
   a. Determine the initial velocity.
   b. When does the ball reach its maximum height?
   c. When does the ball hit the ground?
   d. What is the velocity of the ball when it hits the ground?
   e. What is the acceleration of the ball on the way up? What is its acceleration on the way down?
4. An object is moving horizontally. The object's displacement, s, in metres at t seconds is described by $s(t) = 4t - 7t^2 + 2t^3$.
   a. Determine the velocity and acceleration at $t = 2$.
   b. When is the object stationary? Describe the motion immediately before and after these times.
   c. At what time, to the nearest tenth of a second, is the acceleration equal to 0? Describe the motion at this time.
5. Determine the absolute extreme values of each function on the given interval, using the algorithm for finding maximum and minimum values.
   a. $f(x) = x^3 + 3x^2 + 1, -2 \leq x \leq 2$
   b. $f(x) = (x + 2)^2, -3 \leq x \leq 3$
   c. $f(x) = \frac{1}{x} - \frac{1}{x^3}, x \in [1, 5]$
6. The volume, V, of 1 kg of H_2O at temperature t between 0 °C and 30 °C can be modelled by $V(t) = -0.000\,067t^3 + 0.008\,504\,3t^2 - 0.064\,26t + 999.87$. Volume is measured in cubic centimetres. Determine the temperature at which the volume of water is the greatest in the given interval.

7. Evaluate each of the following:
   a. $f'(3)$ if $f(x) = x^4 - 3x$
   b. $f'(-2)$ if $f(x) = 2x^3 + 4x^2 - 5x + 8$
   c. $f''(1)$ if $f(x) = -3x^2 - 5x + 7$
   d. $f''(-3)$ if $f(x) = 4x^3 - 3x^2 + 2x - 6$
   e. $f'(0)$ if $f(x) = 14x^2 + 3x - 6$
   f. $f''(4)$ if $f(x) = x^4 + x^5 - x^3$
   g. $f''\left(\frac{1}{3}\right)$ if $f(x) = -2x^5 + 2x - 6 - 3x^3$
   h. $f'\left(\frac{3}{4}\right)$ if $f(x) = -3x^3 - 7x^2 + 4x - 11$
8. On the surface of the Moon, an astronaut can jump higher because the force of gravity is less than it is on Earth. When a certain astronaut jumps, his height, in metres above the Moon's surface, can be modelled by $s(t) = t\left(-\frac{5}{6}t + 1\right)$, where t is measured in seconds. What is the acceleration due to gravity on the Moon?
9. The forward motion of a space shuttle, t seconds after touchdown, is described by $s(t) = 189t - t^{\frac{7}{3}}$, where s is measured in metres.
   a. What is the velocity of the shuttle at touchdown?
   b. How much time is required for the shuttle to stop completely?
   c. How far does the shuttle travel from touchdown to a complete stop?
   d. What is the deceleration 8 s after touchdown?
10. In a curling game, one team's skip slides a stone toward the rings at the opposite end of the ice. The stone's position, s, in metres at t seconds, can be modelled by $s(t) = 12t - 4t^{\frac{3}{2}}$. How far does the stone travel before it stops? How long is it moving?
11. After a football is punted, its height, h, in metres above the ground at t seconds, can be modelled by $h(t) = -4.9t^2 + 21t + 0.45$.
   a. Determine the restricted domain of this model.
   b. When does the ball reach its maximum height?
   c. What is the ball's maximum height?

Section 3.3—Optimization Problems

We frequently encounter situations in which we are asked to do the best we can. Such a request is vague unless we are given some conditions. Asking us to minimize the cost of making tables and chairs is not clear. Asking us to make the maximum number of tables and chairs possible, with a given amount of material, so that the costs of production are minimized allows us to construct a function that describes the situation. We can then determine the minimum (or maximum) of the function.

Such a procedure is called **optimization**. To optimize a situation is to realize the best possible outcome, subject to a set of restrictions. Because of these restrictions, the domain of the function is usually restricted. As you have seen earlier, in such situations, the maximum or minimum can be identified through the use of calculus, but might also occur at the ends of the restricted domain.

EXAMPLE 1

Solving a problem involving optimal area

A farmer has 800 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road that is already fenced, so the farmer needs to fence only the remaining three sides of the field. The farmer wants to enclose the maximum possible area and to use all the fencing. How does the farmer determine the dimensions to achieve this goal?

Solution

The farmer can achieve this goal by determining a function that describes the area, subject to the condition that the amount of fencing used is to be exactly 800 m, and by finding the maximum of the function. To do so, the farmer proceeds as follows:

Let the width of the enclosed area be x metres.

Then the length of the rectangular field is $(800 - 2x)$ m. The area of the field can be represented by the function $A(x)$, where

$$A(x) = x(800 - 2x)$$
$$= 800x - 2x^2$$

The domain of the function is $0 < x < 400$, since the amount of fencing is 800 m. To find the minimum and maximum values, determine $A'(x)$: $A'(x) = 800 - 4x$. Setting $A'(x) = 0$, we obtain $800 - 4x = 0$, so $x = 200$.

The minimum and maximum values can occur at $x = 200$ or at the ends of the domain, $x = 0$ and $x = 400$. Evaluating the area function at each of these gives

$$\begin{aligned} A(0) &= 0 \\ A(200) &= 200(800 - 400) \\ &= 80\,000 \\ A(400) &= 400(800 - 800) \\ &= 0 \end{aligned}$$

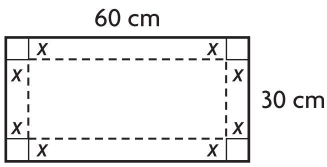
Sometimes, the ends of the domain produce results that are either not possible or unrealistic. In this case, $x = 200$ produces the maximum. The ends of the domain do not result in possible dimensions of a rectangle.

The maximum area that the farmer can enclose is 80 000 m^2, within a field 200 m by 400 m.

EXAMPLE 2

Solving a problem involving optimal volume

A piece of sheet metal, 60 cm by 30 cm, is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box with the largest volume.

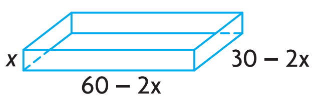


Solution

From the diagram, making the box requires the four corner squares to be cut out and discarded. Folding up the sides creates the box. Let each side of the squares be x centimetres.

$$\begin{aligned} \text{Therefore, height} &= x \\ \text{length} &= 60 - 2x \\ \text{width} &= 30 - 2x \end{aligned}$$

Since all dimensions are positive, $0 < x < 15$.



The volume of the box is the product of its dimensions and is given by the function $V(x)$, where

$$\begin{aligned} V(x) &= x(60 - 2x)(30 - 2x) \\ &= 4x^3 - 180x^2 + 1800x \end{aligned}$$

For extreme values, set $V'(x) = 0$.

$$V'(x) = 12x^2 - 360x + 1800$$
$$= 12(x^2 - 30x + 150)$$

Setting $V'(x) = 0$, we obtain $x^2 - 30x + 150 = 0$. Solving for x using the quadratic formula results in

$$x = \frac{30 \pm \sqrt{300}}{2}$$
$$= 15 \pm 5\sqrt{3}$$
$$x \doteq 23.7 \text{ or } x \doteq 6.3$$

Since $0 < x < 15$, $x = 15 - 5\sqrt{3} \doteq 6.3$. This is the only place within the interval where the derivative is 0.

To find the largest volume, substitute $x = 6.3$ in $V(x) = 4x^3 - 180x^2 + 1800x$.

$$V(6.3) = 4(6.3)^3 - 180(6.3)^2 + 1800(6.3)$$
$$= 5196$$

Notice that the endpoints of the domain did not have to be tested since it is impossible to make a box using the values $x = 0$ or $x = 15$.

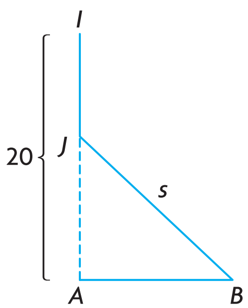
The maximum volume is obtained by cutting out corner squares of side length 6.3 cm. The length of the box is $60 - 2 \times 6.3 = 47.4$ cm, the width is about $30 - 2 \times 6.3 = 17.4$ cm, and the height is about 6.3 cm.

EXAMPLE 3

Solving a problem that minimizes distance

Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 a.m. one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?

Solution



If Ian starts at point I, he reaches point J after time t hours. Then $IJ = 8t$ km, and $JA = (20 - 8t)$ km.

If Ada starts at point A, she reaches point B after t hours, and $AB = 6t$ km. Now the distance they are apart is $s = JB$, and s can be expressed as a function of t by

$$s(t) = \sqrt{JA^2 + AB^2}$$
$$= \sqrt{(20 - 8t)^2 + (6t)^2}$$
$$= \sqrt{100t^2 - 320t + 400}$$
$$= (100t^2 - 320t + 400)^{\frac{1}{2}}$$

The domain for t is $0 \leq t \leq 2.5$.

$$s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)$$

$$= \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$

To obtain a minimum or maximum value, let $s'(t) = 0$.

$$\frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} = 0$$

$$100t - 160 = 0$$

$$t = 1.6$$

Using the algorithm for finding extreme values,

$$s(0) = \sqrt{400} = 20$$

$$s(1.6) = \sqrt{100(1.6)^2 - 320(1.6) + 400} = 12$$

$$s(2.5) = \sqrt{225} = 15$$

Therefore, the minimum value of $s(t)$ is 12 km, which occurs at time 10:36 a.m.

IN SUMMARY

Key Ideas

- In an optimization problem, you must determine the maximum or minimum value of a quantity.
- An optimization problem can be solved using a mathematical model that is developed using information given in the problem. The numerical solution represents the extreme value of the model.

Need to Know

- **Algorithm for Solving Optimization Problems:**
  1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
  2. Whenever possible, draw a diagram, labelling the given and required quantities.
  3. Determine the domain of the function to be optimized, using the information given in the problem.
  4. Use the algorithm for extreme values to find the absolute maximum or minimum value in the domain.
  5. Use your result for step 4 to answer the original problem.

Exercise 3.3

PART A

1. A piece of wire, 100 cm long, needs to be bent to form a rectangle. Determine the dimensions of a rectangle with the maximum area.

C 2. Discuss the result of maximizing the area of a rectangle, given a fixed perimeter.

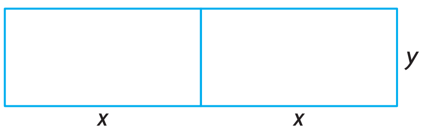
3. A farmer has 600 m of fence and wants to enclose a rectangular field beside a river. Determine the dimensions of the fenced field in which the maximum area is enclosed. (Fencing is required on only three sides: those that aren't next to the river.)

4. A rectangular piece of cardboard, 100 cm by 40 cm, is going to be used to make a rectangular box with an open top by cutting congruent squares from the corners. Calculate the dimensions (to one decimal place) for a box with the largest volume.

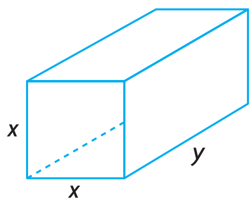
5. A rectangle has a perimeter of 440 cm. What dimensions will maximize the area of the rectangle?

6. What are the dimensions of a rectangle with an area of 64 m^2 and the smallest possible perimeter?

7. A rancher has 1000 m of fencing to enclose two rectangular corrals. The corrals have the same dimensions and one side in common. What dimensions will maximize the enclosed area?



8. A net enclosure for practising golf shots is open at one end, as shown. Find the dimensions that will minimize the amount of netting needed and give a volume of 144 m^2. (Netting is required only on the sides, the top, and the far end.)



PART B

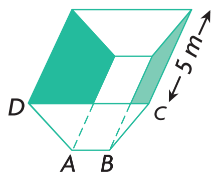
9. The volume of a square-based rectangular cardboard box needs to be 1000 cm^3. Determine the dimensions that require the minimum amount of material to manufacture all six faces. Assume that there will be no waste material. The machinery available cannot fabricate material smaller than 2 cm in length.

10. Determine the area of the largest rectangle that can be inscribed inside a semicircle with a radius of 10 units. Place the length of the rectangle along the diameter.

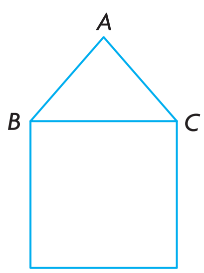
A 11. A cylindrical-shaped tin can must have a capacity of 1000 cm^3.
   a. Determine the dimensions that require the minimum amount of tin for the can. (Assume no waste material.) According to the marketing department, the smallest can that the market will accept has a diameter of 6 cm and a height of 4 cm.
   b. Express your answer for part a. as a ratio of height to diameter. Does this ratio meet the requirements outlined by the marketing department?

12. a. Determine the area of the largest rectangle that can be inscribed in a right triangle if the legs adjacent to the right angle are 5 cm and 12 cm long. The two sides of the rectangle lie along the legs.
   b. Repeat part a. for a right triangle that has sides 8 cm and 15 cm.
   c. Hypothesize a conclusion for any right triangle.

T 13. a. An isosceles trapezoidal drainage gutter is to be made so that the angles at A and B in the cross-section $ABCD$ are each 120°. If the 5 m long sheet of metal that has to be bent to form the open-topped gutter and the width of the sheet of metal is 60 cm, then determine the dimensions so that the cross-sectional area will be a maximum.



   b. Calculate the maximum volume of water that can be held by this gutter.



14. The 6 segments of the window frame shown in the diagram are to be constructed from a piece of window framing material 6m in length. A carpenter wants to build a frame for a rural gothic style window, where $\triangle ABC$ is equilateral. The window must fit inside a space that is 1 m wide and 3 m high.

a. Determine the dimensions that should be used for the six pieces so that the maximum amount of light will be admitted. Assume no waste material for corner cuts and so on.

b. Would the carpenter get more light if the window was built in the shape of an equilateral triangle only? Explain.

15. A train leaves the station at 10:00 a.m. and travels due south at a speed of 60 km/h. Another train has been heading due west at 45 km/h and reaches the same station at 11:00 a.m. At what time were the two trains closest together?

16. A north–south highway intersects an east–west highway at point P. A vehicle crosses P at 1:00 p.m., travelling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at this time.

PART C

17. In question 12, part c., you looked at two specific right triangles and observed that a rectangle with the maximum area that can be inscribed inside the triangle had dimensions equal to half the lengths of the sides adjacent to the rectangle. Prove that this is true for any right triangle.

18. Prove that any cylindrical can of volume k cubic units that is to be made using a minimum amount of material must have the height equal to the diameter.

19. A piece of wire, 100 cm long, is cut into two pieces. One piece is bent to form a square, and the other piece is bent to form a circle. Determine how the wire should be cut so that the total area enclosed is

    a. a maximum

    b. a minimum

20. Determine the minimal distance from point $(-3, 3)$ to the curve given by $y = (x - 3)^2$.

21. A chord joins any two points A and B on the parabola whose equation is $y^2 = 4x$. If C is the midpoint of AB, and CD is drawn parallel to the x-axis to meet the parabola at D, prove that the tangent at D is parallel to chord AB.

22. A rectangle lies in the first quadrant, with one vertex at the origin and two of the sides along the coordinate axes. If the fourth vertex lies on the line defined by $x + 2y - 10 = 0$, find the rectangle with the maximum area.

23. The base of a rectangle lies along the x-axis, and the upper two vertices are on the curve defined by $y = k^2 - x^2$. Determine the dimensions of the rectangle with the maximum area.

Section 3.4—Optimization Problems in Economics and Science

In the world of business, it is extremely important to manage costs effectively. Good control will allow for minimization of costs and maximization of profit. At the same time, there are human considerations. If your company is able to maximize profit but antagonizes customers or employees in the process, there may be problems in the future. For this reason, it may be important that, in addition to any mathematical constraints, you consider other more practical constraints on the domain when you construct a workable function.

The following examples will illustrate economic situations and domain constraints you may encounter.

EXAMPLE 1 Solving a problem to maximize revenue

A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is $7.00 per person. Market research shows that 40 fewer people would ride the train for each $0.10 increase in the fare, and 40 more people would ride the train for each $0.10 decrease. If the capacity of the train is 2600 passengers, and carrying fewer than 1600 passengers means costs exceed revenue, what fare should the railway charge to get the largest possible revenue?

Solution

To maximize revenue, we require a revenue function. We know that revenue $=$ (number of passengers) $\times$ (fare per passenger).

To form a revenue function, the most straightforward choice for the independent variable comes from noticing that both the number of passengers and the fare per passenger change with each $0.10 increase or decrease in the fare. If we let x represent the number of $0.10 increases in the fare (for example, $x = 3$ represents a $0.30 increase in the fare, whereas $x = -1$ represents a $0.10 decrease in the fare), then we can write expressions for both the number of passengers and the fare per passenger in terms of x, as follows:

- the fare per passenger is $7 + 0.10x$
- the number of passengers is $2000 - 40x$

Since the number of passengers must be at least 1600, $2000 - 40x \geq 1600$, and $x \leq 10$. Since the number of passengers cannot exceed 2600, $2000 - 40x \leq 2600$, and $x \geq -15$.

The domain is $-15 \leq x \leq 10$.

The revenue function is

$$R(x) = (7 + 0.10x)(2000 - 40x)$$
$$= -4x^2 - 80x + 14\,000$$

From a practical point of view, we also require x to be an integer, so that the fare only varies by increments of \$0.10. We do not wish to consider fares that are not multiples of 10 cents.

Therefore, we need to find the absolute maximum value of the revenue function $R(x) = -4x^2 - 80x + 14\,000$ on the interval $-15 \le x \le 10$, where x must be an integer.

$$R'(x) = -8x - 80$$

$R'(x) = 0$ when $-8x - 80 = 0$ $x = -10$

$R'(x)$ is never undefined. Notice that $x = -10$, is in the domain. To determine the maximum revenue, we evaluate

$$R(-15) = -4(-15)^2 - 80(-15) + 14\,000$$
$$= 14\,300$$
$$R(-10) = -4(-10)^2 - 80(-10) + 14\,000$$
$$= 14\,400$$
$$R(10) = -4(10)^2 - 80(10) + 14\,000$$
$$= 12\,800$$

Therefore, the maximum revenue occurs when there are -10 fare increases of \$0.10 each, or a fare decrease of $10(0.10) = \$1.00$. At a fare of \$6.00, the daily revenue is \$14 400, and the number of passengers is $2000 - 40(-10) = 2400$.

EXAMPLE 2 **Solving a problem to minimize cost**

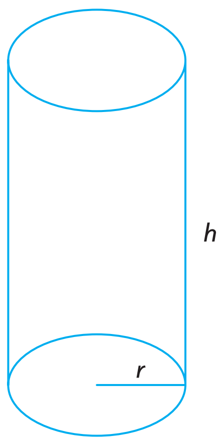
A cylindrical chemical storage tank with a capacity of 1000 m^3 is going to be constructed in a warehouse that is 12 m by 15 m, with a height of 11 m. The specifications call for the base to be made of sheet steel that costs \$100/m^2, the top to be made of sheet steel that costs \$50/m^2, and the wall to be made of sheet steel that costs \$80/m^2.

a. Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it *is* possible, state the restrictions on the radius.
b. If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions and that minimize the cost of the steel for construction.

All calculations should be accurate to two decimal places.

Solution

a. The radius of the tank cannot exceed 6 m, and the maximum height is 11 m.
The volume, using $r = 6$ and $h = 11$, is $V = \pi r^2 h \doteq 1244 \text{ m}^3$.
It is possible to build a tank with a volume of 1000 m^3.
There are limits on the radius and the height. Clearly, $0 < r \leq 6$. Also, if $h = 11$, then $\pi r^2(11) \geq 1000$, so $r \geq 5.38$.
The tank can be constructed to fit in the warehouse. Its radius must be $5.38 \leq r \leq 6$.



b. If the height is h metres and the radius is r metres, then
- the cost of the base is $\$100(\pi r^2)$
- the cost of the top is $\$50(\pi r^2)$
- the cost of the wall is $\$80(2\pi rh)$

The cost of the tank is $C = 150\pi r^2 + 160\pi rh$.
Here we have two variable quantities, r and h.

However, since $V = \pi r^2 h = 1000$, $h = \frac{1000}{\pi r^2}$.

Substituting for h, we have a cost function in terms of r.

$$C(r) = 150\pi r^2 + 160\pi r\left(\frac{1000}{\pi r^2}\right)$$

or $$C(r) = 150\pi r^2 + \frac{160\,000}{r}$$

From part a., we know that the domain is $5.38 \leq r \leq 6$.
To find points where extreme values could occur, set $C'(r) = 0$.

$$300\pi r - \frac{160\,000}{r^2} = 0$$

$$300\pi r = \frac{160\,000}{r^2}$$

$$r^3 = \frac{1600}{3\pi}$$

$$r \doteq 5.54$$

This value is within the given domain, so we use the algorithm for finding maximum and minimum values.

$$C(5.38) = 150\pi(5.38)^2 + \frac{160\,000}{5.38} \doteq 43\,380$$

$$C(5.54) = 150\pi(5.54)^2 + \frac{160\,000}{5.54} \doteq 43\,344$$

$$C(6) = 150\pi(6)^2 + \frac{160\,000}{6} \doteq 43\,631$$

The minimal cost is approximately \$43 344, with a tank of radius 5.54 m and a height of $\frac{1000}{\pi(5.54)^2} = 10.37$ m.

When solving real-life optimization problems, there are often many factors that can affect the required functions and their domains. Such factors may not be obvious from the statement of the problem. We must do research and ask many questions to address all the factors. Solving an entire problem is a series of many steps, and optimization using calculus techniques is only one step in determining a solution.

IN SUMMARY

Key Ideas

- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced or sold.
- Economic situations usually involve minimizing costs or maximizing profits.

Need to Know

- To maximize revenue, we can use the revenue function.
  revenue = total revenue from the sale of x units = (price per unit) $\times$ x.
- Practical constraints, as well as mathematical constraints, must always be considered when constructing a model.
- Once the constraints on the model have been determined—that is the domain of the function—apply the extreme value algorithm to the function over the appropriately defined domain to determine the absolute extrema.

Exercise 3.4

PART A

K 1. The cost, in dollars, to produce x litres of maple syrup for the Elmira Maple Syrup Festival is $C(x) = 75(\sqrt{x} - 10)$, where $x \geq 400$.

a. What is the average cost of producing 625 L?

b. The marginal cost is $C'(x)$, and the marginal revenue is $R'(x)$. Marginal cost at x litres is the expected change in cost if we were to produce one additional litre of syrup. Similarly for marginal revenue. What is the marginal cost at 1225 L?

c. How much production is needed to achieve a marginal cost of \$0.50/L?

2. A sociologist determines that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study.
   a. How many terms are learned between times $t = 2$ and $t = 3$?
   b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$?
   c. What is the maximum rate, in terms per hour, at which the student is learning?
3. A researcher found that the level of antacid in a person's stomach, t minutes after a certain brand of antacid tablet is taken, is $L(t) = \frac{6t}{t^2 + 2t + 1}$.
   a. Determine the value of t for which $L'(t) = 0$.
   b. Determine $L(t)$ for the value you found in part a.
   c. Using your graphing calculator, graph $L(t)$.
   d. From the graph, what can you predict about the level of antacid in a person's stomach after 1 min?
   e. What is happening to the level of antacid in a person's stomach from $2 \le t \le 8$?

PART B

A
4. The operating cost, C, in dollars per hour, for an airplane cruising at a height of h metres and an air speed of 200 km/h is given by $C = 4000 + \frac{h}{15} + \frac{15\,000\,000}{h}$ for the domain $1000 \le h \le 20\,000$. Determine the height at which the operating cost is at a minimum, and find the operating cost per hour at this height.
5. A rectangular piece of land is to be fenced using two kinds of fencing. Two opposite sides will be fenced using standard fencing that costs \$6/m, while the other two sides will require heavy-duty fencing that costs \$9/m. What are the dimensions of the rectangular lot of greatest area that can be fenced for a cost of \$9000?
6. A real estate office manages 50 apartments in a downtown building. When the rent is \$900 per month, all the units are occupied. For every \$25 increase in rent, one unit becomes vacant. On average, all units require \$75 in maintenance and repairs each month. How much rent should the real estate office charge to maximize profits?
7. A bus service carries 10 000 people daily between Ajax and Union Station, and the company has space to serve up to 15 000 people per day. The cost to ride the bus is \$20. Market research shows that if the fare increases by \$0.50, 200 fewer people will ride the bus. What fare should be charged to get the maximum revenue, given that the bus company must have at least \$130 000 in fares a day to cover operating costs?

T 8. The fuel cost per hour for running a ship is approximately one half the cube of the speed (measured in knots) plus additional fixed costs of $216 per hour. Find the most economical speed to run the ship for a 500 M (nautical mile) trip. *Note:* Assume that there are no major disturbances, such as heavy tides or stormy seas.

9. A 20 000 m^3 rectangular cistern is to be made from reinforced concrete such that the interior length will be twice the height. If the cost is $40/$m^2$ for the base, $100/$m^2$ for the side walls, and $200/$m^2$ for the roof, find the interior dimensions (to one decimal place) that will keep the cost to a minimum. To protect the water table, the building code specifies that no excavation can be more than 22 m deep. It also specifies that all cisterns must be at least 1 m deep.

C 10. The cost of producing an ordinary cylindrical tin can is determined by the materials used for the wall and the end pieces. If the end pieces are twice as expensive per square centimetre as the wall, find the dimensions (to the nearest millimetre) to make a 1000 cm^3 can at minimal cost.

11. Your neighbours operate a successful bake shop. One of their specialties is a very rich whipped-cream-covered cake. They buy the cakes from a supplier who charges $6.00 per cake, and they sell 200 cakes weekly at $10.00 each. Research shows that profit from the cake sales can be increased by increasing the price. Unfortunately, for every increase of $0.50 cents, sales will drop by seven cakes.
    a. What is the optimal retail price for a cake to obtain a maximum weekly profit?
    b. The supplier, unhappy with reduced sales, informs the owners that if they purchase fewer than 165 cakes weekly, the cost per cake will increase to $7.50. Now what is the optimal retail price per cake, and what is the bake shop's total weekly profit?
    c. Situations like this occur regularly in retail trade. Discuss the implications of reduced sales with increased total profit versus greater sales with smaller profits. For example, a drop in the number of customers could mean fewer sales of associated products.

12. Sandy is making a closed rectangular jewellery box with a square base from two different woods. The wood for the top and bottom costs $20/$m^2$. The wood for the sides costs $30/$m^2$. Find the dimensions that will minimize the cost of the wood for a volume of 4000 cm^3.

13. An electronics store is selling personal CD players. The regular price for each CD player is $90. During a typical two weeks, the store sells 50 units. Past sales indicate that for every $1 decrease in price, the store sells five more units during two weeks. Calculate the price that will maximize revenue.

14. A professional basketball team plays in an arena that holds 20 000 spectators. Average attendance at each game has been 14 000. The average ticket price is \$75. Market research shows that for each \$5 reduction in the ticket price, attendance increases by 800. Find the price that will maximize revenue.

15. Through market research, a computer manufacturer found that x thousand units of its new laptop will sell at a price of $2000 - 5x$ dollars per unit. The cost, C, in dollars to produce this many units is $C(x) = 15\,000\,000 + 1\,800\,000x + 75x^2$. Determine the level of sales that will maximize profit.

PART C

16. If the cost of producing x items is given by the function $C(x)$, and the total revenue when x items are sold is $R(x)$, then the profit function is $P(x) = R(x) - C(x)$. Show that the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

17. A fuel tank is being designed to contain 200 m^3 of gasoline, but the maximum length of a tank (measured from the tips of each hemisphere) that can be safely transported to clients is 16 m long. The design of the tank calls for a cylindrical part in the middle, with hemispheres at each end. If the hemispheres are twice as expensive per unit area as the cylindrical part, find the radius and height of the cylindrical part so the cost of manufacturing the tank will be minimal. Give your answers correct to the nearest centimetre.

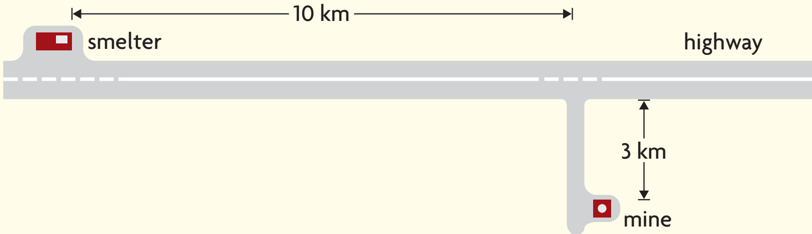
18. A truck crossing the prairies at a constant speed of 110 kilometres per hour gets gas mileage of 8 kilometre per litre. Gas costs \$1.15 per litre. The truck loses 0.10 kilometres per litre in fuel efficiency for each kilometre per hour increase in speed. The driver is paid \$35 per hour in wages and benefits. Fixed costs for running the truck are \$15.50 per hour. If a trip of 450 kilometres is planned, what speed will minimize operating expenses?

19. During a cough, the diameter of the trachea decreases. The velocity, v, of air in the trachea during a cough may be modelled by the formula $v(r) = Ar^2(r_0 - r)$, where A is a constant, r is the radius of the trachea during the cough, and r_0 is the radius of the trachea in a relaxed state. Find the radius of the trachea when the velocity is the greatest, and find the associated maximum velocity of air. Note that the domain for the problem is $0 \leq r \leq r_0$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 3: MAXIMIZING PROFITS

A construction company has been offered a contract for $7.8 million to construct and operate a trucking route for five years to transport ore from a mine site to a smelter. The smelter is located on a major highway, and the mine is 3 km into a heavily forested area off the road.



Construction (capital) costs are estimated as follows:

- Repaving the highway will cost $200 000/km.
- A new gravel road from the mine to the highway will cost $500 000/km.

Operating conditions are as follows:

- There will be 100 round trips each day, for 300 days a year, for each of the five years the mine will be open.
- Operating costs on the gravel road will be $65/h, and the speed limit will be 40 km/h.
- Operating costs on the highway will be $50/h, and the speed limit will be 70 km/h.

Use calculus to determine if the company should accept the contract. Determine the average speeds of the trucks along the paved and gravel roads that produce optimum conditions (maximum profit). What is the maximum profit?

Key Concepts Review

In Chapter 3, you have considered a variety of applications of derivatives on an interval.

You should now be familiar with the following concepts:

- the position, velocity, and acceleration functions $s(t)$, $v(t)$, and $a(t)$, respectively, where $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$
- the algorithm for finding absolute maximum and absolute minimum values
- derivatives that involve cost, revenue, and profit in the social sciences
- optimization problems (remember that you must first create a function to analyze, and that restrictions in the domain may be crucial)

Review Exercise

1. Determine f' and f'', if $f(x) = x^4 - \frac{1}{x^4}$.

2. For $y = x^9 - 7x^3 + 2$, find $\frac{d^2y}{dx^2}$.

3. Determine the velocity and acceleration of an object that moves along a straight line in such a way that its position is $s(t) = t^2 + (2t - 3)^{\frac{1}{2}}$.

4. Determine the velocity and acceleration as functions of time, t, for $s(t) = t - 7 + \frac{5}{t}, t \neq 0$.

5. A pellet is shot into the air. Its position above the ground at any time, t, is defined by $s(t) = 45t - 5t^2$ m. For what values of t, $t \geq 0$, is the upward velocity of the pellet positive? For what values of t is the upward velocity zero and negative? Draw a graph to represent the velocity of the pellet.

6. Determine the maximum and minimum of each function on the given interval.
   a. $f(x) = 2x^3 - 9x^2, -2 \leq x \leq 4$
   b. $f(x) = 12x - x^3, x \in [-3, 5]$
   c. $f(x) = 2x + \frac{18}{x}, 1 \leq x \leq 5$

7. A motorist starts braking for a stop sign. After t seconds, the distance, in metres, from the front of the car to the sign is $s(t) = 62 - 16t + t^2$.
   a. How far was the front of the car from the sign when the driver started braking?
   b. Does the car go beyond the stop sign before stopping?
   c. Explain why it is unlikely that the car would hit another vehicle that is travelling perpendicular to the motorist's road when the car first comes to a stop at the intersection.

8. The position function of an object that moves in a straight line is $s(t) = 1 + 2t - \frac{8}{t^2 + 1}, 0 \leq t \leq 2$. Calculate the maximum and minimum velocities of the object over the given time interval.

9. Suppose that the cost, in dollars, of manufacturing x items is approximated by $C(x) = 625 + 15x + 0.01x^2$, for $1 \leq x \leq 500$. The unit cost (the cost of manufacturing one item) would then be $U(x) = \frac{C(x)}{x}$. How many items should be manufactured to ensure that the unit cost is minimized?

10. For each of the following cost functions, determine

    i. the cost of producing 400 items

    ii. the average cost of each of the first 400 items produced

    iii. the marginal cost when $x = 400$, as well as the cost of producing the 401st item

    a. $C(x) = 3x + 1000$

    b. $C(x) = 0.004x^2 + 40x + 8000$

    c. $C(x) = \sqrt{x} + 5000$

    d. $C(x) = 100x^{-\frac{1}{2}} + 5x + 700$

11. Find the production level that minimizes the average cost per unit for the cost function $C(x) = 0.004x^2 + 40x + 16\,000$. Show that it is a minimum by using a graphing calculator to sketch the graph of the average cost function.

12. a. The position of an object moving along a straight line is described by the function $s(t) = 3t^2 - 10$ for $t \geq 0$. Is the object moving toward or away from its starting position when $t = 3$?

    b. Repeat the problem using $s(t) = -t^3 + 4t^2 - 10$ for $t \geq 0$.

13. A particle moving along a straight line will be s centimetres from a fixed point at time t seconds, where $t > 0$ and $s = 27t^3 + \frac{16}{t} + 10$.

    a. Determine when the velocity will be zero.

    b. Is the particle accelerating? Explain.

14. A box with a square base and no top must have a volume of 10 000 cm^3. If the smallest dimension is 5 cm, determine the dimensions of the box that minimize the amount of material used.

15. An animal breeder wishes to create five adjacent rectangular pens, each with an area of 2400 m^2. To ensure that the pens are large enough for grazing, the minimum for either dimension must be 10 m. Find the dimensions required for the pens to keep the amount of fencing used to a minimum.

16. You are given a piece of sheet metal that is twice as long as it is wide and has an area of 800 m^2. Find the dimensions of the rectangular box that would contain a maximum volume if it were constructed from this piece of metal by cutting out squares of equal area at all four corners and folding up the sides. The box will not have a lid. Give your answer correct to one decimal place.

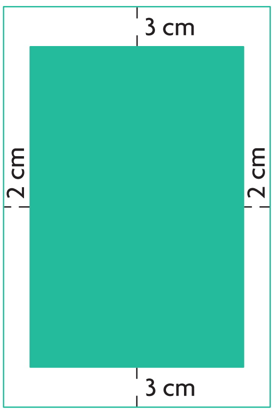
17. A cylindrical can needs to hold 500 cm^3 of apple juice. The height of the can must be between 6 cm and 15 cm, inclusive. How should the can be constructed so that a minimum amount of material will be used in the construction? (Assume that there will be no waste.)

18. In oil pipeline construction, the cost of pipe to go underwater is 60% more than the cost of pipe used in dry-land situations. A pipeline comes to a river that is 1 km wide at point A and must be extended to a refinery, R, on the other side, 8 km down the river. Find the best way to cross the river (assuming it is straight) so that the total cost of the pipe is kept to a minimum. (Give your answer correct to one decimal place.)

19. A train leaves the station at 10:00 p.m. and travels due north at a speed of 100 km/h. Another train has been heading due west at 120 km/h and reaches the same station at 11:00 p.m. At what time were the two trains closest together?

20. A store sells portable MP3 players for $100 each and, at this price, sells 120 MP3 players every month. The owner of the store wishes to increase his profit, and he estimates that, for every $2 increase in the price of MP3 players, one less MP3 player will be sold each month. If each MP3 player costs the store $70, at what price should the store sell the MP3 players to maximize profit?

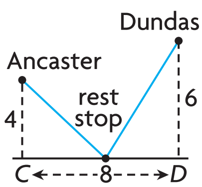
21. An offshore oil well, P, is located in the ocean 5 km from the nearest point on the shore, A. A pipeline is to be built to take oil from P to a refinery that is 20 km along the straight shoreline from A. If it costs $100 000 per kilometre to lay pipe underwater and only $75 000 per kilometre to lay pipe on land, what route from the well to the refinery will be the cheapest? (Give your answer correct to one decimal place.)



22. The printed area of a page in a book will be 81 cm^2. The margins at the top and bottom of the page will each be 3 cm deep. The margins at the sides of the page will each be 2 cm wide. What page dimensions will minimize the amount of paper?

23. A rectangular rose garden will be surrounded by a brick wall on three sides and by a fence on the fourth side. The area of the garden will be 1000 m^2. The cost of the brick wall is $192/m. The cost of the fencing is $48/m. Find the dimensions of the garden so that the cost of the materials will be as low as possible.

24. A boat leaves a dock at 2:00 p.m., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 p.m. When were the boats closest to each other?



25. Two towns, Ancaster and Dundas, are 4 km and 6 km, respectively, from an old railroad line that has been made into a bike trail. Points C and D on the trail are the closest points to the two towns, respectively. These points are 8 km apart. Where should a rest stop be built to minimize the length of new trail that must be built from both towns to the rest stop?

26. Find the absolute maximum and minimum values.

a. $f(x) = x^2 - 2x + 6, -1 \leq x \leq 7$

b. $f(x) = x^3 + x^2, -3 \leq x \leq 3$

c. $f(x) = x^3 - 12x + 2, -5 \leq x \leq 5$

d. $f(x) = 3x^5 - 5x^3, -2 \leq x \leq 4$

27. Sam applies the brakes steadily to stop his car, which is travelling at 20 m/s. The position of the car, s, in metres at t seconds, is given by $s(t) = 20t - 0.3t^3$. Determine

a. the stopping distance
b. the stopping time
c. the deceleration at 2 s

28. Calculate each of the following:

a. $f''(2)$ if $f(x) = 5x^3 - x$

b. $f''(-1)$ if $f(x) = -2x^{-3} + x^2$

c. $f''(0)$ if $f(x) = (4x - 1)^4$

d. $f''(1)$ if $f(x) = \dfrac{2x}{x - 5}$

e. $f''(4)$ if $f(x) = \sqrt{x + 5}$

f. $f''(8)$ if $f(x) = \sqrt[3]{x^2}$

29. An object moves along a straight line. The object's position at time t is given by $s(t)$. Find the position, velocity, acceleration, and speed at the specified time.

a. $s(t) = \dfrac{2t}{t + 3}, t = 3$
b. $s(t) = t + \dfrac{5}{t + 2}, t = 1$

30. The function $s(t) = (t^2 + t)^{\frac{2}{3}}, t \geq 0$, represents the displacement, s, in metres, of a particle moving along a straight line after t seconds.

a. Determine $v(t)$ and $a(t)$.

b. Find the average velocity during the first 5 s.

c. Determine the velocity at exactly 5 s.

d. Find the average acceleration during the first 5 s.

e. Determine the acceleration at exactly 5 s.

Chapter 3 Test

1. Determine the second derivative of each of the following:
   a. $y = 7x^2 - 9x + 22$
   b. $f(x) = -9x^5 - 4x^3 + 6x - 12$
   c. $y = 5x^{-3} + 10x^3$
   d. $f(x) = (4x - 8)^3$
2. For each of the following displacement functions, calculate the velocity and acceleration at the indicated time:
   a. $s(t) = -3t^3 + 5t^2 - 6t, t = 3$
   b. $s(t) = (2t - 5)^3, t = 2$
3. The position function of an object moving horizontally along a straight line as a function of time is $s(t) = t^2 - 3t + 2, t \geq 0$, in metres, at time t, in seconds.
   a. Determine the velocity and acceleration of the object.
   b. Determine the position of the object when the velocity is 0.
   c. Determine the speed of the object when the position is 0.
   d. When does the object move to the left?
   e. Determine the average velocity from $t = 2$ to $t = 5$.
4. Determine the maximum and minimum of each function on the given interval.
   a. $f(x) = x^3 - 12x + 2, -5 \leq x \leq 5$
   b. $f(x) = x + \frac{9}{x}, x \in [1, 6]$
5. After a football is punted, its height, h, in metres above the ground at t seconds, can be modelled by $h(t) = -4.9t^2 + 21t + 0.45, t \geq 0$.
   a. When does the football reach its maximum height?
   b. What is the football's maximum height?
6. A man purchased 2000 m of used wire fencing at an auction. He and his wife want to use the fencing to create three adjacent rectangular paddocks. Find the dimensions of the paddocks so that the fence encloses the largest possible area.
7. An engineer working on a new generation of computer called The Beaver is using compact VLSI circuits. The container design for the CPU is to be determined by marketing considerations and must be rectangular in shape. It must contain exactly 10 000 cm^3 of interior space, and the length must be twice the height. If the cost of the base is $\$0.02/cm^2$, the cost of the side walls is $\$0.05/cm^2$, and the cost of the upper face is $\$0.10/cm^2$, find the dimensions to the nearest millimetre that will keep the cost of the container to a minimum.
8. The landlord of a 50-unit apartment building is planning to increase the rent. Currently, residents pay \$850 per month, and all the units are occupied. A real estate agency advises that every \$100 increase in rent will result in 10 vacant units. What rent should the landlord charge to maximize revenue?

3. $1 - 2x$

4. a. $x^2 + 15x^{-6}$
b. $60(2x - 9)^4$
c. $-x^{-\frac{3}{2}} + \frac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$
d. $\frac{5(x^2 + 6)^4(3x^2 + 8x - 18)}{(3x + 4)^6}$
e. $2x(6x^2 - 7)^{-\frac{2}{3}}(8x^2 - 7)$
f. $\frac{4x^5 - 18x + 8}{x^5}$

5. 14

6. $-\frac{40}{3}$

7. $60x + y - 61 = 0$

8. $\frac{75}{32}$ ppm/year

9. $\left(-\frac{1}{4}, \frac{1}{256}\right)$

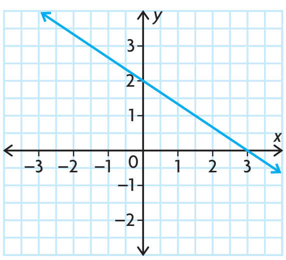
10. $\left(-\frac{1}{3}, \frac{32}{27}\right)$, $(1, 0)$

11. $a = 1, b = -1$

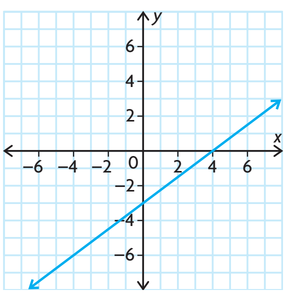
Chapter 3

Review of Prerequisite Skills, pp. 116–117

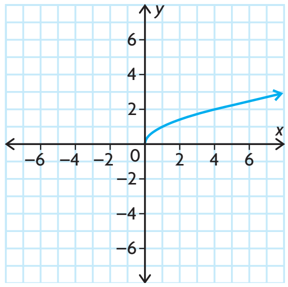
1. a.



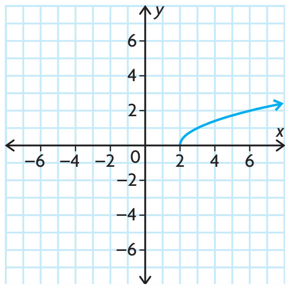
b.



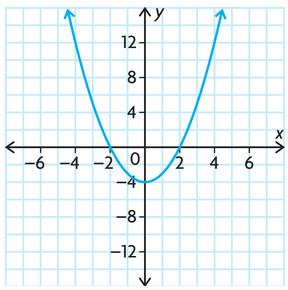
c.



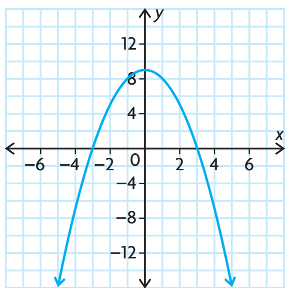
d.



e.



f.



2. a. $x = \frac{14}{5}$
b. $x = -13$
c. $t = 3$ or $t = 1$
d. $t = -\frac{1}{2}$ or $t = 3$
e. $t = 3$ or $t = 6$
f. $x = 0$ or $x = -3$ or $x = 1$
g. $x = 0$ or $x = 4$
h. $t = -3$ or $t = \frac{1}{2}$ or $t = -\frac{1}{2}$
i. $t = \pm\frac{9}{4}$ or $t = \pm 1$

3. a. $x > 3$
b. $x < 0$ or $x > 3$
c. $0 < x < 4$

4. a. 25 cm^2
b. 48 cm^2
c. $49\pi \text{ cm}^2$
d. $36\pi \text{ cm}^2$

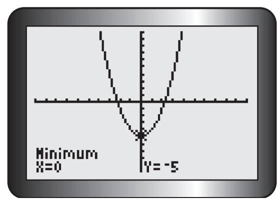
5. a. $SA = 56\pi \text{ cm}^2$, $V = 48\pi \text{ cm}^3$
b. $h = 6$ cm, $SA = 80\pi \text{ cm}^2$
c. $r = 6$ cm, $SA = 144\pi \text{ cm}^2$
d. $h = 7$ cm, $V = 175\pi \text{ cm}^3$

6. a. $SA = 54 \text{ cm}^2$, $V = 27 \text{ cm}^3$
b. $SA = 30 \text{ cm}^2$, $V = 5\sqrt{5} \text{ cm}^3$
c. $SA = 72 \text{ cm}^2$, $V = 24\sqrt{3} \text{ cm}^3$
d. $SA = 24k^2 \text{ cm}^2$, $V = 8k^3 \text{ cm}^3$

7. a. $(3, \infty)$
b. $(-\infty, -2]$
c. $(-\infty, 0)$
d. $[-5, \infty)$
e. $(-2, 8]$
f. $(-4, 4)$

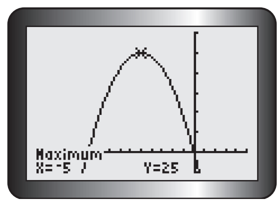
8. a. $\{x \in \mathbf{R} \mid x > 5\}$
b. $\{x \in \mathbf{R} \mid x \le -1\}$
c. $\{x \in \mathbf{R}\}$
d. $\{x \in \mathbf{R} \mid -10 \le x \le 12\}$
e. $\{x \in \mathbf{R} \mid -1 < x < 3\}$
f. $\{x \in \mathbf{R} \mid 2 \le x < 20\}$

9. a.



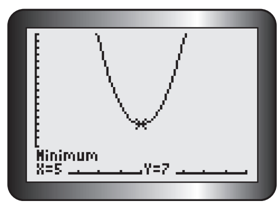
The function has a minimum value of −5 and no maximum value.

b.



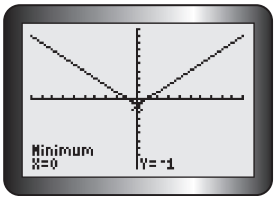
The function has a maximum value of 25 and no minimum value.

c.



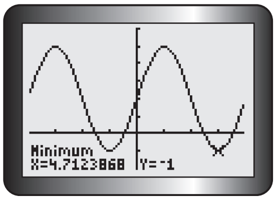
The function has a minimum value of 7 and no maximum value.

d.

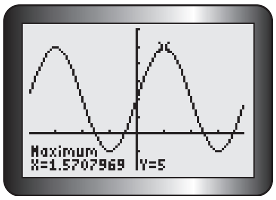


The function has a minimum value of -1 and no maximum value.

e.

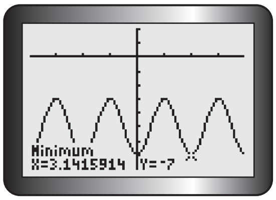


The function has a minimum value of -1.

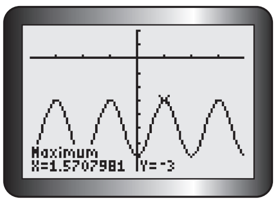


The function has a maximum value of 5.

f.



The function has a minimum value of -7.



The function has a maximum value of -3.

Section 3.1, pp. 127–129

1. At $t = 1$, the velocity is positive; this means that the object is moving in whatever is the positive direction for the scenario. At $t = 5$, the velocity is negative; this means that the object is moving in whatever is the negative direction for the scenario.

2. **a.** $y'' = 90x^8 + 90x^4$

b. $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

c. $y'' = 2$

d. $h''(x) = 36x^2 - 24x - 6$

e. $y'' = \frac{3}{\sqrt{x}} - \frac{6}{x^4}$

f. $f''(x) = \frac{-4x - 4}{(x + 1)^4}$

g. $y'' = 2 + \frac{6}{x^4}$

h. $g''(x) = -\frac{9}{4(3x - 6)^{\frac{3}{2}}}$

i. $y'' = 48x + 96$

j. $h''(x) = \frac{10}{9x^{\frac{1}{3}}}$

3. **a.** $v(t) = 10t - 3$,
$a(t) = 10$

b. $v(t) = 6t^2 + 36$,
$a(t) = 12t$

c. $v(t) = 1 - 6t^{-2}$,
$a(t) = 12t^{-3}$

d. $v(t) = 2(t - 3)$,
$a(t) = 2$

e. $v(t) = \frac{1}{2}(t + 1)^{\frac{1}{2}}$,
$a(t) = -\frac{1}{4}(t + 1)^{\frac{3}{2}}$

f. $v(t) = \frac{27}{(t + 3)^2}$,
$a(t) = -54(t + 3)^{-3}$

4. **a.** **i.** $t = 3$
ii. $1 < t < 3$
iii. $3 < t < 5$

b. **i.** $t = 3, t = 7$
ii. $1 < t < 3, 7 < t < 9$
iii. $3 < t < 7$

5. **a.** $v(t) = t^2 - 4t + 3$,
$a(t) = 2t - 4$

b. at $t = 1$ and $t = 3$

c. after 3 s

6. **a.** For $t = 1$, moving in a positive direction.
For $t = 4$, moving in a negative direction.

b. For $t = 1$, the object is stationary.
For $t = 4$, the object is moving in a positive direction.

c. For $t = 1$, the object is moving in a negative direction.
For $t = 4$, the object is moving in a positive direction.

7. **a.** $v(t) = 2t - 6$

b. $t = 3$ s

8. **a.** $t = 4$ s

b. $s(4) = 80$ m

9. **a.** $v(5) = 3$ m/s

b. $a(5) = 2$ m/s^2

10. **a.** $v(t) = \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$,
$a(t) = \frac{105}{2}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$

b. 5 s

c. 5 s

d. $0 < t < 6$ s

e. after 7 s

11. **a.** 25 m/s

b. 31.25 m

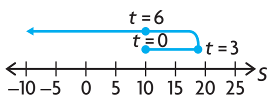
c. -25 m/s

12. **a.** $v(8) = 98$ m/s,
$a(8) = 12$ m/s^2

b. 38 m/s

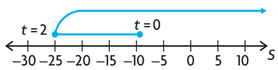
13. **a.** $s = 10 + 6t - t^2$
$v = 6 - 2t$
$= 2(3 - t)$
$a = -2$

The object moves to the right from its initial position of 10 m from the origin, 0, to the 19 m mark, slowing down at a rate of 2 m/s^2. It stops at the 19 m mark, then moves to the left, accelerating at 2 m/s^2 as it goes on its journey into the universe. It passes the origin after $(3 + \sqrt{19})$ s.



b. $s = t^3 - 12t - 9$
$v = 3t^2 - 12$
$= 3(t^2 - 4)$
$= 3(t - 2)(t + 2)$
$a = 6t$

The object begins at 9 m to the left of the origin, 0, and slows down to a stop after 2 s when it is 25 m to the left of the origin. Then, the object moves to the right, accelerating at faster rates as time increases. It passes the origin just before 4 s (approximately 3.7915) and continues to accelerate as time goes by on its journey into space.



14. $t = 1$ s; away

15. a. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$

$v(t) = 2kt + (6k^2 - 10k)$

$a(t) = 2k + 0$

$= 2k$

Since $k \neq 0$ and $k \in \mathbf{R}$, then $a(t) = 2k \neq 0$ and an element of the real numbers. Therefore, the acceleration is constant.

b. $t = 5 - 3k$, $-9k^3 + 30k^2 - 23k$.

16. a. The acceleration is continuous at $t = 0$ if $\lim_{t \to 0} a(t) = a(0)$.

For $t \geq 0$,

$$s(t) = \frac{t^3}{t^2 + 1}$$

$$\text{and } v(t) = \frac{3t^2(t^2 + 1) - 2t(t^3)}{(t^2 + 1)^2} = \frac{t^4 + 3t^2}{(t^2 + 1)^2}$$

$$\text{and } a(t) = \frac{(4t^3 + 6t)(t^2 + 1)^2}{(t^2 + 1)^2} - \frac{2(t^2 + 1)(2t)(t^4 + 3t^2)}{(t^2 + 1)^2}$$

$$= \frac{(4t^3 + 6t)(t^2 + 1)}{(t^2 + 1)^3} - \frac{4t(t^4 + 3t^2)}{(t^2 + 1)^3}$$

$$= \frac{4t^5 + 6t^3 + 4t^3}{(t^2 + 1)^3} + \frac{6t - 4t^5 - 12t^3}{(t^2 + 1)^3}$$

$$= \frac{-2t^3 + 6t}{(t^2 + 1)^3}$$

Therefore,

$$a(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{-2t^3 + 6t}{(t^2 + 1)^3}, \text{ if } t \geq 0 \end{cases}$$

and

$$v(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^4 + 3t^2}{(t^2 + 1)^2}, \text{ if } t \geq 0 \end{cases}$$

$$\lim_{t \to 0^-} a(t) = 0, \quad \lim_{t \to 0^+} a(t) = \frac{0}{1} = 0$$

Thus, $\lim_{t \to 0} a(t) = 0$.

Also, $a(0) = \dfrac{0}{1} = 0$

Therefore, $\lim_{t \to 0} a(t) = a(0)$.

Thus, the acceleration is continuous at $t = 0$.

b. velocity approaches 1, acceleration approaches 0

17.

$$v = \sqrt{b^2 + 2gs}$$

$$v = (b^2 + 2gs)^{\frac{1}{2}}$$

$$\frac{dv}{dt} = \frac{1}{2}(b^2 + 2gs)^{\frac{1}{2}} \times \left(0 + 2g\frac{ds}{dt}\right)$$

$$a = \frac{1}{2v} \times 2gv$$

$$a = g$$

Since g is a constant, a is a constant, as required.

Note: $\dfrac{ds}{dt} = v$

$\dfrac{dv}{dt} = a$

18. $F = m_0 \dfrac{d}{dt}\left(\dfrac{v}{\sqrt{1 - (\frac{v}{c})^2}}\right)$

Using the quotient rule,

$$= \frac{m_0 \frac{dv}{dt}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{1 - \frac{v^2}{c^2}} - \frac{\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{2v\frac{dv}{dt}}{c^2}\right) \times v}{1 - \frac{v^2}{c^2}}$$

Since $\dfrac{dv}{dt} = a$,

$$= \frac{m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left[a\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2 a}{c^2}\right]}{1 - \frac{v^2}{c^2}}$$

$$= \frac{m_0\left[\frac{ac^2 - av^2}{c^2} + \frac{v^2 a}{c^2}\right]}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 a c^2}{c^2(1 - \frac{v^2}{c^2})^{\frac{3}{2}}}$$

$$= \frac{m_0 a}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}}, \text{ as required.}$$

Section 3.2, pp. 135–138

1. a. The algorithm can be used; the function is continuous.

b. The algorithm cannot be used; the function is discontinuous at $x = 2$.

c. The algorithm cannot be used; the function is discontinuous at $x = 2$.

d. The algorithm can be used; the function is continuous on the given domain.

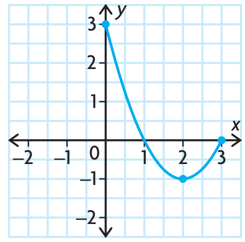
2. a. max: 8, min: -12

b. max: 30, min: -5

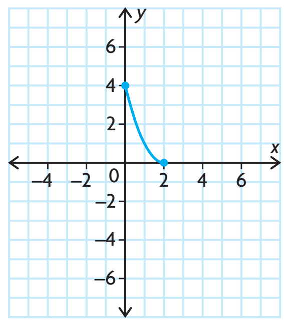
c. max: 100, min: -100

d. max: 30, min: -20

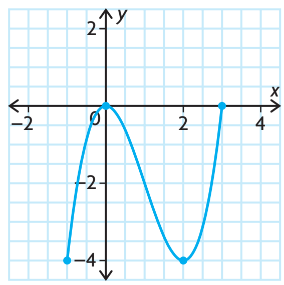
3. a. max is 3 at $x = 0$, min is -1 at $x = 2$



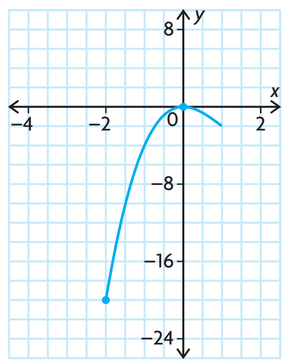
b. max is 4 at $x = 0$, min is 0 at $x = 2$



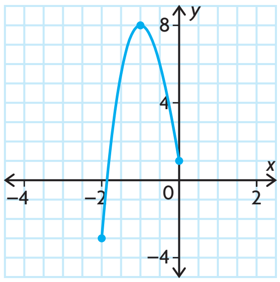
c. min is -4 at $x = -1, 2$, max is 0 at $x = 0, 3$



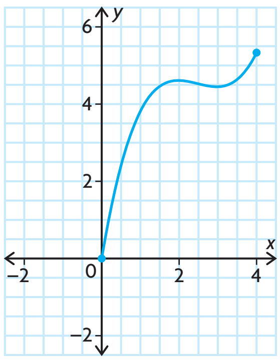
d. max is 0 at $x = 0$, min is -20 at $x = -2$



e. max is 8 at $x = -1$,
min is -3 at $x = -2$



f. max is $\frac{16}{3}$ at $x = 4$,
min is 0 at $x = 0$



4. **a.** min value of 4 when $x = 2$,
max value of 10.4 when $x = 10$
b. min value of 3 when $x = 9$,
max value of 4 when $x = 4$
c. max value of 1 when $x = 1$,
min value of $\frac{1}{2}$ when $x = 0, 2$
d. min value of -169 when $x = 3$,
max value of 47 when $x = -3$
e. max value of 2 when $x = 1$,
min value of -2 when $x = -1$
f. min value of 0.94 when $x = 4$,
max value of 1.6 when $x = 2$

5. **a.** max velocity is $\frac{4}{3}$ m/s,
min velocity is $\frac{4}{5}$ m/s
b. min velocity is 0 m/s, no maximum velocity, but $v(t) \rightarrow 4$ as $t \rightarrow \infty$

6. 20 bacteria/cm^3

7. **a.** 80 km/h
b. 50 km/h
c. $0 \le v < 80$
d. $80 < v \le 100$

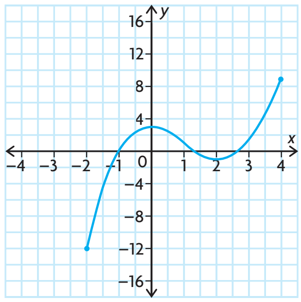
8. min concentration is at $t = 1$
max concentration is at $t = 3$

9. 0.05 years or approximately 18 days

10. 70 km/h; \$31.50

11. absolute max value = 42,
absolute min value = 10

12. **a.**



b. $-2 \le x \le 4$
c. increasing: $-2 \le x < 0$
$2 < x \le 4$
decreasing: $0 < x < 2$

13. Absolute max: Compare all local maxima and values of $f(a)$ and $f(b)$ when the domain of $f(x)$ is $a \le x \le b$. The one with the highest value is the absolute maximum.
Absolute min: We need to consider all local minima and the value of $f(a)$ and $f(b)$ when the domain of $f(x)$ is $a \le x \le b$. Compare them, and the one with the lowest value is the absolute minimum.
You need to check the endpoints because they are not necessarily critical points.

14. 245 units

15. 300 units

Mid-Chapter Review, pp. 139–140

1. **a.** $h''(x) = 36x^2 - 24x - 6$
b. $f''(x) = 48x - 120$
c. $y'' = \dfrac{30}{(x+3)^3}$
d. $g''(x) = -\dfrac{x^2}{(x^2+1)^{\frac{3}{2}}} + \dfrac{1}{(x^2+1)^{\frac{1}{2}}}$

2. **a.** 108 m
b. -45 m/s
c. -18 m/s^2

3. **a.** 6 m/s
b. $t \doteq 0.61$ s
c. $t = 1.50$ s
d. -8.67 m/s
e. -9.8 m/s^2, -9.8 m/s^2

4. **a.** Velocity is 0 m/s
Acceleration is 10 m/s
b. Object is stationary at time $t = \frac{1}{3}$ s and $t = 2$ s.
Before $t = \frac{1}{3}$, $v(t)$ is positive and therefore the object is moving to the right.
Between $t = \frac{1}{3}$ and $t = 2$, $v(t)$ is negative and therefore the object is moving to the left.
After $t = 2$, $v(t)$ is positive and therefore the object is moving to the right.
c. $t \doteq 1.2$ s; At that time, the object is neither accelerating nor decelerating.

5. **a.** min value is 1 when $x = 0$,
max value is 21 when $x = 2$
b. min value is 0 when $x = -2$,
max value is 25 when $x = 3$
c. min value is 0 when $x = 1$,
max value is 0.38 when $x = \sqrt{3}$

6. 3.96 °C

7. **a.** 105 **b.** 3 **c.** -6 **d.** -78 **e.** 3 **f.** 1448 **g.** $-\dfrac{202}{27}$ **h.** $-\dfrac{185}{6}$

8. -1.7 m/s^2

9. **a.** 189 m/s
b. 27 s
c. 2916 m
d. 6.2 m/s^2

10. 16 m; 4 s

11. **a.** $0 \le t \le 4.31$
b. 2.14 s
c. 22.95 m

Section 3.3, pp. 145–147

1. 25 cm by 25 cm

2. If the perimeter is fixed, then the figure will be a square.

3. 150 m by 300 m

4. height 8.8 cm, length 8.24 cm, and width 22.4 cm

5. 110 cm by 110 cm

6. 8 m by 8 m

7. 125 m by 166.67 m

8. 4 m by 6 m by 6 m

9. base 10 cm by 10 cm, height 10 cm

10. 100 square units when $5\sqrt{2}$

11. **a.** $r = 5.42$, $h = 10.84$
b. $\dfrac{h}{d} = \dfrac{1}{1}$; yes

12. **a.** 15 cm^2 when $W = 2.5$ cm and $L = 6$ cm
b. 30 cm^2 when $W = 4$ cm and $L = 7.5$ cm
c. The largest area occurs when the length and width are each equal to one-half of the sides adjacent to the right angle.

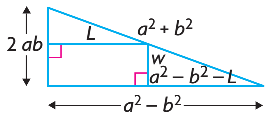
13. **a.** base is 20 cm and each side is 20 cm
b. approximately 260 000 cm^3

14. **a.** triangle side length 0.96 cm, rectangle 0.96 cm by 1.09 cm
b. Yes. All the wood would be used for the outer frame.

15. 0.36 h after the first train left the station

16. 1:02 p.m.; 3 km

17.



$$\frac{a^2 - b^2 - L}{a^2 - b^2} = \frac{W}{2ab}$$

$$W = \frac{2ab}{a^2 - b^2}(a^2 - b^2 - L)$$

$$A = LW = \frac{2ab}{a^2 - b^2}[a^2L - b^2L - L^2]$$

Let $\frac{dA}{dL} = a^2 - b^2 - 2L = 0,$

$$L = \frac{a^2 - b^2}{2} \text{ and}$$

$$W = \frac{2ab}{a^2 - b^2}\left[a^2 - b^2 - \frac{a^2 - b^2}{2}\right]$$

$$= ab$$

The hypothesis is proven.

18. Let the height be h and the radius r.

Then, $\pi r^2 h = k$, $h = \frac{k}{\pi r^2}$.

Let M represent the amount of material,

$$M = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2k}{r}, 0 \le r \le \infty$$

Using the max min Algorithm,

$$\frac{dM}{dr} = 4\pi r - \frac{2k}{r^2}$$

Let $\frac{dM}{dr} = 0$, $r^3 = \frac{k}{2\pi}$, $r \ne 0$ or

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}.$$

When $r \to 0$, $M \to \infty$

$r \to \infty$, $M \to \infty$

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$d = 2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{k}{\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}}} = \frac{k}{\pi} \cdot \frac{(2\pi)^{\frac{2}{3}}}{k^{\frac{2}{3}}} = \frac{k^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} \times 2^{\frac{2}{3}}$$

Min amount of material is

$$M = 2\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}} + 2k\left(\frac{2\pi}{k}\right)^{\frac{1}{3}}.$$

Ratio $\frac{h}{d} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \times 2^{\frac{2}{3}}}{2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \times 2^{\frac{2}{3}}}{2^{\frac{2}{3}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}} = \frac{1}{1}$

19. **a.** no cut
b. 44 cm for circle; 56 cm for square

20. $\sqrt{17}$

21. Let point A have coordinates $(a^2, 2a)$. (Note that the x-coordinate of any point on the curve is positive, but that the y-coordinate can be positive or negative. By letting the x-coordinate be a^2, we eliminate this concern.)

Similarly, let B have coordinates $(b^2, 2b)$. The slope of AB is $\frac{2a - 2b}{a^2 - b^2} = \frac{2}{a + b}$.

Using the mid-point property, C has coordinates $\left(\frac{a^2 + b^2}{2}, a + b\right)$.

Since CD is parallel to the x-axis, the y-coordinate of D is also $a + b$. The slope of the tangent at D is given by $\frac{dy}{dx}$ for the expression $y^2 = 4x$.

Differentiating,

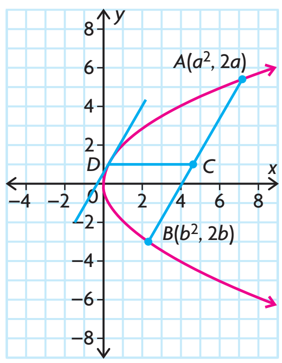
$$2y\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{2}{y}$$

And since at point D, $y = a + b$,

$$\frac{dy}{dx} = \frac{2}{a + b}$$

But this is the same as the slope of AB. Then, the tangent at D is parallel to the chord AB.



22. when P is at the point (5, 2.5)

23. $\frac{2k}{\sqrt{3}}$ by $\frac{2}{3}k^2$

Section 3.4, pp. 151–154

1. **a.** \$1.80
b. \$1.07
c. 5625 L

2. **a.** 15 terms
b. 16 terms/h
c. 20 terms/h

3. **a.** $t = 1$
b. 1.5
c.

d. The level will be a maximum.
e. The level is decreasing.

4. \$6000/h when plane is flying at 15 000 m

5. 250 m by 375 m

6. \$1100 or \$1125

7. \$22.50

8. 6 nautical miles/h

9. 20.4 m by 40.8 m by 24.0 m

10. $r = 4.3$ cm, $h = 17.2$ cm

11. **a.** \$15
b. \$12.50, \$825
c. If you increase the price, the number sold will decrease. Profit in situations like this will increase for several price increases and then it will decrease because too many customers stop buying.

12. 12.1 cm by 18.2 cm by 18.2 cm

13. \$50

14. \$81.25

15. 19 704 units

16. $P(x) = R(x) - C(x)$
Marginal Revenue $= R'(x)$.
Marginal Cost $= C'(x)$.
Now $P'(x) = R'(x) - C'(x)$.
The critical point occurs when $P'(x) = 0$.
If $R'(x) = C'(x)$, then
$P'(x) = R'(x) - C'(x)$
$= 0$
Therefore, the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

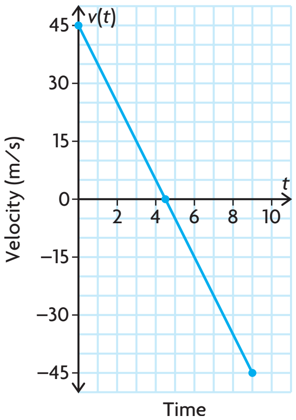
17. $r = 230$ cm and h is about 900 cm

18. 128.4 km/h

19. maximum velocity: $\frac{4}{27}r_0A$, radius: $\frac{2r_0}{3}$.

Review Exercise, pp. 156–159

1. $f'(x) = 4x^3 + 4x^{-5}$,
   $f''(x) = 12x^2 - 20x^{-6}$
2. $\frac{d^2y}{dx^2} = 72x^7 - 42x$
3. $v = 2t + (2t - 3)^{\frac{1}{2}}$,
   $a = 2 - (2t - 3)^{\frac{3}{2}}$
4. $v(t) = 1 - 5t^{-2}$,
   $a(t) = 10t^{-3}$
5. The upward velocity is positive for $0 \le t \le 4.5$ s, zero for $t = 4.5$ s, and negative for $t > 4.5$ s.



6. **a.** min: −52,
   max: 0
   b. min: −65,
   max: 16
   c. min: 12,
   max: 20
7. **a.** 62 m
   b. Yes, 2 m beyond the stop sign
   c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.
8. min is 2, max is $2 + 3\sqrt{3}$
9. 250
10. **a.** **i.** \$2200
    ii. \$5.50
    iii. \$3.00; \$3.00
    b. **i.** \$24 640
    ii. \$61.60
    iii. \$43.20; \$43.21
    c. **i.** \$5020
    ii. \$12.55
    iii. \$0.03; \$0.03
    d. **i.** \$2705
    ii. \$6.88
    iii. \$5.01; \$5.01
11. 2000
12. **a.** moving away from its starting point
    b. moving away from the origin and towards its starting position
13. **a.** $t = \frac{2}{3}$ **b.** yes
14. 27.14 cm by 27.14 cm for the base and height 13.57 cm
15. length 190 m, width approximately 63 m
16. 31.6 cm by 11.6 cm by 4.2 cm
17. radius 4.3 cm, height 8.6 cm
18. Run the pipe 7.2 km along the river shore and then cross diagonally to the refinery.
19. 10:35 p.m.
20. \$204 or \$206
21. The pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.
22. 11.35 cm by 17.02 cm
23. 34.4 m by 29.1 m
24. 2:23 p.m.
25. 3.2 km from point C
26. **a.** absolute maximum: $f(7) = 41$,
    absolute minimum: $f(1) = 5$
    b. absolute maximum: $f(3) = 36$,
    absolute minimum: $f(-3) = -18$
    c. absolute maximum: $f(5) = 67$,
    absolute minimum: $f(-5) = -63$
    d. absolute maximum: $f(4) = 2752$,
    absolute minimum: $f(-2) = -56$
27. **a.** 62.9 m **c.** 3.6 m/s^2
    b. 4.7 s
28. **a.** $f''(2) = 60$ **d.** $f''(1) = -\frac{5}{16}$
    b. $f''(-1) = 26$ **e.** $f''(4) = -\frac{1}{108}$
    c. $f''(0) = 192$ **f.** $f''(8) = -\frac{1}{72}$
29. **a.** position: 1, velocity: $\frac{1}{6}$,
    acceleration: $-\left(\frac{1}{18}\right)$, speed: $\frac{1}{6}$
    b. position: $\frac{8}{3}$, velocity: $\frac{4}{9}$,
    acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
30. **a.** $v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$,
    $a(t) = \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$
    b. 1.931 m/s
    c. 2.36 m/s
    d. undefined
    e. 0.141 m/s^2

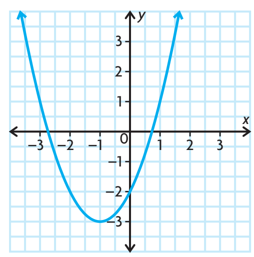
Chapter 3 Test, p. 160

1. **a.** $y'' = 14$
   b. $f''(x) = -180x^3 - 24x$
   c. $y'' = 60x^{-5} + 60x$
   d. $f''(x) = 96(4x - 8)$
2. **a.** $v(3) = -57$,
   $a(3) = -44$
   b. $v(2) = 6$,
   $a(2) = -24$
3. **a.** $v(t) = 2t - 3$,
   $a(t) = 2$
   b. −0.25 m
   c. 1 m/s, 1 m/s
   d. between $t = 0$ s and $t = 1.5$ s
   e. 2 m/s^2
4. **a.** min: −63, max: 67
   b. min: 7.5, max: 10
5. **a.** 2.1 s
   b. about 22.9
6. 250 m by 166.7 m
7. 162 mm by 324 mm by 190 mm
8. \$850/month

Chapter 4

Review of Prerequisite Skills, pp. 162–163

1. **a.** $y = -\frac{3}{2}$ or $y = 1$
   b. $x = 7$ or $x = -2$
   c. $x = -\frac{5}{2}$
   d. $y = 1$ or $y = -3$ or $y = -2$
2. **a.** $x < -\frac{7}{3}$
   b. $x \le 2$
   c. $-1 < t < 3$
   d. $x < -4$ or $x > 1$
3. **a.**



b.