Chapter 4

CURVE SKETCHING

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at this point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century. In this chapter, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine properties of the graphs of polynomial and rational functions, **Sections 4.1, 4.3, 4.5**
- describe key features of a given graph of a function, **Sections 4.1, 4.2, 4.4**
- determine intercepts and positions of the asymptotes of a graph, **Section 4.3**
- determine the values of a function near its asymptotes, **Section 4.3**
- determine key features of the graph of a function, **Section 4.5, Career Link**
- sketch, by hand, the graph of the derivative of a given graph, **Section 4.2**
- determine, from the equation of a simple combination of polynomial or rational functions $\left(\text{such as } f(x) = x^2 + \frac{1}{x}\right)$, the key features of the graph of the function, using the techniques of differential calculus, and sketch the graph by hand, **Section 4.4**

Review of Prerequisite Skills

There are many features that we can analyze to help us sketch the graph of a function. For example, we can try to determine the x- and y-intercepts of the graph, we can test for horizontal and vertical asymptotes using limits, and we can use our knowledge of certain kinds of functions to help us determine domains, ranges, and possible symmetries.

In this chapter, we will use the derivatives of functions, in conjunction with the features mentioned above, to analyze functions and their graphs. Before you begin, you should

- be able to solve simple equations and inequalities
- know how to sketch graphs of parent functions and simple transformations of these graphs (including quadratic, cubic, and root functions)
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits
- be able to determine the derivatives of functions using all known rules

Exercise

1. Solve each equation.

a. $2y^2 + y - 3 = 0$

b. $x^2 - 5x + 3 = 17$

c. $4x^2 + 20x + 25 = 0$

d. $y^3 + 4y^2 + y - 6 = 0$

2. Solve each inequality.

a. $3x + 9 < 2$

b. $5(3 - x) \geq 3x - 1$

c. $t^2 - 2t < 3$

d. $x^2 + 3x - 4 > 0$

3. Sketch the graph of each function.

a. $f(x) = (x + 1)^2 - 3$

b. $f(x) = x^2 - 5x - 6$

c. $f(x) = \dfrac{2x - 4}{x + 2}$

d. $f(x) = \sqrt{x - 2}$

4. Evaluate each limit.

a. $\lim\limits_{x\to 2^-} (x^2 - 4)$

b. $\lim\limits_{x\to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

c. $\lim\limits_{x\to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x\to 4^+} \sqrt{2x + 1}$

5. Determine the derivative of each function.

a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$

c. $f(x) = (3x^2 - 6x)^2$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$

6. Divide, and then write your answer in the form $ax + b + \frac{r}{q(x)}$. For example, $(x^2 + 4x - 5) \div (x - 2) = x + 6 + \frac{7}{x - 2}$.

a. $(x^2 - 5x + 4) \div (x + 3)$

b. $(x^2 + 6x - 9) \div (x - 1)$

7. Determine the points where the tangent is horizontal to $f(x) = x^3 + 0.5x^2 - 2x + 3$.

8. State each differentiation rule in your own words.

a. power rule

b. constant rule

c. product rule

d. quotient rule

e. chain rule

f. power of a function rule

9. Describe the end behaviour of each function as $x \to \infty$ and $x \to -\infty$.

a. $f(x) = 2x^2 - 3x + 4$

b. $f(x) = -2x^3 + 4x - 1$

c. $f(x) = -5x^4 + 2x^3 - 6x^2 + 7x - 1$

d. $f(x) = 6x^5 - 4x - 7$

10. For each function, determine the reciprocal, $y = \frac{1}{f(x)}$, and the equations of the vertical asymptotes of $y = \frac{1}{f(x)}$. Verify your results using graphing technology.

a. $f(x) = 2x$

b. $f(x) = -x + 3$

c. $f(x) = (x + 4)^2 + 1$

d. $f(x) = (x + 3)^2$

11. State the equation of the horizontal asymptote of each function.

a. $y = \frac{5}{x + 1}$

b. $y = \frac{4x}{x - 2}$

c. $y = \frac{3x - 5}{6x - 3}$

d. $y = \frac{10x - 4}{5x}$

12. For each function in question 11, determine the following:

a. the x- and y-intercepts

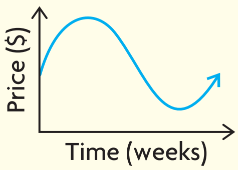
b. the domain and range

CAREER LINK Investigate

CHAPTER 4: PREDICTING STOCK VALUES

Stock-market analysts collect and interpret vast amounts of information and then predict trends in stock values. Stock analysts are classified into two main groups: the fundamentalists who predict stock values based on analysis of the companies' economic situations, and the technical analysts who predict stock values based on trends and patterns in the market. Technical analysts spend a significant amount of their time constructing and interpreting graphical models to find undervalued stocks that will give returns in excess of what the market predicts. In this chapter, your skills in producing and analyzing graphical models will be extended through the use of differential calculus.

Case Study: Technical Stock Analyst



To raise money for expansion, many privately owned companies give the public a chance to own part of their company through purchasing stock. Those who buy ownership expect to obtain a share in the future profits of the company. Some technical analysts believe that the greatest profits to be had in the stock market are through buying brand new stocks and selling them quickly. A technical analyst predicts that a stock's price over its first several weeks on the market will follow the pattern shown on the graph. The technical analyst is advising a person who purchased the stock the day it went on sale.

DISCUSSION QUESTIONS

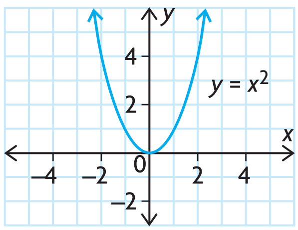
Make a rough sketch of the graph, and answer the following questions:

1. When would you recommend that the owner sell her shares? Label this point S on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
2. When would you recommend that the owner get back into the company and buy shares again? Label this point B on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
3. A concave-down section of a graph opens in a downward direction, and a concave-up section opens upward. On your graph, find the point where the concavity changes from concave down to concave up, and label this point C. Another analyst says that a change in concavity from concave down to concave up is a signal that a selling opportunity will soon occur. Do you agree with the analyst? Explain.

At the end of this chapter, you will have an opportunity to apply the tools of curve sketching to create, evaluate, and apply a model that could be used to advise clients on when to buy, sell, and hold new stocks.

Section 4.1—Increasing and Decreasing Functions

The graph of the quadratic function $f(x) = x^2$ is a parabola. If we imagine a particle moving along this parabola from left to right, we can see that, while the x-coordinates of the ordered pairs steadily increase, the y-coordinates of the ordered pairs along the particle's path first decrease and then increase. Determining the intervals in which a function increases and decreases is extremely useful for understanding the behaviour of the function. The following statements give a clear picture:



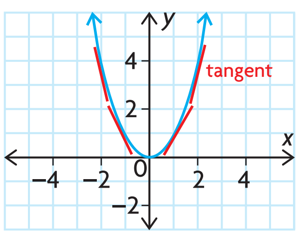
Intervals of Increase and Decrease

We say that a function f *is decreasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) > f(x_2)$.

Similarly, we say that a function f *is increasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) < f(x_2)$.

For the parabola with the equation $y = x^2$, the change from decreasing y-values to increasing y-values occurs at $(0, 0)$, the vertex of the parabola. The function $f(x) = x^2$ is decreasing on the interval $x < 0$ and is increasing on the interval $x > 0$.

If we examine tangents to the parabola anywhere on the interval where the y-values are decreasing (that is, on $x < 0$), we see that all of these tangents have negative slopes. Similarly, the slopes of tangents to the graph on the interval where the y-values are increasing are all positive.



For functions that are both continuous and differentiable, we can determine intervals of increasing and decreasing y-values using the derivative of the function. In the case of $y = x^2$, $\frac{dy}{dx} = 2x$. For $x < 0$, $\frac{dy}{dx} < 0$, and the slopes of the tangents are negative. The interval $x < 0$ corresponds to the decreasing portion of the graph of the parabola. For $x > 0$, $\frac{dy}{dx} > 0$, and the slopes of the tangents are positive on the interval where the graph is increasing.

We summarize this as follows: For a continuous and differentiable function, f, the function values (y-values) are increasing for all x-values where $f'(x) > 0$, and the function values (y-values) are decreasing for all x-values where $f'(x) < 0$.

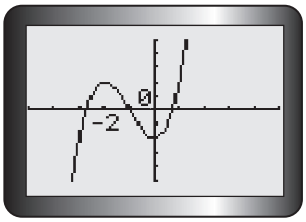
EXAMPLE 1 **Using the derivative to reason about intervals of increase and decrease**

Use your calculator to graph the following functions. Use the graph to estimate the values of x for which the function values (y-values) are increasing, and the values of x for which the y-values are decreasing. Verify your estimates with an algebraic solution.

a. $y = x^3 + 3x^2 - 2$ b. $y = \frac{x}{x^2 + 1}$

Solution

a. Using a calculator, we obtain the graph of $y = x^3 + 3x^2 - 2$. Using the TRACE key on the calculator, we estimate that the function values are increasing on $x < -2$, decreasing on $-2 < x < 0$, and increasing again on $x > 0$. To verify these estimates with an algebraic solution, we consider the slopes of the tangents.



The slope of a general tangent to the graph of $y = x^3 + 3x^2 - 2$ is given by $\frac{dy}{dx} = 3x^2 + 6x$. We first determine the values of x for which $\frac{dy}{dx} = 0$. These values tell us where the function has a **local maximum** or **local minimum** value. These are greatest and least values respectively of a function in relation to its neighbouring values.

Setting $\frac{dy}{dx} = 0$, we obtain $3x^2 + 6x = 0$

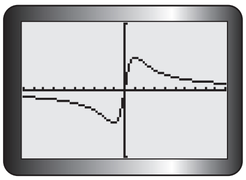
$$3x(x + 2) = 0$$
$$x = 0, x = -2$$

These values of x locate points on the graph where the slope of the tangent is zero (that is, where the tangent is horizontal).

Since this is a polynomial function it is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ only at $x = -2$ and $x = 0$, the derivative must be either positive or negative for all other values of x. We consider the intervals $x < -2$, $-2 < x < 0$, and $x > 0$.

Value of *x*	$x < -2$	$-2 < x < 0$	$x > 0$
Sign of $\frac{dy}{dx} = 3x(x + 2)$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$
Slope of Tangents	positive	negative	positive
Values of *y* Increasing or Decreasing	increasing	decreasing	increasing

So $y = x^3 + 3x^2 - 2$ is increasing on the intervals $x < -2$ and $x > 0$ and is decreasing on the interval $-2 < x < 0$.



b. Using a calculator, we obtain the graph of $y = \frac{x}{x^2 + 1}$. Using the TRACE key on the calculator, we estimate that the function values (y-values) are decreasing on $x < -1$, increasing on $-1 < x < 1$, and decreasing again on $x > 1$.

We analyze the intervals of increasing/decreasing y-values for the function by determining where $\frac{dy}{dx}$ is positive and where it is negative.

$$y = \frac{x}{x^2 + 1} \qquad \text{(Express as a product)}$$

$$= x(x^2 + 1)^{-1}$$

$$\frac{dy}{dx} = 1(x^2 + 1)^{-1} + x(-1)(x^2 + 1)^{-2}(2x) \qquad \text{(Product and chain rules)}$$

$$= \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \qquad \text{(Simplify)}$$

$$= \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Setting $\frac{dy}{dx} = 0$, we obtain $\frac{-x^2 + 1}{(x^2 + 1)^2} = 0$ (Solve)

$$-x^2 + 1 = 0$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$

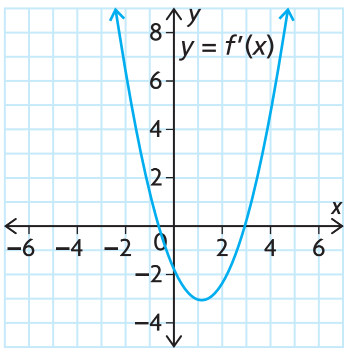
These values of x locate the points on the graph where the slope of the tangent is 0. Since the denominator of this rational function can never be 0, this function is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ at $x = -1$ and $x = 1$, we consider the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Value of *x*	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $\frac{dy}{dx} = \frac{-x^2 + 1}{(x^2 + 1)^2}$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$
Slope of Tangents	negative	positive	negative
Values of *y* Increasing or Decreasing	decreasing	increasing	decreasing

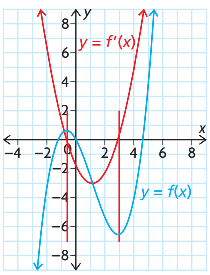
Then $y = \frac{x}{x^2 + 1}$ is increasing on the interval $(-1, 1)$ and is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$.

EXAMPLE 2 Graphing a function given the graph of the derivative

Consider the graph of $f'(x)$. Graph $f(x)$.



Solution



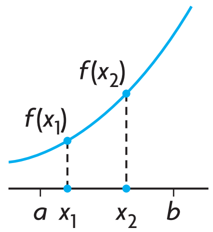
When the derivative, $f'(x)$, is positive, the graph of $f(x)$ is rising. When the derivative is negative, the graph is falling. In this example, the derivative changes sign from positive to negative at $x \doteq -0.6$. This indicates that the graph of $f(x)$ changes from increasing to decreasing, resulting in a local maximum for this value of x. The derivative changes sign from negative to positive at $x = 2.9$, indicating the graph of $f(x)$ changes from decreasing to increasing resulting in a local minimum for this value of x.

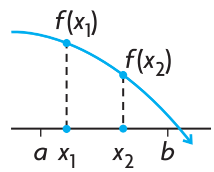
One possible graph of $f(x)$ is shown.

IN SUMMARY

Key Ideas

- A function f is **increasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.
- A function f is **decreasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.





- For a function f that is continuous and differentiable on an interval I
  - $f(x)$ is **increasing** on I if $f'(x) > 0$ for all values of x in I
  - $f(x)$ is **decreasing** on I if $f'(x) < 0$ for all values of x in I

Need to Know

- A function increases on an interval if the graph rises from left to right.
- A function decreases on an interval if the graph falls from left to right.
- The slope of the tangent at a point on a section of a curve that is increasing is always positive.
- The slope of the tangent at a point on a section of a curve that is decreasing is always negative.

Exercise 4.1

PART A

K 1. Determine the points at which $f'(x) = 0$ for each of the following functions:

a. $f(x) = x^3 + 6x^2 + 1$

b. $f(x) = \sqrt{x^2 + 4}$

c. $f(x) = (2x - 1)^2(x^2 - 9)$

d. $f(x) = \dfrac{5x}{x^2 + 1}$

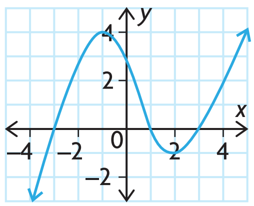
C 2. Explain how you would determine when a function is increasing or decreasing.

3. For each of the following graphs, state

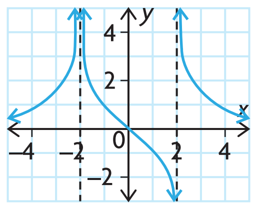
i. the intervals where the function is increasing

ii. the intervals where the function is decreasing

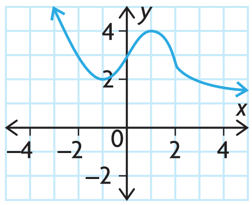
iii. the points where the tangent to the function is horizontal a.



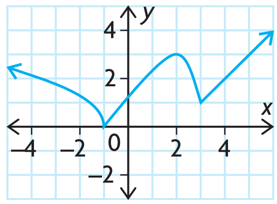
c.



b.



d.



4. Use a calculator to graph each of the following functions. Inspect the graph to estimate where the function is increasing and where it is decreasing. Verify your estimates with algebraic solutions.

a. $f(x) = x^3 + 3x^2 + 1$

b. $f(x) = x^5 - 5x^4 + 100$

c. $f(x) = x + \frac{1}{x}$

d. $f(x) = \frac{x-1}{x^2+3}$

e. $f(x) = 3x^4 + 4x^3 - 12x^2$

f. $f(x) = x^4 + x^2 - 1$

PART B

5. Suppose that f is a differentiable function with the derivative $f'(x) = (x-1)(x+2)(x+3)$. Determine the values of x for which the function f is increasing and the values of x for which the function is decreasing.

A 6. Sketch a graph of a function that is differentiable on the interval $-2 \le x \le 5$ and that satisfies the following conditions:

- The graph of f passes through the points $(-1, 0)$ and $(2, 5)$.
- The function f is decreasing on $-2 < x < -1$, increasing on $-1 < x < 2$, and decreasing again on $2 < x < 5$.

7. Find constants a, b, and c such that the graph of $f(x) = x^3 + ax^2 + bx + c$ will increase to the point $(-3, 18)$, decrease to the point $(1, -14)$, and then continue increasing.

8. Sketch a graph of a function f that is differentiable and that satisfies the following conditions:

- $f'(x) > 0$, when $x < -5$
- $f'(x) < 0$, when $-5 < x < 1$ and when $x > 1$
- $f'(-5) = 0$ and $f'(1) = 0$
- $f(-5) = 6$ and $f(1) = 2$

9. Each of the following graphs represents the derivative function $f'(x)$ of a function $f(x)$. Determine

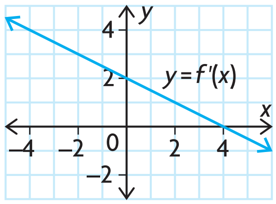
   i. the intervals where $f(x)$ is increasing

   ii. the intervals where $f(x)$ is decreasing

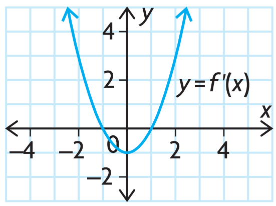
   iii. the x-coordinate for all local extrema of $f(x)$

   Assuming that $f(0) = 2$, make a rough sketch of the graph of each function.

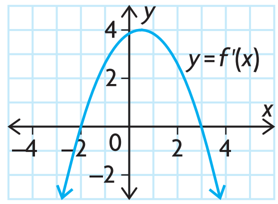
a.



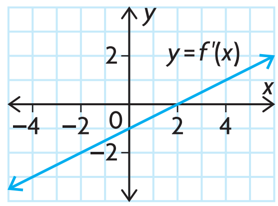
b.



c.



d.



10. Use the derivative to show that the graph of the quadratic function $f(x) = ax^2 + bx + c$, $a > 0$, is decreasing on the interval $x < -\frac{b}{2a}$ and increasing on the interval $x > -\frac{b}{2a}$.

11. For $f(x) = x^4 - 32x + 4$, find where $f'(x) = 0$, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.

12. Sketch a graph of the function g that is differentiable on the interval $-2 \le x \le 5$, decreases on $0 < x < 3$, and increases elsewhere on the domain. The absolute maximum of g is 7, and the absolute minimum is -3. The graph of g has local extrema at $(0, 4)$ and $(3, -1)$.

PART C

T 13. Let f and g be continuous and differentiable functions on the interval $a \le x \le b$. If f and g are both increasing on $a \le x \le b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \le x \le b$, show that the product fg is also increasing on $a \le x \le b$.

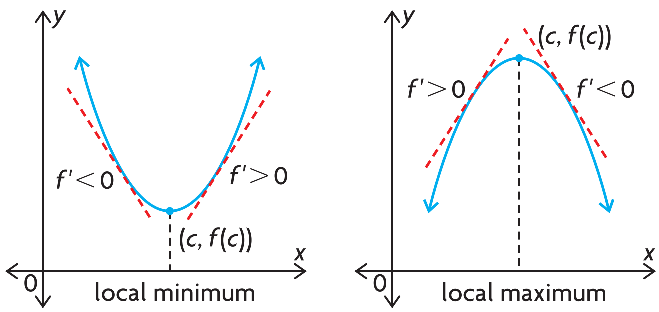
14. Let f and g be continuous and differentiable functions on the interval $a \le x \le b$. If f and g are both increasing on $a \le x \le b$, and if $f(x) < 0$ and $g(x) < 0$ on $a \le x \le b$, is the product fg increasing on $a \le x \le b$, decreasing, or neither?

Section 4.2—Critical Points, Local Maxima, and Local Minima

In Chapter 3, we learned that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c) = 0$. It is also possible that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c)$ is undefined. Since these points help to define the shape of the function's graph, they are called **critical points** and the values of c are called **critical numbers**. Combining this with the properties of increasing and decreasing functions, we have a **first derivative test** for local extrema.

The First Derivative Test

Test for local minimum and local maximum points. Let $f'(c) = 0$.



When moving left to right through x-values:

- if $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point.
- if $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

$f'(c) = 0$ may imply something other than the existence of a maximum or a minimum at $x = c$. There are also simple functions for which the derivative does not exist at certain points. In Chapter 2, we demonstrated three different ways that this could happen. For example, extrema could occur at points that correspond to cusps and corners on a function's graph and in these cases the derivative is undefined.

EXAMPLE 1 **Connecting the first derivative test to local extrema of a polynomial function**

For the function $y = x^4 - 8x^3 + 18x^2$, determine all the critical numbers. Determine whether each of these values of x gives a local maximum, a local minimum, or neither for the function.

Solution

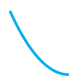
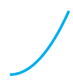
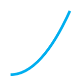
First determine $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$$
$$= 4x(x^2 - 6x + 9)$$
$$= 4x(x - 3)^2$$

For critical numbers, let $\frac{dy}{dx} = 0$.

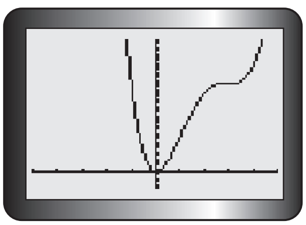
$$4x(x - 3)^2 = 0$$
$$x = 0 \text{ or } x = 3$$

Both values of x are in the domain of the function. There is a horizontal tangent at each of these values of x. To determine which of these values of x yield local maximum or minimum values of the function, we use a table to analyze the behaviour of $\frac{dy}{dx}$ and $y = x^4 - 8x^3 + 18x^2$.

Interval	$x < 0$	$0 < x < 3$	$x > 3$
4*x*	$-$	$+$	$+$
$(x - 3)^2$	$+$	$+$	$+$
$4x(x - 3)^2$	$(-)(+) = -$	$(+)(+) = +$	$(+)(+) = +$
$\frac{dy}{dx}$	< 0	> 0	> 0
$y = x^4 - 8x^3 + 18x^2$	decreasing	increasing	increasing
Shape of the Curve			

Using the information from the table, we see that there is a local minimum value of the function at $x = 0$, since the function values are decreasing before $x = 0$ and increasing after $x = 0$. We can also tell that there is neither a local maximum nor minimum value at $x = 3$, since the function values increase toward this point and increase away from it.

A calculator gives the following graph for $y = x^4 - 8x^3 + 18x^2$, which verifies our solution:



EXAMPLE 2 Reasoning about the significance of horizontal tangents

Determine whether or not the function $f(x) = x^3$ has a maximum or minimum at $(c, f(c))$, where $f'(c) = 0$.

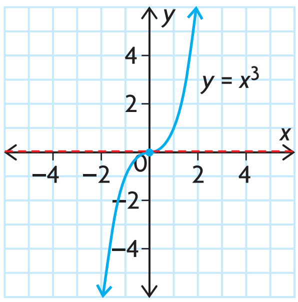
Solution

The derivative is $f'(x) = 3x^2$.

Setting $f'(x) = 0$ gives

$$3x^2 = 0$$
$$x = 0$$

$f(x)$ has a horizontal tangent at $(0, 0)$.



From the graph, it is clear that $(0, 0)$ is neither a maximum nor a minimum value since the values of this function are always increasing. Note that $f'(x) > 0$ for all values of x other than 0.

From this example, we can see that it is possible for a horizontal tangent to exist when $f'(c) = 0$, but that $(c, f(c))$ is neither a maximum nor a minimum. In the next example you will see that it is possible for a maximum or minimum to occur at a point at which the derivative does not exist.

EXAMPLE 3 Reasoning about the significance of a cusp

For the function $f(x) = (x + 2)^{\frac{2}{3}}$, determine the critical numbers. Use your calculator to sketch a graph of the function.

Solution

First determine $f'(x)$.

$$f'(x) = \frac{2}{3}(x + 2)^{-\frac{1}{3}}$$
$$= \frac{2}{3(x + 2)^{\frac{1}{3}}}$$

Note that there is no value of x for which $f'(x) = 0$ since the numerator is always positive. However, $f'(x)$ is undefined for $x = -2$.

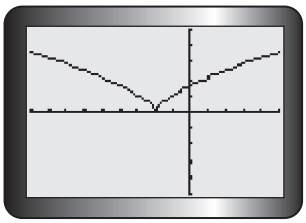
Since $f(-2) = (-2 + 2)^{\frac{2}{3}} = 0$, $x = -2$ is in the domain of $f(x) = (x + 2)^{\frac{2}{3}}$. We determine the slopes of tangents for x-values close to -2.

x	$f'(x) = \frac{2}{3(x+2)^{\frac{1}{3}}}$	x	$f'(x) = \frac{2}{3(x+2)^{\frac{1}{3}}}$
−2.1	−1.436 29	−1.9	1.436 29
−2.01	−3.094 39	−1.99	3.094 39
−2.001	−6.666 67	−1.999	6.666 67
−2.000 01	−30.943 9	−1.999 99	30.943 9

The slope of the tangent is undefined at the point $(-2, 0)$. Therefore, the function has one critical point, when $x = -2$.

In this example, the slopes of the tangents to the left of $x = -2$ are approaching $-\infty$, while the slopes to the right of $x = -2$ are approaching $+\infty$. Since the slopes on opposite sides of $x = -2$ are not approaching the same value, there is no tangent at $x = -2$ even though there is a point on the graph.

A calculator gives the following graph of $f(x) = (x + 2)^{\frac{2}{3}}$. There is a cusp at $(-2, 0)$.



If a value c is in the domain of a function $f(x)$, and if this value is such that $f'(c) = 0$ or $f'(c)$ is undefined, then $(c, f(c))$ is a critical point of the function f and c is called a critical number for f''.

In summary, critical points that occur when $\frac{dy}{dx} = 0$ give the locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists). Besides giving the location of interesting tangents (or lack thereof), critical points also determine other interesting features of the graph of a function.

Critical Numbers and Local Extrema

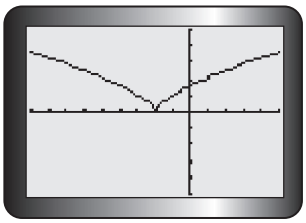
The critical number c determines the location of a local minimum value for a function f if $f(c) < f(x)$ for all values of x near c.

Similarly, the critical number c determines the location of a local maximum value for a function f if $f(c) > f(x)$ for all values of x near c.

Together, local maximum and minimum values of a function are called local extrema.

As mentioned earlier, a local minimum value of a function does not have to be the smallest value in the entire domain, just the smallest value in its neighbourhood. Similarly, a local maximum value of a function does not have to be the largest value in the entire domain, just the largest value in its neighbourhood. Local extrema occur graphically as peaks or valleys. The peaks and valleys can be either smooth or sharp.

To apply this reasoning, let's reconsider the graph of $f(x) = (x + 2)^{\frac{2}{3}}$.



The function $f(x) = (x + 2)^{\frac{2}{3}}$ has a local minimum value at $x = -2$, which also happens to be a critical value of the function.

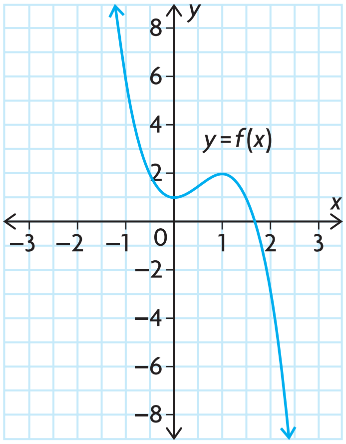
Every local maximum or minimum value of a function occurs at a critical point of the function.

In simple terms, peaks or valleys occur on the graph of a function at places where the tangent to the graph is horizontal, vertical, or does not exist.

How do we determine whether a critical point yields a local maximum or minimum value of a function without examining the graph of the function? We use the first derivative test to analyze whether the function is increasing or decreasing on either side of the critical point.

EXAMPLE 4 Graphing the derivative given the graph of a polynomial function

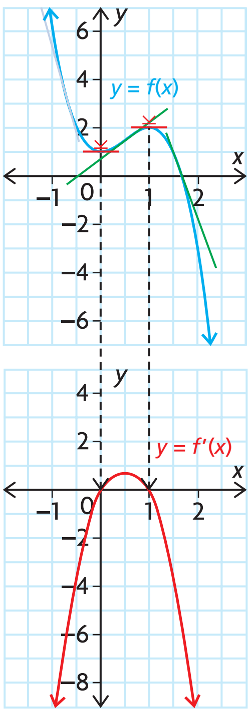
Given the graph of a polynomial function $y = f(x)$, graph $y = f'(x)$.



Solution

A polynomial function f is continuous for all values of x in the domain of f. The derivative of f, f', is also continuous for all values of x in the domain of f.

To graph $y = f'(x)$ using the graph of $y = f(x)$, first determine the slopes of the tangent lines, $f'(x_i)$, at certain x-values, x_i. These x-values include zeros, critical numbers, and numbers in each interval where f is increasing or decreasing. Then plot the corresponding ordered pairs on a graph. Draw a smooth curve through these points to complete the graph.



The given graph has a local minimum at (0, 1) and a local maximum at (1, 2). At these points, the tangents are horizontal. Therefore, $f'(0) = 0$ and $f'(1) = 0$.

At $x = \frac{1}{2}$, which is halfway between $x = 0$ and $x = 1$, the slope of the tangent is about $\frac{2}{3}$. So $f'\left(\frac{1}{2}\right) \doteq \frac{2}{3}$

The function $f(x)$ is decreasing when $f'(x) < 0$. The tangent lines show that $f'(x) < 0$ when $x < 0$ and when $x > 1$. Similarly, $f(x)$ is increasing when $f'(x) > 0$. The tangent lines show that $f'(x) > 0$ when $0 < x < 1$.

The shape of the graph of $f(x)$ suggests that $f(x)$ is a cubic polynomial with a negative leading coefficient. Assume that this is true. The derivative, $f'(x)$, may be a quadratic function with a negative leading coefficient. If it is, the graph of $f'(x)$ is a parabola that opens down.

Plot (0, 0), (1, 0), and $\left(\frac{1}{2}, \frac{2}{3}\right)$ on the graph of $f'(x)$. The graph of $f'(x)$ is a parabola that opens down and passes through these points.

IN SUMMARY

Key Idea

For a function $f(x)$, a **critical number** is a number, c, in the domain of $f(x)$ such that $f'(x) = 0$ or $f'(x)$ is undefined. As a result $(c, f(c))$ is called a critical point and usually corresponds to local or absolute extrema.

Need to Know

First Derivative Test

Let c be a critical number of a function f.

When moving through x-values from left to right:

- if $f'(x)$ changes from negative to positive at c, then $(c, f(c))$ is a **local minimum** of f.
- if $f'(x)$ changes from positive to negative at c, then $(c, f(c))$ is a **local maximum** of f.
- if $f'(x)$ does not change its sign at c, then $(c, f(c))$ is neither a local minimum or a local maximum.

Algorithm for Finding Local Maximum and Minimum Values of a Function *f*

1. Find critical numbers of the function (that is, determine where $f'(x) = 0$ and where $f'(x)$ is undefined) for all x-values in the domain of f.
2. Use the first derivative to analyze whether f is increasing or decreasing on either side of each critical number.
3. Based upon your findings in step 2., conclude whether each critical number locates a local maximum value of the function f, a local minimum value, or neither.

Exercise 4.2

PART A

C 1. Explain what it means to determine the critical points of the graph of a given function.

2. a. For the function $y = x^3 - 6x^2$, explain how you would find the critical points.

   b. Determine the critical points for $y = x^3 - 6x^2$, and then sketch the graph.

3. Find the critical points for each function. Use the first derivative test to determine whether the critical point is a local maximum, local minimum, or neither.

   a. $y = x^4 - 8x^2$ b. $f(x) = \dfrac{2x}{x^2 + 9}$ c. $y = x^3 + 3x^2 + 1$

4. Find the x- and y-intercepts of each function in question 3, and then sketch the curve.

5. Determine the critical points for each function. Determine whether the critical point is a local maximum or minimum, and whether or not the tangent is parallel to the horizontal axis.

   a. $h(x) = -6x^3 + 18x^2 + 3$
   b. $g(t) = t^5 + t^3$
   c. $y = (x - 5)^{\frac{1}{3}}$
   d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

6. Use graphing technology to graph the functions in question 5 and verify your results.

PART B

K 7. Determine the critical points for each of the following functions, and determine whether the function has a local maximum value, a local minimum value, or neither at the critical points. Sketch the graph of each function.

   a. $f(x) = -2x^2 + 8x + 13$
   b. $f(x) = \frac{1}{3}x^3 - 9x + 2$
   c. $f(x) = 2x^3 + 9x^2 + 12x$
   d. $f(x) = -3x^3 - 5x$
   e. $f(x) = \sqrt{x^2 - 2x + 2}$
   f. $f(x) = 3x^4 - 4x^3$

A 8. Suppose that f is a differentiable function with the derivative $f'(x) = (x + 1)(x - 2)(x + 6)$. Find all the critical numbers of f, and determine whether each corresponds to a local maximum, a local minimum, or neither.

9. Sketch a graph of a function f that is differentiable on the interval $-3 \leq x \leq 4$ and that satisfies the following conditions:

   - The function f is decreasing on $-1 < x < 3$ and increasing elsewhere on $-3 \leq x \leq 4$.
   - The largest value of f is 6, and the smallest value is 0.
   - The graph of f has local extrema at $(-1, 6)$ and $(3, 1)$.

10. Determine values of a, b, and c such that the graph of $y = ax^2 + bx + c$ has a relative maximum at $(3, 12)$ and crosses the y-axis at $(0, 1)$.

11. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of f on the interval $0 \leq x \leq 2$. Is this extremum a maximum value or a minimum value? Explain.

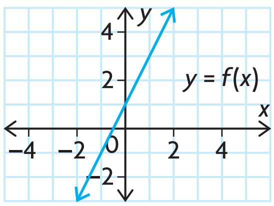
12. For $f(x) = x^3 - kx$, where $k \in \mathbf{R}$, find the values of k such that f has

    a. no critical numbers
    b. one critical number
    c. two critical numbers

T 13. Find values of a, b, c, and d such that $g(x) = ax^3 + bx^2 + cx + d$ has a local maximum at $(2, 4)$ and a local minimum at $(0, 0)$.

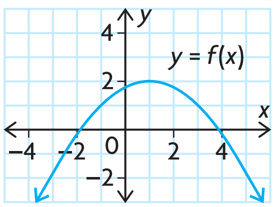
14. For each of the following graphs of the function $y = f(x)$, make a rough sketch of the derivative function $f'(x)$. By comparing the graphs of $f(x)$ and $f'(x)$, show that the intervals for which $f(x)$ is increasing correspond to the intervals where $f'(x)$ is positive. Also show that the intervals where $f(x)$ is decreasing correspond to the intervals for which $f'(x)$ is negative.

a.



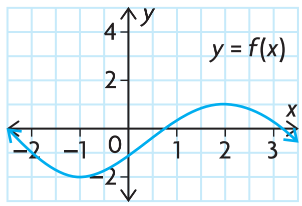
$f(x)$ is a linear function.

b.



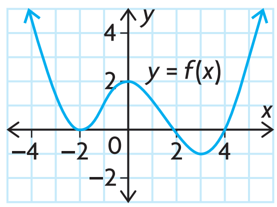
$f(x)$ is a quadratic function.

c.



$f(x)$ is a cubic function.

d.



$f(x)$ is a quartic function.

15. Consider the function $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$.
    a. Find constants a, b, c, and d such that the graph of f will have horizontal tangents at $(-2, -73)$ and $(0, -9)$.
    b. There is a third point that has a horizontal tangent. Find this point.
    c. For all three points, determine whether each corresponds to a local maximum, a local minimum, or neither.

PART C

16. For each of the following polynomials, find the local extrema and the direction that the curve is opening for $x = 100$. Use this information to make a quick sketch of the curve.
    a. $y = 4 - 3x^2 - x^4$
    b. $y = 3x^5 - 5x^3 - 30x$

17. Suppose that $f(x)$ and $g(x)$ are positive functions (functions where $f(x) > 0$ and $g(x) > 0$) such that $f(x)$ has a local maximum and $g(x)$ has a local minimum at $x = c$. Show that the function $h(x) = \frac{f(x)}{g(x)}$ has a local maximum at $x = c$.

Section 4.3—Vertical and Horizontal Asymptotes

Adding, subtracting, or multiplying two polynomial functions yields another polynomial function. Dividing two polynomial functions results in a function that is not a polynomial. The quotient is a **rational function**. Asymptotes are among the special features of rational functions, and they play a significant role in curve sketching. In this section, we will consider vertical and horizontal asymptotes of rational functions.

INVESTIGATION

The purpose of this investigation is to examine the occurrence of vertical asymptotes for rational functions.

A. Use your graphing calculator to obtain the graph of $f(x) = \frac{1}{x - k}$ and the table of values for each of the following: $k = 3, 1, 0, -2, -4$, and -5.

B. Describe the behaviour of each graph as x approaches k from the right and from the left.

C. Repeat parts A and B for the function $f(x) = \frac{x + 3}{x - k}$ using the same values of k.

D. Repeat parts A and B for the function $f(x) = \frac{1}{x^2 + x - k}$ using the following values: $k = 2, 6$, and 12.

E. Make a general statement about the existence of a vertical asymptote for a rational function of the form $y = \frac{p(x)}{q(x)}$ if there is a value c such that $q(c) = 0$ and $p(c) \neq 0$.

Vertical Asymptotes and Rational Functions

Recall that the notation $x \to c^+$ means that x approaches c from the right. Similarly, $x \to c^-$ means that x approaches c from the left.

You can see from this investigation that as $x \to c$ from either side, the function values get increasingly large and either positive or negative depending on the value of $p(c)$. We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers. They are symbols that represent the behaviour of a function that increases or decreases without limit.

Because the symbol ∞ is not a number, the limits $\lim_{x \to c^+} \frac{1}{x - c}$ and $\lim_{x \to c^-} \frac{1}{x - c}$ *do not exist*. For convenience, however, we use the notation $\lim_{x \to c^+} \frac{1}{x - c} = +\infty$ and $\lim_{x \to c^-} \frac{1}{x - c} = -\infty$.

These limits form the basis for determining the asymptotes of simple functions.

Vertical Asymptotes of Rational Functions

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

EXAMPLE 1 Reasoning about the behaviour of a rational function near its vertical asymptotes

Determine any vertical asymptotes of the function $f(x) = \frac{x}{x^2 + x - 2}$, and describe the behaviour of the graph of the function for values of x near the asymptotes.

Solution

First, determine the values of x for which $f(x)$ is undefined by solving the following:

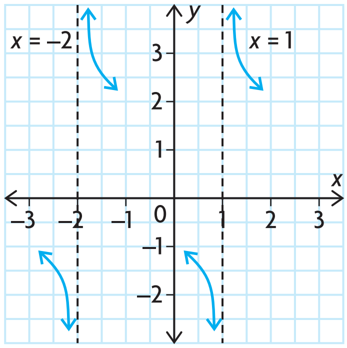
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Neither of these values of x makes the numerator zero, so both of these values give vertical asymptotes. The equations of the asymptotes are $x = -2$ and $x = 1$.

To determine the behaviour of the graph near the asymptotes, it can be helpful to use a chart.

Values of x	x	$x + 2$	$x - 1$	$f(x) = \frac{x}{(x + 2)(x - 1)}$	$f(x) \to ?$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

The behaviour of the graph can be illustrated as follows:



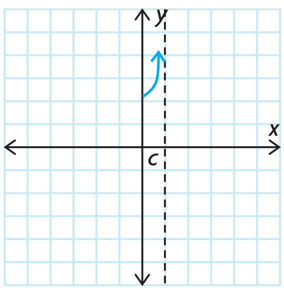
To proceed beyond this point, we require additional information.

Vertical Asymptotes and Infinite Limits

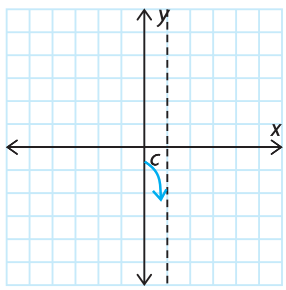
The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty, \quad \lim_{x \to c^-} f(x) = -\infty, \quad \lim_{x \to c^+} f(x) = +\infty \text{ or } \lim_{x \to c^+} f(x) = -\infty$$

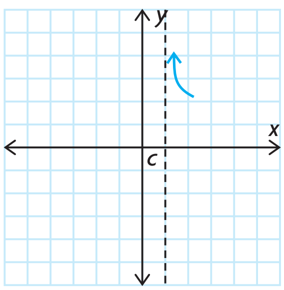
The following graphs correspond to each limit statement above:



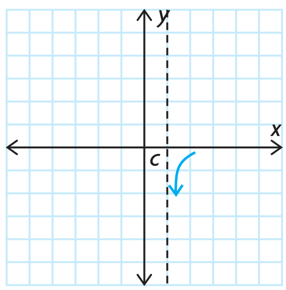
$\lim_{x \to c^-} f(x) = +\infty$



$\lim_{x \to c^-} f(x) = -\infty$



$\lim_{x \to c^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

Horizontal Asymptotes and Rational Functions

Consider the behaviour of rational functions $f(x) = \frac{p(x)}{q(x)}$ as x increases without bound in both the positive and negative directions. The following notation is used to describe this behaviour:

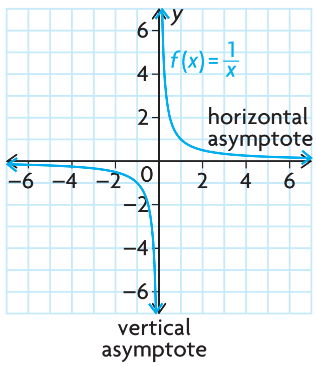
$$\lim_{x \to +\infty} f(x) \text{ and } \lim_{x \to -\infty} f(x)$$

The notation $x \to +\infty$ is read "x tends to positive infinity" and means that the values of x are positive and growing in magnitude without bound. Similarly, the notation $x \to -\infty$ is read "x tends to negative infinity" and means that the values of x are negative and growing in magnitude without bound.

The values of these limits can be determined by making two observations. The first observation is a list of simple limits, similar to those used for determining vertical asymptotes.

The Reciprocal Function and Limits at Infinity

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x} = 0$$



The second observation is that a polynomial can always be written so the term of highest degree is a factor.

EXAMPLE 2 Expressing a polynomial function in an equivalent form

Write each function so the term of highest degree is a factor.

a. $p(x) = x^2 + 4x + 1$

b. $q(x) = 3x^2 - 4x + 5$

Solution

a. $$p(x) = x^2 + 4x + 1 = x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)$$

b. $$q(x) = 3x^2 - 4x + 5 = 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right)$$

The value of writing a polynomial in this form is clear. It is easy to see that as x becomes large (either positive or negative), the value of the second factor always approaches 1.

We can now determine the limit of a rational function in which the degree of $p(x)$ is equal to or less than the degree of $q(x)$.

EXAMPLE 3 Selecting a strategy to evaluate limits at infinity

Determine the value of each of the following:

a. $\lim\limits_{x \to +\infty} \dfrac{2x - 3}{x + 1}$

b. $\lim\limits_{x \to -\infty} \dfrac{x}{x^2 + 1}$

c. $\lim\limits_{x \to +\infty} \dfrac{2x^2 + 3}{3x^2 - x + 4}$

Solution

a. $$f(x) = \frac{2x - 3}{x + 1} = \frac{2x\left(1 - \frac{3}{2x}\right)}{x\left(1 + \frac{1}{x}\right)}$$ (Factor and simplify)

$$= \frac{2\left(1 - \frac{3}{2x}\right)}{1 + \frac{1}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{2\left[\lim\limits_{x \to +\infty}\left(1 - \frac{3}{2x}\right)\right]}{\lim\limits_{x \to +\infty}\left(1 + \frac{1}{x}\right)}$$ (Apply limit properties)

$$= \frac{2(1 - 0)}{1 + 0}$$ (Evaluate)

$$= 2$$

b.

$$g(x) = \frac{x}{x^2 + 1} \qquad \text{(Factor)}$$

$$= \frac{x(1)}{x^2\left(1 + \frac{1}{x^2}\right)} \qquad \text{(Simplify)}$$

$$= \frac{1}{x\left(1 + \frac{1}{x^2}\right)}$$

$$\lim_{x \to -\infty} g(x) = \frac{1}{\lim_{x \to -\infty} x \times \lim_{x \to -\infty}\left(1 + \frac{1}{x^2}\right)} \qquad \text{(Apply limit properties)}$$

$$= \frac{1}{\lim_{x \to -\infty} x \times (1)}$$

$$= \lim_{x \to -\infty} \frac{1}{x} \qquad \text{(Evaluate)}$$

$$= 0$$

c. To evaluate this limit, we can use the technique of dividing the numerator and denominator by the highest power of x in the denominator.

$$p(x) = \frac{2x^2 + 3}{3x^2 - x + 4} \qquad \text{(Divide by } x^2\text{)}$$

$$= \frac{(2x^2 + 3) \div x^2}{(3x^2 - x + 4) \div x^2} \qquad \text{(Simplify)}$$

$$= \frac{2 + \frac{3}{x^2}}{3 - \frac{1}{x} + \frac{4}{x^2}}$$

$$\lim_{x \to +\infty} p(x) = \frac{\lim_{x \to +\infty}\left(2 + \frac{3}{x^2}\right)}{\lim_{x \to +\infty}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)} \qquad \text{(Apply limit properties)}$$

$$= \frac{2 + 0}{3 - 0 + 0} \qquad \text{(Evaluate)}$$

$$= \frac{2}{3}$$

When $\lim_{x\to+\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$, the graph of the function is approaching the line $y = k$. This line is a horizontal asymptote of the function. In Example 3, part a, $y = 2$ is a horizontal asymptote of $f(x) = \frac{2x - 3}{x + 1}$. Therefore, for large positive x-values, the y-values approach 2. This is also the case for large negative x-values.

To sketch the graph of the function, we need to know whether the curve approaches the horizontal asymptote from above or below. To find out, we need to consider $f(x) - k$, where k is the limit we just determined. This is illustrated in the following examples.

EXAMPLE 4 Reasoning about the end behaviours of a rational function

Determine the equations of any horizontal asymptotes of the function $f(x) = \frac{3x + 5}{2x - 1}$. State whether the graph approaches the asymptote from above or below.

Solution

$$f(x) = \frac{3x + 5}{2x - 1} = \frac{(3x + 5) \div x}{(2x - 1) \div x}$$ (Divide by x)

$$= \frac{3 + \frac{5}{x}}{2 - \frac{1}{x}}$$

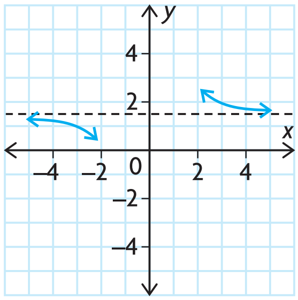
$$\lim_{x\to+\infty} f(x) = \frac{\lim_{x\to+\infty}\left(3 + \frac{5}{x}\right)}{\lim_{x\to+\infty}\left(2 - \frac{1}{x}\right)}$$ (Evaluate)

$$= \frac{3}{2}$$

Similarly, we can show that $\lim_{x\to-\infty} f(x) = \frac{3}{2}$. So, $y = \frac{3}{2}$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x. To determine whether the graph approaches the asymptote from above or below, we consider very large positive and negative values of x.

If x is large and positive (for example, if $x = 1000$), $f(x) = \frac{3005}{1999}$, which is greater than $\frac{3}{2}$. Therefore, the graph approaches the asymptote $y = \frac{3}{2}$ from above.

If x is large and negative (for example, if $x = -1000$), $f(x) = \frac{-2995}{-20\,001}$, which is less than $\frac{3}{2}$. This part of the graph approaches the asymptote $y = \frac{3}{2}$ from below, as illustrated in the diagram.



EXAMPLE 5 **Selecting a limit strategy to analyze the behaviour of a rational function near its asymptotes**

For the function $f(x) = \frac{3x}{x^2 - x - 6}$, determine the equations of all horizontal or vertical asymptotes. Illustrate the behaviour of the graph as it approaches the asymptotes.

Solution

For vertical asymptotes,

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \text{ or } x = -2$$

There are two vertical asymptotes, at $x = 3$ and $x = -2$.

Values of x	x	$x - 3$	$x + 2$	$f(x)$	$f(x) \to ?$
$x \to 3^-$	> 0	< 0	> 0	< 0	$-\infty$
$x \to 3^+$	> 0	> 0	> 0	> 0	$+\infty$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	> 0	$+\infty$

For horizontal asymptotes,

$$f(x) = \frac{3x}{x^2 - x - 6} \qquad \text{(Factor)}$$

$$= \frac{3x}{x^2\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \qquad \text{(Simplify)}$$

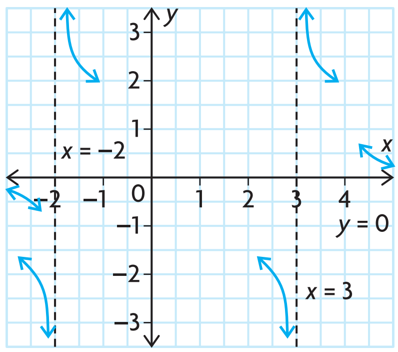
$$= \frac{3}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{3}{x} = 0$$

Similarly, we can show $\lim_{x \to -\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x.

As x becomes large positively, $f(x) > 0$, so the graph is above the horizontal asymptote. As x becomes large negatively, $f(x) < 0$, so the graph is below the horizontal asymptote.

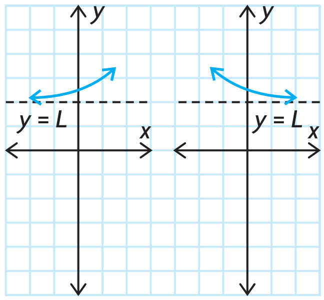
This diagram illustrates the behaviour of the graph as it nears the asymptotes:



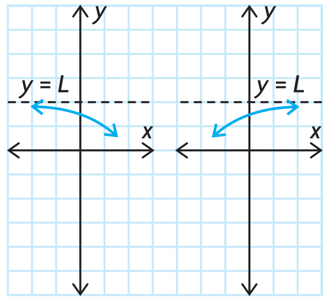
Horizontal Asymptotes and Limits at Infinity

If $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote of the graph of $f(x)$.

The following graphs illustrate some typical situations:



$f(x) > L$, so the graph approaches from above.



$f(x) < L$, so the graph approaches from below.

In addition to vertical and horizontal asymptotes, it is possible for a graph to have **oblique asymptotes**. These are straight lines that are slanted and to which the curve becomes increasingly close. They occur with rational functions in which the degree of the numerator exceeds the degree of the denominator by exactly one. This is illustrated in the following example.

EXAMPLE 6 Reasoning about oblique asymptotes

Determine the equations of all asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.

Solution

Since $x + 1 = 0$ for $x = -1$, and $2x^2 + 3x - 1 \neq 0$ for $x = -1$, $x = -1$ is a vertical asymptote.

$$\text{Now } \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} = \lim_{x\to\infty} \frac{2x^2\left(1 + \frac{3}{2x} - \frac{1}{2x^2}\right)}{x\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} 2x$$

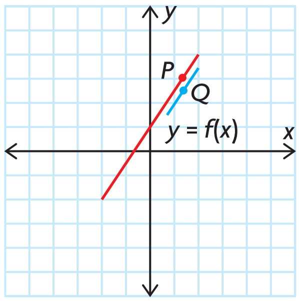
This limit does not exist, and, by a similar calculation, $\lim_{x\to-\infty} f(x)$ does not exist, so there is no horizontal asymptote.

Dividing the numerator by the denominator,

$$\begin{array}{r|l} & 2x + 1 \\ x + 1 & \overline{2x^2 + 3x - 1} \\ & \underline{2x^2 + 2x} \\ & \quad x - 1 \\ & \quad \underline{x + 1} \\ & \quad -2 \end{array}$$

Thus, we can write $f(x)$ in the form $f(x) = 2x + 1 - \frac{2}{x + 1}$.

Now let's consider the straight line $y = 2x + 1$ and the graph of $y = f(x)$. For any value of x, we can determine point $P(x, 2x + 1)$ on the line and point $Q\left(x, 2x + 1 - \frac{2}{x + 1}\right)$ on the curve.



Then the vertical distance QP from the curve to the line is

$$QP = 2x + 1 - \left(2x + 1 - \frac{2}{x + 1}\right)$$
$$= \frac{2}{x + 1}$$

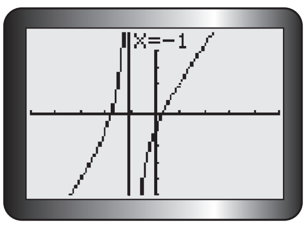
$$\lim_{x \to \infty} QP = \lim_{x \to \infty} \frac{2}{x + 1}$$
$$= 0$$

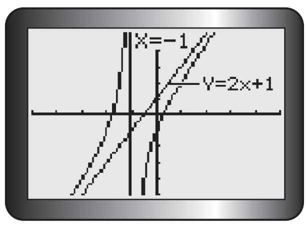
That is, as x gets very large, the curve approaches the line but never touches it. Therefore, the line $y = 2x + 1$ is an asymptote of the curve.

Since $\lim_{x \to -\infty} \frac{2}{x + 1} = 0$, the line is also an asymptote for large negative values of x.

In conclusion, there are two asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$. They are $y = 2x + 1$ and $x = -1$.

Use a graphing calculator to obtain the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.





Note that the vertical asymptote $x = -1$ appears on the graph on the left, but the oblique asymptote $y = 2x + 1$ does not. Use the Y2 function to graph the oblique asymptote $y = 2x + 1$.

IN SUMMARY

Key Ideas

- The graph of $f(x)$ has a **vertical asymptote** $x = c$ if any of the following is true:
  $\lim_{x\to c^-} f(x) = +\infty$ $\qquad$ $\lim_{x\to c^-} f(x) = -\infty$
  $\lim_{x\to c^+} f(x) = +\infty$ $\qquad$ $\lim_{x\to c^+} f(x) = -\infty$
- The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x\to +\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.
- In a rational function, an **oblique asymptote** occurs when the degree of the numerator is exactly one greater than the degree of the denominator**.**

Need to Know

The techniques for curve sketching developed to this point are described in the following algorithm. As we develop new ideas, the algorithm will be extended.

Algorithm for Curve Sketching (so far)

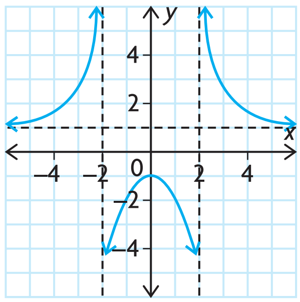
To sketch a curve, apply these steps in the order given.

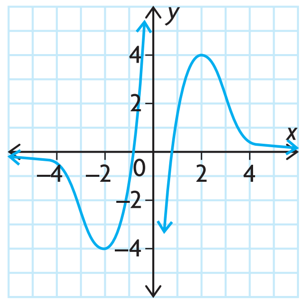
1. Check for any discontinuities in the domain. Determine if there are vertical asymptotes at these discontinuities, and determine the direction from which the curve approaches these asymptotes.
2. Find **both intercepts**.
3. Find any critical points.
4. Use the first derivative test to determine the type of critical points that may be present.
5. **Test end behaviour** by determining $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$.
6. Construct an interval of increase/decrease table and identify all local or absolute extrema.
7. Sketch the curve.

Exercise 4.3

PART A

1. State the equations of the vertical and horizontal asymptotes of the curves shown.

a. 

b. 

C 2. Under what conditions does a rational function have vertical, horizontal, and oblique asymptotes?

3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, using the symbol "∞" when appropriate.

a. $f(x) = \dfrac{2x + 3}{x - 1}$

b. $f(x) = \dfrac{5x^2 - 3}{x^2 + 2}$

c. $f(x) = \dfrac{-5x^2 + 3x}{2x^2 - 5}$

d. $f(x) = \dfrac{2x^5 - 3x^2 + 5}{3x^4 + 5x - 4}$

4. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \dfrac{x}{x + 5}$

b. $f(x) = \dfrac{x + 2}{x - 2}$

c. $s = \dfrac{1}{(t - 3)^2}$

d. $y = \dfrac{x^2 - x - 6}{x - 3}$

e. $f(x) = \dfrac{6}{(x + 3)(x - 1)}$

f. $y = \dfrac{x^2}{x^2 - 1}$

5. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.

a. $y = \dfrac{x}{x + 4}$

b. $f(x) = \dfrac{2x}{x^2 - 1}$

c. $g(t) = \dfrac{3t^2 + 4}{t^2 - 1}$

d. $y = \dfrac{3x^2 - 8x - 7}{x - 4}$

PART B

K 6. For each of the following, check for discontinuities and then use at least two other tests to make a rough sketch of the curve. Verify using a calculator.

a. $y = \frac{x - 3}{x + 5}$

b. $f(x) = \frac{5}{(x + 2)^2}$

c. $g(t) = \frac{t^2 - 2t - 15}{t - 5}$

d. $y = \frac{(2 + x)(3 - 2x)}{(x^2 - 3x)}$

7. Determine the equation of the oblique asymptote for each of the following:

a. $f(x) = \frac{3x^2 - 2x - 17}{x - 3}$

b. $f(x) = \frac{2x^2 + 9x + 2}{2x + 3}$

c. $f(x) = \frac{x^3 - 1}{x^2 + 2x}$

d. $f(x) = \frac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

8. a. For question 7, part a., determine whether the curve approaches the asymptote from above or below.

b. For question 7, part b., determine the direction from which the curve approaches the asymptote.

9. For each function, determine any vertical or horizontal asymptotes and describe its behaviour on each side of any vertical asymptote.

a. $f(x) = \frac{3x - 1}{x + 5}$

b. $g(x) = \frac{x^2 + 3x - 2}{(x - 1)^2}$

c. $h(x) = \frac{x^2 + x - 6}{x^2 - 4}$

d. $m(x) = \frac{5x^2 - 3x + 2}{x - 2}$

A 10. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = \frac{3 - x}{2x + 5}$

b. $h(t) = 2t^3 - 15t^2 + 36t - 10$

c. $y = \frac{20}{x^2 + 4}$

d. $s(t) = t + \frac{1}{t}$

e. $g(x) = \frac{2x^2 + 5x + 2}{x + 3}$

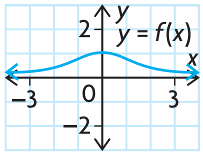
f. $s(t) = \frac{t^2 + 4t - 21}{t - 3}, t \geq -7$

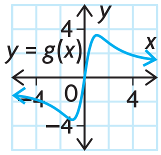
11. Consider the function $y = \frac{ax + b}{cx + d}$, where a, b, c, and d are constants, $a \neq 0, c \neq 0$.

a. Determine the horizontal asymptote of the graph.

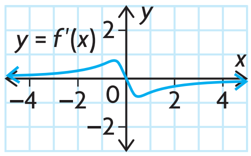
b. Determine the vertical asymptote of the graph.

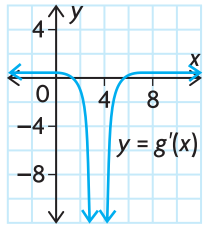
12. Use the features of each function's graph to sketch the graph of its first derivative.

a. 

b. 

13. A function's derivative is shown in each graph. Use the graph to sketch a possible graph for the original function.

a. 

b. 

14. Let $f(x) = \frac{-x - 3}{x^2 - 5x - 14}$, $g(x) = \frac{x - x^3}{x - 3}$, $h(x) = \frac{x^3 - 1}{x^2 + 4}$, and $r(x) = \frac{x^2 + x - 6}{x^2 - 16}$. How can you can tell from its equation which of these functions has

a. a horizontal asymptote?

b. an oblique asymptote?

c. no vertical asymptote?

Explain. Determine the equations of all asymptote(s) for each function. Describe the behaviour of each function close to its asymptotes.

PART C

T 15. Find constants a and b such that the graph of the function defined by $f(x) = \frac{ax + 5}{3 - bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

16. To understand why we cannot work with the symbol ∞ as though it were a real number, consider the functions $f(x) = \frac{x^2 + 1}{x + 1}$ and $g(x) = \frac{x^2 + 2x + 1}{x + 1}$.

a. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$.

b. Evaluate $\lim_{x \to +\infty} [f(x) - g(x)]$, and show that the limit is not zero.

17. Use the algorithm for curve sketching to sketch the graph of the function $f(x) = \frac{2x^2 - 2x}{x^2 - 9}$.

Mid-Chapter Review

1. Use a graphing calculator or graphing software to graph each of the following functions. Inspect the graph to determine where the function is increasing and where it is decreasing.
   a. $y = 3x^2 - 12x + 7$
   b. $y = 4x^3 - 12x^2 + 8$
   c. $f(x) = \frac{x + 2}{x + 3}$
   d. $f(x) = \frac{x^2 - 1}{x^2 + 3}$
2. Determine where $g(x) = 2x^3 - 3x^2 - 12x + 15$ is increasing and where it is decreasing.
3. Graph $f(x)$ if $f'(x) < 0$ when $x < -2$ and $x > 3$, $f'(x) > 0$ when $-2 < x < 3$, $f(-2) = 0$, and $f(3) = 5$.
4. Find all the critical numbers of each function.
   a. $y = -2x^2 + 16x - 31$
   b. $y = x^3 - 27x$
   c. $y = x^4 - 4x^2$
   d. $y = 3x^5 - 25x^3 + 60x$
   e. $y = \frac{x^2 - 1}{x^2 + 1}$
   f. $y = \frac{x}{x^2 + 2}$
5. For each function, find the critical numbers. Use the first derivative test to identify the local maximum and minimum values.
   a. $g(x) = 2x^3 - 9x^2 + 12x$
   b. $g(x) = x^3 - 2x^2 - 4x$
6. Find a value of k that gives $f(x) = x^2 + kx + 2$ a local minimum value of 1.
7. For $f(x) = x^4 - 32x + 4$, find the critical numbers, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.
8. Find the vertical asymptote(s) of the graph of each function. Describe the behaviour of $f(x)$ to the left and right of each asymptote.
   a. $f(x) = \frac{x - 1}{x + 2}$
   b. $f(x) = \frac{1}{9 - x^2}$
   c. $f(x) = \frac{x^2 - 4}{3x + 9}$
   d. $f(x) = \frac{2 - x}{3x^2 - 13x - 10}$
9. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.
   a. $y = \frac{3x - 1}{x + 5}$
   b. $f(x) = \frac{x^2 + 3x - 2}{(x - 1)^2}$
10. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.
   a. $f(x) = \frac{x}{(x - 5)^2}$
   b. $f(x) = \frac{5}{x^2 + 9}$
   c. $f(x) = \frac{x - 2}{x^2 - 12x + 12}$

11. a. What does $f'(x) > 0$ imply about $f(x)$?

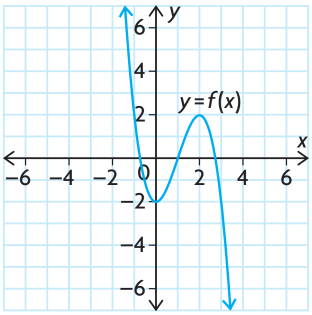
b. What does $f'(x) < 0$ imply about $f(x)$?

12. A diver dives from the 3 m springboard. The diver's height above the water, in metres, at t seconds is $h(t) = -4.9t^2 + 9.5t + 2.2$.

a. When is the height of the diver increasing? When is it decreasing?

b. When is the velocity of the diver increasing? When is it decreasing?

13. The concentration, C, of a drug injected into the bloodstream t hours after injection can be modelled by $C(t) = \frac{t}{4} + 2t^{-2}$. Determine when the concentration of the drug is increasing and when it is decreasing.



14. Graph $y = f'(x)$ for the function shown at the left.

15. For each function $f(x)$,

i. find the critical numbers

ii. determine where the function increases and decreases

iii. determine whether each critical number is at a local maximum, a local minimum, or neither

iv. use all the information to sketch the graph

a. $f(x) = x^2 - 7x - 18$

b. $f(x) = -2x^3 + 9x^2 + 3$

c. $f(x) = 2x^4 - 4x^2 + 2$

d. $f(x) = x^5 - 5x$

16. Determine the equations of any vertical or horizontal asymptotes for each function. Describe the behaviour of the function on each side of any vertical or horizontal asymptote.

a. $f(x) = \dfrac{x - 5}{2x + 1}$

b. $g(x) = \dfrac{x^2 - 4x - 5}{(x + 2)^2}$

c. $h(x) = \dfrac{x^2 + 2x - 15}{9 - x^2}$

d. $m(x) = \dfrac{2x^2 + x + 1}{x + 4}$

17. Find each limit.

a. $\lim\limits_{x \to \infty} \dfrac{3 - 2x}{3x}$

b. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 5}{6x^2 + 2x - 1}$

c. $\lim\limits_{x \to \infty} \dfrac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$

d. $\lim\limits_{x \to \infty} \dfrac{5 - 2x^3}{x^4 - 4x}$

e. $\lim\limits_{x \to \infty} \dfrac{2x^5 - 1}{3x^4 - x^2 - 2}$

f. $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x - 18}{(x - 3)^2}$

g. $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x - 5}{x^2 - 1}$

h. $\lim\limits_{x \to \infty} \left(5x + 4 - \dfrac{7}{x + 3}\right)$

Section 4.4—Concavity and Points of Inflection

In Chapter 3, you saw that the second derivative of a function has applications in problems involving velocity and acceleration or in general rates-of-change problems. Here we examine the use of the second derivative of a function in curve sketching.

INVESTIGATION 1 The purpose of this investigation is to examine the relationship between slopes of tangents and the second derivative of a function.

A. Sketch the graph of $f(x) = x^2$.

B. Determine $f'(x)$. Use $f'(x)$ to calculate the slope of the tangent to the curve at the points with the following x-coordinates: $x = -4, -3, -2, -1, 0, 1, 2, 3,$ and 4. Sketch each of these tangents.

C. Are these tangents above or below the graph of $y = f(x)$?

D. Describe the change in the slopes as x increases.

E. Determine $f''(x)$. How does the value of $f''(x)$ relate to the way in which the curve opens? How does the value of $f''(x)$ relate to the way $f'(x)$ changes as x increases?

F. Repeat parts B, C, and D for the graph of $f(x) = -x^2$.

G. How does the value of $f''(x)$ relate to the way in which the curve opens?

INVESTIGATION 2 The purpose of this investigation is to extend the results of Investigation 1 to other functions.

A. Sketch the graph of $f(x) = x^3$.

B. Determine all the values of x for which $f'(x) = 0$.

C. Determine intervals on the domain of the function such that $f''(x) < 0$, $f''(x) = 0$, and $f''(x) > 0$.

D. For values of x such that $f''(x) < 0$, how does the shape of the curve compare with your conclusions in Investigation 1?

E. Repeat part D for values of x such that $f''(x) > 0$.

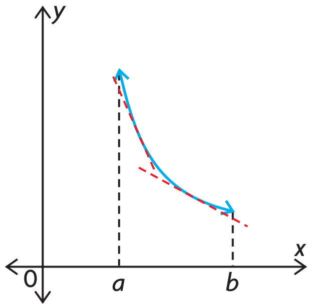
F. What happens when $f''(x) = 0$?

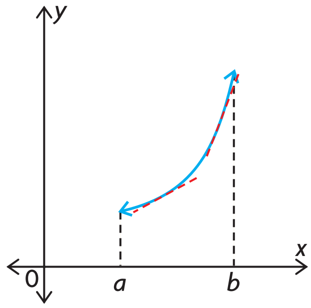
G. Using your observations from this investigation, sketch the graph of $y = x^3 - 12x$.

From these investigations, we can make a summary of the behaviour of the graphs.

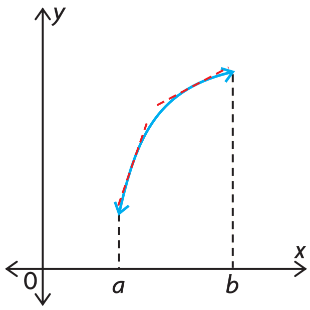
Concavity and the Second Derivative

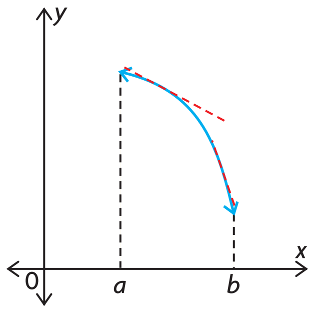
1. The graph of $y = f(x)$ is **concave up** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are increasing. On this interval, $f''(x)$ exists and $f''(x) > 0$. The graph of the function is above the tangent at every point on the interval.



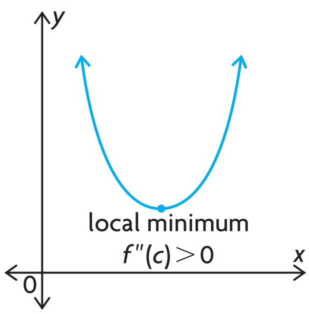


2. The graph of $y = f(x)$ is **concave down** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are decreasing. On this interval, $f''(x)$ exists and $f''(x) < 0$. The graph of the function is below the tangent at every point on the interval.

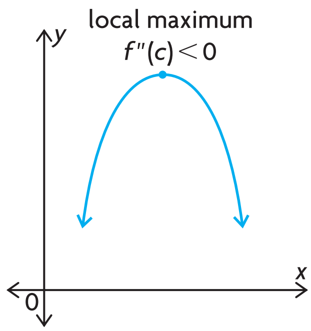




3. If $y = f(x)$ has a critical point at $x = c$, with $f'(c) = 0$, then the behaviour of $f(x)$ at $x = c$ can be analyzed through the use of the **second derivative test** by analyzing $f''(c)$, as follows:
   a. The graph is concave up, and $x = c$ is the location of a local minimum value of the function, if $f''(c) > 0$.

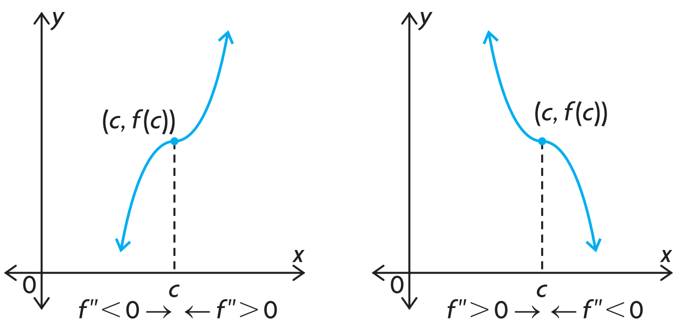


b. The graph is concave down, and $x = c$ is the location of a local maximum value of the function, if $f''(c) < 0$.



c. If $f''(c) = 0$, the nature of the critical point cannot be determined without further work.

4. A **point of inflection** occurs at $(c, f(c))$ on the graph of $y = f(x)$ if $f''(x)$ changes sign at $x = c$. That is, the curve changes from concave down to concave up, or vice versa.



5. All points of inflection on the graph of $y = f(x)$ must occur either where $\frac{d^2y}{dx^2}$ equals zero or where $\frac{d^2y}{dx^2}$ is undefined.

In the following examples, we will use these properties to sketch graphs of other functions.

EXAMPLE 1 **Using the first and second derivatives to analyze a cubic function**

Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$.

Solution

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Setting $\frac{dy}{dx} = 0$, we obtain

$$3(x^2 - 2x - 3) = 0$$
$$3(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

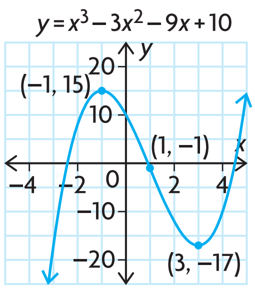
$$\frac{d^2y}{dx^2} = 6x - 6$$

Setting $\frac{d^2y}{dx^2} = 0$, we obtain $6x - 6 = 0$ or $x = 1$.

Now determine the sign of $f''(x)$ in the intervals determined by $x = 1$.

Interval	$x < 1$	$x = 1$	$x > 1$
$f''(x)$	< 0	0	> 0
Graph of $f(x)$	concave down	point of inflection	concave up
Sketch of $f(x)$			

Applying the second derivative test, at $x = 3$, we obtain the local minimum point, $(3, -17)$ and at $x = -1$, we obtain the local maximum point, $(-1, 15)$. The point of inflection occurs at $x = 1$ where $f(1) = -1$.
The graph can now be sketched.



EXAMPLE 2 Using the first and second derivatives to analyze a quartic function

Sketch the graph of $f(x) = x^4$.

Solution

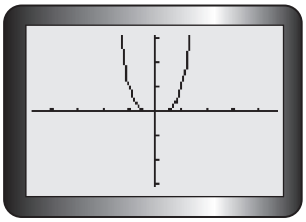
The first and second derivatives of $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$.
Setting $f''(x) = 0$, we obtain
$12x^2 = 0$ or $x = 0$

But $x = 0$ is also obtained from $f'(x) = 0$.

Now determine the sign of $f''(x)$ on the intervals determined by $x = 0$.

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	> 0
Graph of $f(x)$	concave up	?	concave up
Sketch of $f(x)$			

We conclude that the point $(0, 0)$ is *not* an inflection point because $f''(x)$ does not change sign at $x = 0$. However, since $x = 0$ is a critical number and $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$, $(0, 0)$ is an absolute minimum.



EXAMPLE 3

Using the first and second derivatives to analyze a root function

Sketch the graph of the function $f(x) = x^{\frac{1}{3}}$.

Solution

The derivative of $f(x)$ is

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$= \frac{1}{3x^{\frac{2}{3}}}$$

Note that $f'(0)$ does not exist, so $x = 0$ is a critical number of $f(x)$. It is important to determine the behaviour of $f'(x)$ as $x \to 0$. Since $f'(x) > 0$ for all values of $x \neq 0$, and the denominator of $f'(x)$ is zero when $x = 0$, we have $\lim_{x \to 0} f'(x) = +\infty$. This means that there is a vertical tangent at $x = 0$. In addition, $f(x)$ is increasing for $x < 0$ and $x > 0$. As a result this graph has no local extrema.

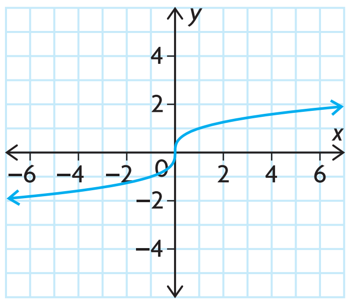
The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$
$$= -\frac{2}{9x^{\frac{5}{3}}}$$

Since $x^{\frac{5}{3}} > 0$ if $x > 0$, and $x^{\frac{5}{3}} < 0$ if $x < 0$, we obtain the following table:

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$\frac{-}{-} = +$	does not exist	$\frac{-}{+} = -$
$f(x)$			

The graph has a point of inflection when $x = 0$, even though $f'(0)$ and $f''(0)$ do not exist. Note that the curve crosses its tangent at $x = 0$.



EXAMPLE 4 Reasoning about points of inflection

Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2 + 3}$.

Solution

The derivative of $f(x) = \frac{1}{x^2 + 3} = (x^2 + 3)^{-1}$ is $f'(x) = -2x(x^2 + 3)^{-2}$.
The second derivative is

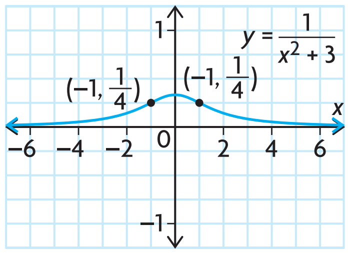
$$f''(x) = -2(x^2 + 3)^{-2} + 4x(x^2 + 3)^{-3}(2x)$$
$$= \frac{-2}{(x^2 + 3)^2} + \frac{8x^2}{(x^2 + 3)^3}$$
$$= \frac{-2(x^2 + 3) + 8x^2}{(x^2 + 3)^3}$$
$$= \frac{6x^2 - 6}{(x^2 + 3)^3}$$

Setting $f''(x) = 0$ gives $6x^2 - 6 = 0$ or $x = \pm 1$.

Determine the sign of $f''(x)$ on the intervals determined by $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	concave up	point of inflection	concave down	point of inflection	concave up

Therefore, $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$ are points of inflection on the graph of $f(x)$.



IN SUMMARY

Key Ideas

- The graph of a function $f(x)$ is **concave up** on an interval if $f'(x)$ is increasing on the interval. The graph of a function $f(x)$ is **concave down** on an interval if $f'(x)$ is decreasing on the interval.
- A point of inflection is a point on the graph of $f(x)$ where the function changes from concave up to concave down, or vice versa. $f''(c) = 0$ or is undefined if $(c, f(c))$ is a point of inflection on the graph of $f(x)$.

Need to Know

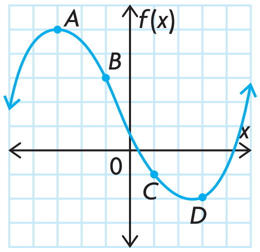
- **Test for concavity:** If $f(x)$ is a differentiable function whose second derivative exists on an open interval I, then
  - the graph of $f(x)$ is concave up on I if $f''(x) > 0$ for all values of x in I
  - the graph of $f(x)$ is concave down on I if $f''(x) < 0$ for all values of x in I
- **The second derivative test:** Suppose that $f(x)$ is a function for which $f''(c) = 0$, and the second derivative of $f(x)$ exists on an interval containing c.
  - If $f''(c) > 0$, then $f(c)$ is a local minimum value.
  - If $f''(c) < 0$, then $f(c)$ is a local maximum value.
  - If $f''(c) = 0$, then the test fails. Use the first derivative test.

Exercise 4.4

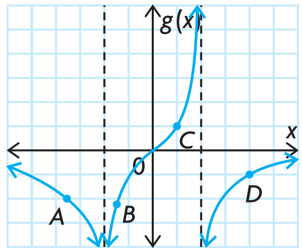
PART A

K 1. For each function, state whether the value of the second derivative is positive or negative at each of points A, B, C, and D.

a.



b.



2. Determine the critical points for each function, and use the second derivative test to decide if the point is a local maximum, a local minimum, or neither.

a. $y = x^3 - 6x^2 - 15x + 10$

b. $y = \dfrac{25}{x^2 + 48}$

c. $s = t + t^{-1}$

d. $y = (x - 3)^3 + 8$

3. Determine the points of inflection for each function in question 2. Then conduct a test to determine the change of sign in the second derivative.

4. Determine the value of the second derivative at the value indicated. State whether the curve lies above or below the tangent at this point.

a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

b. $g(x) = x^2 - \dfrac{1}{x}$ at $x = -1$

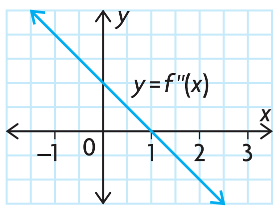
c. $p(w) = \dfrac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

d. $s(t) = \dfrac{2t}{t - 4}$ at $t = -2$

PART B

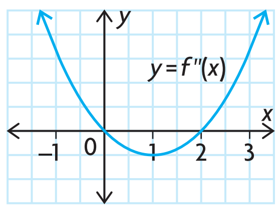
5. Each of the following graphs represents the second derivative, $f''(x)$, of a function $f(x)$:

a.



$f''(x)$ is a linear function.

b.



$f''(x)$ is a quadratic function.

For each of the graphs above, answer the following questions:

i. On which intervals is the graph of $f(x)$ concave up? On which intervals is the graph concave down?

ii. List the x-coordinates of all the points of inflection.

iii. Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 2$.

C 6. Describe how you would use the second derivative to determine a local minimum or maximum.

7. In the algorithm for curve sketching in Section 4.3, reword step 4 to include the use of the second derivative to test for local minimum or maximum values.

8. For each of the following functions,

i. determine any points of inflection

ii. use the results of part i, along with the revised algorithm, to sketch each function.

a. $f(x) = x^4 + 4x$

b. $g(w) = \dfrac{4w^2 - 3}{w^3}$

A 9. Sketch the graph of a function with the following properties:

- $f'(x) > 0$ when $x < 2$ and when $2 < x < 5$
- $f'(x) < 0$ when $x > 5$
- $f'(2) = 0$ and $f'(5) = 0$
- $f''(x) < 0$ when $x < 2$ and when $4 < x < 7$
- $f''(x) > 0$ when $2 < x < 4$ and when $x > 7$
- $f(0) = -4$

10. Find constants a, b, and c such that the function $f(x) = ax^3 + bx^2 + c$ will have a local extremum at $(2, 11)$ and a point of inflection at $(1, 5)$. Sketch the graph of $y = f(x)$.

PART C

11. Find the value of the constant b such that the function $f(x) = \sqrt{x + 1} + \frac{b}{x}$ has a point of inflection at $x = 3$.

T 12. Show that the graph of $f(x) = ax^4 + bx^3$ has two points of inflection. Show that the x-coordinate of one of these points lies midway between the x-intercepts.

13. a. Use the algorithm for curve sketching to sketch the function $y = \dfrac{x^3 - 2x^2 + 4x}{x^2 - 4}$.

b. Explain why it is difficult to determine the oblique asymptote using a graphing calculator.

14. Find the inflection points, if any exist, for the graph of $f(x) = (x - c)^n$, for $n = 1, 2, 3$, and 4. What conclusion can you draw about the value of n and the existence of inflection points on the graph of f?

Section 4.5—An Algorithm for Curve Sketching

You now have the necessary skills to sketch the graphs of most elementary functions. However, you might be wondering why you should spend time developing techniques for sketching graphs when you have a graphing calculator. The answer is that, in doing so, you develop an understanding of the qualitative features of the functions you are analyzing. Also, for certain functions, maximum/minimum/inflection points are not obvious if the window setting is not optimal. In this section, you will combine the skills you have developed. Some of them use the calculus properties. Others were learned earlier. Putting all the skills together will allow you to develop an approach that leads to simple, yet accurate, sketches of the graphs of functions.

An Algorithm for Sketching the Graph of $y = f(x)$

Note: As each piece of information is obtained, use it to build the sketch.

1: Determine any discontinuities or limitations in the domain. For discontinuities, investigate the function's values on either side of the discontinuity.
2: Determine any vertical asymptotes.
3: Determine any intercepts.
4: Determine any critical numbers by finding where $\frac{dy}{dx} = 0$ or where $\frac{dy}{dx}$ is undefined.
5: Determine the intervals of increase/decrease, and then test critical points to see whether they are local maxima, local minima, or neither.
6: Determine the behaviour of the function for large positive and large negative values of x. This will identify horizontal asymptotes, if they exist. Identify if the functions values approach the horizontal asymptote from above or below.
7: Determine $\frac{d^2y}{dx^2}$ and test for points of inflection using the intervals of concavity.
8: Determine any oblique asymptotes. Identify if the functions values approach the obliques asymptote from above or below.
9: Complete the sketch using the above information.

When using this algorithm, keep two things in mind:

1. You will not use all the steps in every situation. Use only the steps that are essential.

2. You are familiar with the basic shapes of many functions. Use this knowledge when possible.

INVESTIGATION Use the algorithm for curve sketching to sketch the graph of each of the following functions. After completing your sketch, use graphing technology to verify your results.

a. $y = x^4 - 3x^2 + 2x$ b. $y = \frac{x}{x^2 - 1}$

EXAMPLE 1 Sketching an accurate graph of a polynomial function

Use the algorithm for curve sketching to sketch the graph of $f(x) = -3x^3 - 2x^2 + 5x$.

Solution

This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$. **Analyze $f(x)$.** Determine any intercepts.

x-intercept, $y = 0$

$$\begin{aligned} -3x^3 - 2x^2 + 5x &= 0 \\ -x(3x^2 + 2x - 5) &= 0 \\ -x(3x + 5)(x - 1) &= 0 \end{aligned}$$

$$x = 0, x = -\frac{5}{3}, x = 1$$

$$(0, 0), \left(-\frac{5}{3}, 0\right), (1, 0)$$

y-intercept, $x = 0$

$$y = 0$$

$$(0, 0)$$

Now determine the critical points.

Analyze $f'(x)$.

$f'(x) = -9x^2 - 4x + 5$

Setting $f'(x) = 0$, we obtain

$$\begin{aligned} -9x^2 - 4x + 5 &= 0 \\ -(9x^2 + 4x - 5) &= 0 \\ -(9x - 5)(x + 1) &= 0 \end{aligned}$$

$$x = \frac{5}{9} \text{ or } x = -1$$

When we sketch the function, we can use approximate values $x = 0.6$ and $y = 1.6$ for $x = \frac{5}{9}$ and $f\left(\frac{5}{9}\right)$.

Analyze $f''(x)$.

$f''(x) = -18x - 4$

At $x = \frac{5}{9}$,

$$\begin{aligned} f''\left(\frac{5}{9}\right) &= -18\left(\frac{5}{9}\right) - 4 \\ &= -10 - 4 \\ &= -14 \\ &< 0 \end{aligned}$$

At $x = -1$,

$$\begin{aligned} f''(-1) &= -18(-1) - 4 \\ &= 18 - 4 \\ &= 14 \\ &> 0 \end{aligned}$$

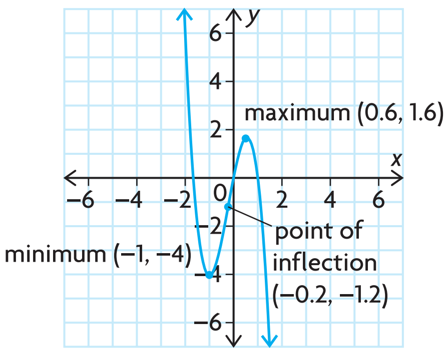
Therefore, by the second derivative test $x = \frac{5}{9}$ gives a local maximum and $x = -1$ gives a local minimum. Since this is a polynomial function, $f(x)$ must be decreasing when $x < -1$, increasing when $-1 < x < \frac{5}{9}$ and decreasing when $x > \frac{5}{9}$. For a point of inflection, $f''(x) = 0$ and changes sign.

$-18x - 4 = 0$ or $x = -\frac{2}{9}$

Now we determine the sign of $f''(x)$ in, the intervals determined by $x = -\frac{2}{9}$.
A point of inflection occurs at about $(-0.2, -1.2)$.

We can now draw our sketch.

Interval	$x < -\frac{2}{9}$	$x = -\frac{2}{9}$	$x > -\frac{2}{9}$
$f''(x)$	> 0	0	< 0
Graph of $f(x)$	concave up	point of inflection	concave down



EXAMPLE 2 Sketching an accurate graph of a rational function

Sketch the graph of $f(x) = \frac{x - 4}{x^2 - x - 2}$.

Solution

Analyze $f(x)$.

$f(x)$ is a rational function.

Determine any intercepts.

x-intercept, $y = 0$

$$\frac{x - 4}{x^2 - x - 2} = 0$$

$$x - 4 = 0$$

$$x = 4$$

$(4, 0)$

y-intercept, $x = 0$

$$y = \frac{0 - 4}{0 - 0 - 2}$$

$$y = \frac{-4}{-2}$$

$$y = 2$$

$(0, 2)$

Determine any asymptotes.

The function is not defined if

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \text{ and } x = -1$$

The domain is $\{x \in \mathbf{R} \mid x \neq 2 \text{ and } x \neq -1\}$.

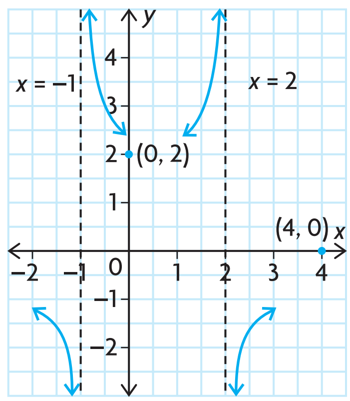
There are vertical asymptotes at $x = 2$ and $x = -1$.

Using $f(x) = \frac{x - 4}{x^2 - x - 2}$, we examine function values near the asymptotes.

$\lim_{x \to -1^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to -1^+} f(x) = +\infty$

$\lim_{x \to 2^-} f(x) = +\infty$ $\qquad$ $\lim_{x \to 2^+} f(x) = -\infty$

Sketch the information you have so far, as shown.



Analyze $f'(x)$. Now determine the critical points.

$$f(x) = (x - 4)(x^2 - x - 2)^{-1}$$
$$f'(x) = (1)(x^2 - x - 2)^{-1} + (x - 4)(-1)(x^2 - x - 2)^{-2}(2x - 1)$$
$$= \frac{1}{x^2 - x - 2} - \frac{(x - 4)(2x - 1)}{(x^2 - x - 2)^2}$$
$$= \frac{(x^2 - x - 2)}{(x^2 - x - 2)^2} - \frac{(2x^2 - 9x + 4)}{(x^2 - x - 2)^2}$$
$$= \frac{-x^2 + 8x - 6}{(x^2 - x - 2)^2}$$
$$f'(x) = 0 \text{ if } -x^2 + 8x - 6 = 0$$
$$x = \frac{8 \pm 2\sqrt{10}}{2}$$
$$x = 4 \pm \sqrt{10}$$

Since we are sketching, approximate values 7.2 and 0.8 are acceptable. These values give the approximate points (7.2, 0.1) and (0.8, 1.5).

Interval	$(-\infty, -1)$	$(-1, 0.8)$	$(0.8, 2)$	$(2, 7.2)$	$(7.2, \infty)$
$-x^2 + 8x - 6$	−	−	+	+	−
$(x^2 - x - 2)^2$	+	+	+	+	+
$f'(x)$	< 0	< 0	> 0	> 0	< 0
$f(x)$	decreasing	decreasing	increasing	increasing	decreasing

From the information obtained, we can see that $(7.2, 0.1)$ is likely a local maximum and $(0.8, 1.5)$ is likely a local minimum. To verify this using the second derivative test is a difficult computational task. Instead, verify using the first derivative test, as follows.
$x \doteq 0.8$ gives the local minimum. $x \doteq 7.2$ gives the local maximum.
Now check the end behaviour of the function.

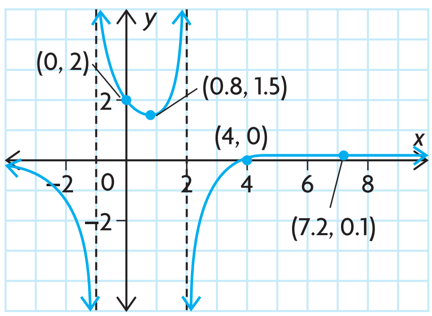
$\lim_{x \to +\infty} f(x) = 0$ but $y > 0$ always.

$\lim_{x \to -\infty} f(x) = 0$ but $y < 0$ always.

Therefore, $y = 0$ is a horizontal asymptote. The curve approaches from above on the right and below on the left.

There is a point of inflection beyond $x = 7.2$, since the curve opens down at that point but changes as x becomes larger. The amount of work necessary to

determine the point is greater than the information we gain, so we leave it undone. (If you wish to check it, it occurs for $x \doteq 10.4$.) The finished sketch is given below and, because it is a sketch, it is not to scale.



IN SUMMARY

Key Idea

- The first and second derivatives of a function give information about the shape of the graph of the function.

Need to Know

Sketching the Graph of a Polynomial or Rational Function

1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of y that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation! Use only the steps that are necessary to give you a good idea of what the graph will look like.

Exercise 4.5

PART A

1. If a polynomial function of degree three has a local minimum, explain how the function's values behave as $x \to +\infty$ and as $x \to -\infty$. Consider all cases.

C 2. How many local maximum and local minimum values are possible for a polynomial function of degree three, four, or n? Explain.

3. Determine whether each function has vertical asymptotes. If it does, state the equations of the asymptotes.

a. $y = \dfrac{x}{x^2 + 4x + 3}$ b. $y = \dfrac{5x - 4}{x^2 - 6x + 12}$ c. $y = \dfrac{3x + 2}{x^2 - 6x + 9}$

PART B

K 4. Use the algorithm for curve sketching to sketch the following:

a. $y = x^3 - 9x^2 + 15x + 30$

b. $f(x) = -4x^3 + 18x^2 + 3$

c. $y = 3 + \frac{1}{(x + 2)^2}$

d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

e. $y = \frac{2x}{x^2 - 25}$

f. $f(x) = \frac{1}{x^2 - 4x}$

g. $y = \frac{6x^2 - 2}{x^3}$

h. $f(x) = \frac{x + 3}{x^2 - 4}$

i. $y = \frac{x^2 - 3x + 6}{x - 1}$

j. $f(x) = (x - 4)^{\frac{2}{3}}$

5. Verify your results for question 4 using graphing technology.

A 6. Determine the constants a, b, c, and d so that the curve defined by $y = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(2, 4)$ and a point of inflection at the origin. Sketch the curve.

7. Given the following results of the analysis of a function, sketch a possible graph for the function:

a. $f(0) = 0$, the horizontal asymptote is $y = 2$, the vertical asymptote is $x = 3$, and $f'(x) < 0$ and $f''(x) < 0$ for $x < 3$; $f'(x) < 0$ and $f''(x) > 0$ for $x > 3$.

b. $f(0) = 6, f(-2) = 0$ the horizontal asymptote is $y = 7$, the vertical asymptote is $x = -4$, and $f'(x) > 0$ and $f''(x) > 0$ for $x < -4$; $f'(x) > 0$ and $f''(x) < 0$ for $x > -4$.

PART C

8. Sketch the graph of $f(x) = \frac{k - x}{k^2 + x^2}$, where k is any positive constant.

9. Sketch the curve defined by $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$.

10. Find the horizontal asymptotes for each of the following:

a. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

T 11. Show that, for any cubic function of the form $y = ax^3 + bx^2 + cx + d$, there is a single point of inflection, and the slope of the curve at that point is $c - \frac{b^2}{3a}$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 4: PREDICTING STOCK VALUES

In the Career Link earlier in the chapter, you investigated a graphical model used to predict stock values for a new stock. A brand new stock is also called an initial public offering, or IPO. Remember that, in this model, the period immediately after the stock is issued offers excess returns on the stock—that is, the stock is selling for more than it is really worth.

One such model for a class of Internet IPOs predicts the percent overvaluation of a stock as a function of time as $R(t) = 250\left(\frac{t^2}{(2.718)^{3t}}\right)$, where $R(t)$ is the overvaluation in percent and t is the time in months after the initial issue.

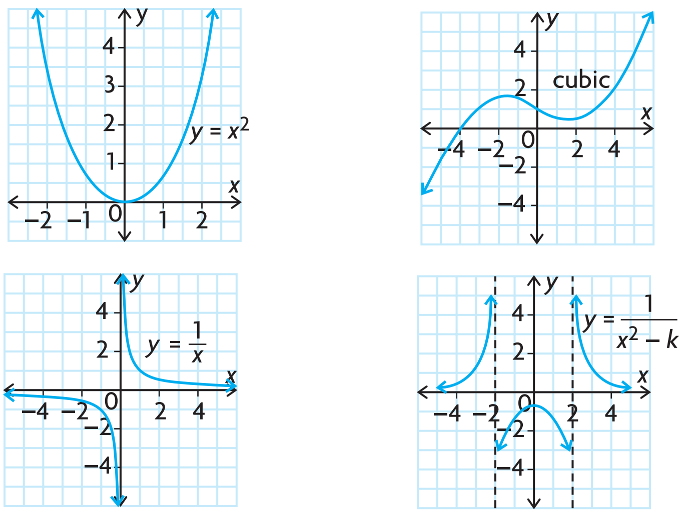
a. Use the information provided by the first derivative, second derivative, and asymptotes to prepare advice for clients as to when they should expect a signal to prepare to buy or sell (inflection point), the exact time when they should buy or sell (local maximum/minimum), and any false signals prior to a horizontal asymptote. Explain your reasoning.

b. Make a sketch of the function without using a graphing calculator.

Key Concepts Review

In this chapter, you saw that calculus can help you sketch graphs of polynomial and rational functions. Remember that concepts you learned in earlier studies are useful, and that calculus techniques help with sketching. Basic shapes should always be kept in mind. Use these, together with the algorithm for curve sketching, and always use your accumulated knowledge.

Basic Shapes to Remember



Sketching the Graph of a Polynomial or Rational Function

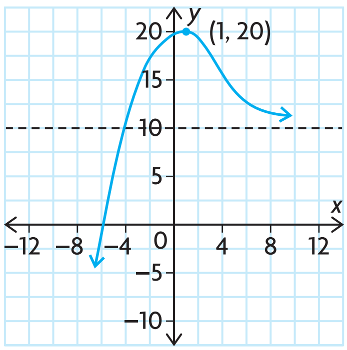
1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

   The second derivative can also be used to identify local maxima and minima.
4. Calculate the values of y that correspond to critical points and points of inflection. Use the information above to sketch the graph.

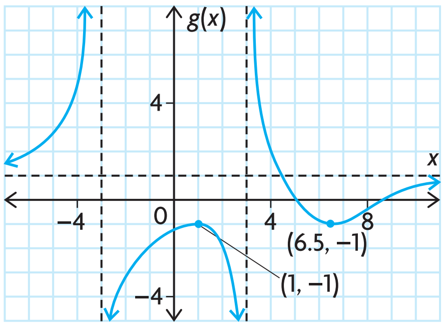
Review Exercise

1. For each of the following graphs, state
   i. the intervals where the function is increasing
   ii. the intervals where the function is decreasing
   iii. the points where the tangent to the function is horizontal

a.



b.



2. Is it always true that an increasing function is concave up in shape? Explain.

3. Determine the critical points for each function. Determine whether the critical point is a local maximum or local minimum and whether or not the tangent is parallel to the x-axis.

   a. $f(x) = -2x^3 + 9x^2 + 20$

   b. $f(x) = x^4 - 8x^3 + 18x^2 + 6$

   c. $h(x) = \dfrac{x - 3}{x^2 + 7}$

   d. $g(x) = (x - 1)^{\frac{1}{3}}$

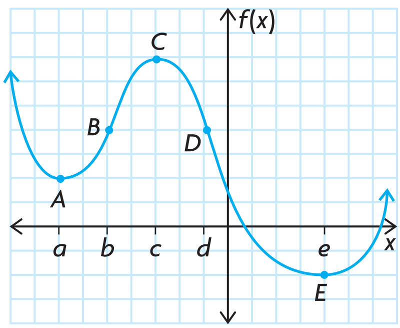
4. The graph of the function $y = f(x)$ has local extrema at points A, C, and E and points of inflection at B and D. If a, b, c, d, and e are the x-coordinates of the points, state the intervals on which the following conditions are true:

   a. $f'(x) > 0$ and $f''(x) > 0$

   b. $f'(x) > 0$ and $f''(x) < 0$

   c. $f'(x) < 0$ and $f''(x) > 0$

   d. $f'(x) < 0$ and $f''(x) < 0$



5. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \dfrac{2x}{x - 3}$

b. $g(x) = \dfrac{x - 5}{x + 5}$

c. $f(x) = \dfrac{x^2 - 2x - 15}{x + 3}$

d. $g(x) = \dfrac{5}{x^2 - x - 20}$

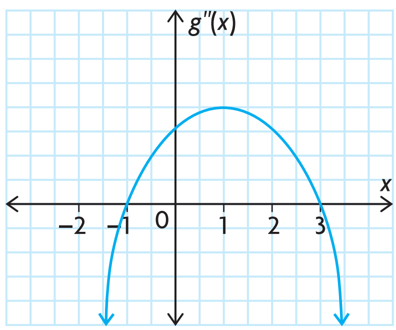
6. Determine the point of inflection on the curve defined by $y = x^3 + 5$. Show that the tangent line at this point crosses the curve.

7. Sketch a graph of a function that is differentiable on the interval $-3 \le x \le 5$ and satisfies the following conditions:

- There are local maxima at $(-2, 10)$ and $(3, 4)$.
- The function f is decreasing on the intervals $-2 < x < 1$ and $3 \le x \le 5$.
- The derivative $f'(x)$ is positive for $-3 \le x < -2$ and for $1 < x < 3$.
- $f(1) = -6$

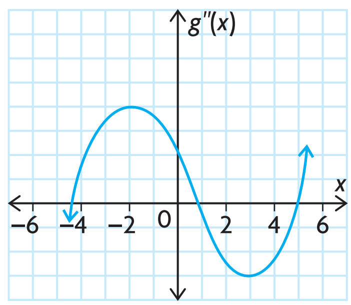
8. Each of the following graphs represents the second derivative, $g''(x)$, of a function $g(x)$:

a.



$g''(x)$ is a quadratic function.

b.



$g''(x)$ is a cubic function.

i. On what intervals is the graph of $g(x)$ concave up? On what intervals is the graph concave down?

ii. List the x-coordinates of the points of inflection.

iii. Make a rough sketch of a possible graph for $g(x)$, assuming that $g(0) = -3$.

9. a. If the graph of the function $g(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent at point $(2, -1)$, determine the values of a and b.
   b. Sketch the function g.

10. Sketch each function using suitable techniques.
   a. $y = x^4 - 8x^2 + 7$
   b. $f(x) = \dfrac{3x - 1}{x + 1}$
   c. $g(x) = \dfrac{x^2 + 1}{4x^2 - 9}$
   d. $y = x(x - 4)^3$
   e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$
   f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$

11. a. Determine the conditions on parameter k such that the function $f(x) = \frac{2x + 4}{x^2 - k^2}$ will have critical points.
   b. Select a value for k that satisfies the constraint established in part a, and sketch the section of the curve that lies in the domain $|x| \leq k$.

12. Determine the equation of the oblique asymptote in the form $y = mx + b$ for each function, and then show that $\lim\limits_{x \to +\infty} [y - f(x)] = 0$.
   a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$
   b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

13. Determine the critical numbers and the intervals on which $g(x) = (x^2 - 4)^2$ is increasing or decreasing.

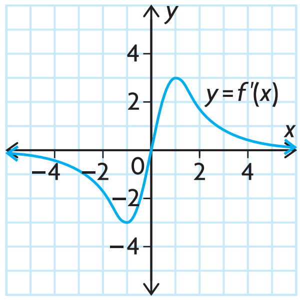
14. Use the second derivative test to identify all maximum and minimum values of $f(x) = x^3 + \frac{3}{2}x^2 - 7x + 5$ on the interval $-4 \leq x \leq 3$.

15. Use the y-intercept, local extrema, intervals of concavity, and points of inflection to graph $f(x) = 4x^3 + 6x^2 - 24x - 2$.

16. Let $p(x) = \frac{3x^3 - 5}{4x^2 + 1}$, $q(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $r(x) = \frac{x^2 - 2x - 8}{x^2 - 1}$, and $s(x) = \frac{x^3 + 2x}{x - 2}$.
   a. Determine the asymptotes for each function, and identify their type (vertical, horizontal, or oblique).
   b. Graph $y = r(x)$, showing clearly the asymptotes and the intercepts.

17. If $f(x) = \frac{x^3 + 8}{x}$, determine the domain, intercepts, asymptotes, intervals of increase and decrease, and concavity. Locate any critical points and points of inflection. Use this information to sketch the graph of $f(x)$.

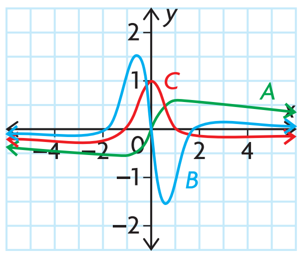
18. Explain how you can use this graph of $y = f'(x)$ to sketch a possible graph of the original function, $y = f(x)$.



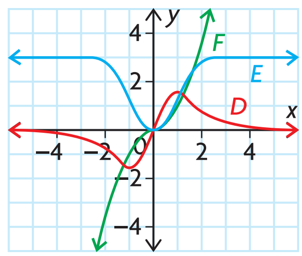
19. For $f(x) = \frac{5x}{(x-1)^2}$, show that $f'(x) = \frac{-5(x+1)}{(x-1)^3}$ and $f''(x) = \frac{100(x+2)}{(x-1)^4}$. Use the function and its derivatives to determine the domain, intercepts, asymptotes, intervals of increase and decrease, and concavity, and to locate any local extrema and points of inflection. Use this information to sketch the graph of f.

20. The graphs of a function and its derivatives, $y = f(x)$, $y = f'(x)$, and $y = f''(x)$, are shown on each pair of axes. Which is which? Explain how you can tell.

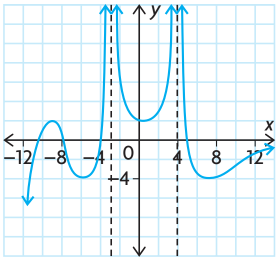
a.



b.



Chapter 4 Test



1. The graph of function $y = f(x)$ is shown at the left.
   a. Estimate the intervals where the function is increasing.
   b. Estimate the intervals where $f'(x) < 0$.
   c. Estimate the coordinates of the critical points.
   d. Estimate the equations of any vertical asymptotes.
   e. What is the value of $f''(x)$ on the interval $-4 < x < 4$?
   f. If $x \geq -6$, estimate the intervals where $f'(x) < 0$ and $f''(x) > 0$.
   g. Identify a point of inflection, and state the approximate ordered pair for the point.
2. a. Determine the critical points of the function $g(x) = 2x^4 - 8x^3 - x^2 + 6x$.
   b. Classify each critical point in part a.
3. Sketch the graph of a function with the following properties:
   - There are local extrema at $(-1, 7)$ and $(3, 2)$.
   - There is a point of inflection at $(1, 4)$.
   - The graph is concave down only when $x < 1$.
   - The x-intercept is -4, and the y-intercept is 6.
4. Check the function $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$ for discontinuities. Conduct appropriate tests to determine if asymptotes exist at the discontinuity values. State the equations of any asymptotes and the domain of $g(x)$.
5. Sketch a graph of a function f with all of the following properties:
   - The graph is increasing when $x < -2$ and when $-2 < x < 4$.
   - The graph is decreasing when $x > 4$.
   - $f'(-2) = 0, f'(4) = 0$
   - The graph is concave down when $x < -2$ and when $3 < x < 9$.
   - The graph is concave up when $-2 < x < 3$ and when $x > 9$.
6. Use at least five curve-sketching techniques to explain how to sketch the graph of the function $f(x) = \dfrac{2x + 10}{x^2 - 9}$. Sketch the graph on graph paper.
7. The function $f(x) = x^3 + bx^2 + c$ has a critical point at $(-2, 6)$.
   a. Find the constants b and c.
   b. Sketch the graph of $f(x)$ using only the critical points and the second derivative test.

Review Exercise, pp. 156–159

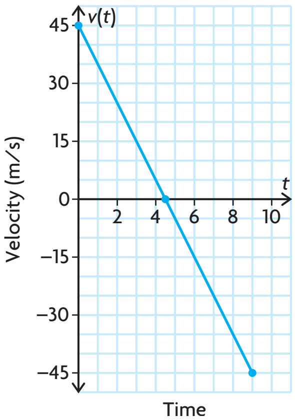
1. $f'(x) = 4x^3 + 4x^{-5}$,
$f''(x) = 12x^2 - 20x^{-6}$

2. $\frac{d^2y}{dx^2} = 72x^7 - 42x$

3. $v = 2t + (2t - 3)^{\frac{1}{2}}$,
$a = 2 - (2t - 3)^{\frac{3}{2}}$

4. $v(t) = 1 - 5t^{-2}$,
$a(t) = 10t^{-3}$

5. The upward velocity is positive for $0 \le t \le 4.5$ s, zero for $t = 4.5$ s, and negative for $t > 4.5$ s.



6. **a.** min: −52,
max: 0
b. min: −65,
max: 16
c. min: 12,
max: 20

7. **a.** 62 m
b. Yes, 2 m beyond the stop sign
c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.

8. min is 2, max is $2 + 3\sqrt{3}$

9. 250

10. **a.** **i.** \$2200
ii. \$5.50
iii. \$3.00; \$3.00
b. **i.** \$24 640
ii. \$61.60
iii. \$43.20; \$43.21
c. **i.** \$5020
ii. \$12.55
iii. \$0.03; \$0.03
d. **i.** \$2705
ii. \$6.88
iii. \$5.01; \$5.01

11. 2000

12. **a.** moving away from its starting point
b. moving away from the origin and towards its starting position

13. **a.** $t = \frac{2}{3}$ **b.** yes

14. 27.14 cm by 27.14 cm for the base and height 13.57 cm

15. length 190 m, width approximately 63 m

16. 31.6 cm by 11.6 cm by 4.2 cm

17. radius 4.3 cm, height 8.6 cm

18. Run the pipe 7.2 km along the river shore and then cross diagonally to the refinery.

19. 10:35 p.m.

20. \$204 or \$206

21. The pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.

22. 11.35 cm by 17.02 cm

23. 34.4 m by 29.1 m

24. 2:23 p.m.

25. 3.2 km from point C

26. **a.** absolute maximum: $f(7) = 41$,
absolute minimum: $f(1) = 5$
b. absolute maximum: $f(3) = 36$,
absolute minimum: $f(-3) = -18$
c. absolute maximum: $f(5) = 67$,
absolute minimum: $f(-5) = -63$
d. absolute maximum: $f(4) = 2752$,
absolute minimum: $f(-2) = -56$

27. **a.** 62.9 m **c.** 3.6 m/s^2
b. 4.7 s

28. **a.** $f''(2) = 60$ **d.** $f''(1) = -\frac{5}{16}$
b. $f''(-1) = 26$ **e.** $f''(4) = -\frac{1}{108}$
c. $f''(0) = 192$ **f.** $f''(8) = -\frac{1}{72}$

29. **a.** position: 1, velocity: $\frac{1}{6}$,
acceleration: $-\left(\frac{1}{18}\right)$, speed: $\frac{1}{6}$
b. position: $\frac{8}{3}$, velocity: $\frac{4}{9}$,
acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$

30. **a.** $v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$,
$a(t) = \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$
b. 1.931 m/s
c. 2.36 m/s
d. undefined
e. 0.141 m/s^2

Chapter 3 Test, p. 160

1. **a.** $y'' = 14$
b. $f''(x) = -180x^3 - 24x$
c. $y'' = 60x^{-5} + 60x$
d. $f''(x) = 96(4x - 8)$

2. **a.** $v(3) = -57$,
$a(3) = -44$
b. $v(2) = 6$,
$a(2) = -24$

3. **a.** $v(t) = 2t - 3$,
$a(t) = 2$
b. −0.25 m
c. 1 m/s, 1 m/s
d. between $t = 0$ s and $t = 1.5$ s
e. 2 m/s^2

4. **a.** min: −63, max: 67
b. min: 7.5, max: 10

5. **a.** 2.1 s
b. about 22.9

6. 250 m by 166.7 m

7. 162 mm by 324 mm by 190 mm

8. \$850/month

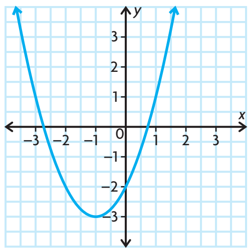
Chapter 4

Review of Prerequisite Skills, pp. 162–163

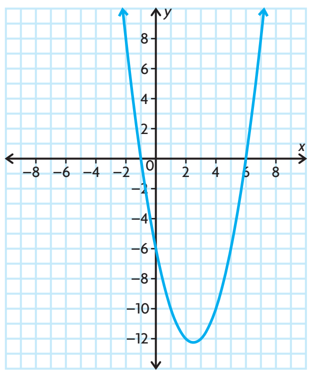
1. **a.** $y = -\frac{3}{2}$ or $y = 1$
b. $x = 7$ or $x = -2$
c. $x = -\frac{5}{2}$
d. $y = 1$ or $y = -3$ or $y = -2$

2. **a.** $x < -\frac{7}{3}$
b. $x \le 2$
c. $-1 < t < 3$
d. $x < -4$ or $x > 1$

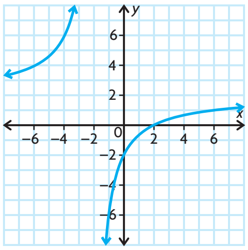
3. **a.**



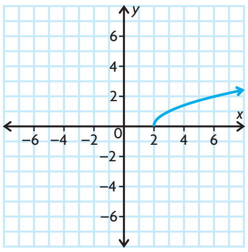
b.



c.



d.



4. **a.** 0
b. 7
c. 27
d. 3

5. **a.** $x^3 + 6x + x^{-2}$
b. $-\dfrac{x^2 + 2x + 3}{(x^2 - 3)^2}$
c. $2(3x^2 - 6x)(6x - 6)$
d. $\dfrac{t - 8}{(t - 4)^{\frac{3}{2}}}$

6. **a.** $x - 8 + \dfrac{28}{x + 3}$
b. $x + 7 - \dfrac{2}{x - 1}$

7. $\left(\dfrac{2}{3}, 2.19\right), (-1, 4.5)$

8. **a.** If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
b. If $f(x) = k$, where k is a constant, then $f'(x) = 0$.
c. If $k(x) = f(x)g(x)$, then $k'(x) = f'(x)g(x) - f(x)g'(x)$
d. If $h(x) = \dfrac{f(x)}{g(x)}$, then
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2},$$
$g(x) \neq 0$.
e. If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x)) \cdot g'(x)$.
f. If u is a function of x, and n is a positive integer, then $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$.

9. **a.** As $x \to \pm\infty, f(x) \to \infty$.
b. As $x \to -\infty, f(x) \to -\infty$.
As $x \to \infty, f(x) \to \infty$.
c. As $x \to -\infty, f(x) \to -\infty$.
As $x \to \infty, f(x) \to -\infty$.

10. **a.** $\dfrac{1}{2x}$; $x = 0$
b. $\dfrac{1}{-x + 3}$; $x = 3$
c. $\dfrac{1}{(x + 4)^2 + 1}$; no vertical asymptote
d. $\dfrac{1}{(x + 3)^2}$; $x = -3$

11. **a.** $y = 0$
b. $y = 4$
c. $y = \dfrac{1}{2}$
d. $y = 2$

12. **a.** **i.** no x-intercept; (0, 5)
ii. (0, 0); (0, 0)
iii. $\left(\dfrac{5}{3}, 0\right)$; $\left(0, \dfrac{5}{3}\right)$
iv. $\left(\dfrac{2}{5}, 0\right)$; no y-intercept
b. **i.** Domain: $\{x \in \mathbf{R} \mid x \neq -1\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 0\}$
ii. Domain: $\{x \in \mathbf{R} \mid x \neq 2\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 4\}$
iii. Domain: $\left\{x \in \mathbf{R} \mid x \neq \dfrac{1}{2}\right\}$,
Range: $\left\{y \in \mathbf{R} \mid y \neq \dfrac{1}{2}\right\}$
iv. Domain: $\{x \in \mathbf{R} \mid x \neq 0\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 2\}$

Section 4.1, pp. 169–171

1. **a.** (0, 1), (−4, 33)
b. (0, 2)
c. $\left(\dfrac{1}{2}, 0\right)$, (2.25, −48.2), (−2, −125)
d. $\left(1, \dfrac{5}{2}\right), \left(-1, -\dfrac{5}{2}\right)$

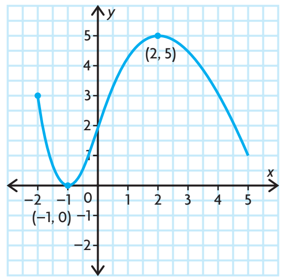
2. A function is increasing when $f'(x) > 0$ and is decreasing when $f'(x) < 0$.

3. **a.** **i.** $x < -1, x > 2$
ii. $-1 < x < 2$
iii. (−1, 4), (2, −1)
b. **i.** $-1 < x < 1$
ii. $x < -1, x > 1$
iii. (−1, 2), (2, 4)
c. **i.** $x < -2$
ii. $-2 < x < 2, 2 < x$
iii. none
d. **i.** $-1 < x < 2, 3 < x$
ii. $x < -1, 2 < x < 3$
iii. (2, 3)

4. **a.** increasing: $x < -2, x > 0$;
decreasing: $-2 < x < 0$
b. increasing: $x < 0, x > 4$;
decreasing: $0 < x < 4$
c. increasing: $x < -1, x > 1$;
decreasing: $-1 < x < 0$,
$0 < x < 1$
d. increasing: $-1 < x < 3$;
decreasing: $x < -1, x > 3$
e. increasing: $-2 < x < 0, x > 1$;
decreasing: $x < -2, 0 < x < 1$
f. increasing: $x > 0$;
decreasing: $x < 0$

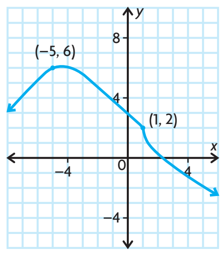
5. increasing: $-3 < x < -2, x > 1$;
decreasing: $x < -3, -2 < x < 1$

6.

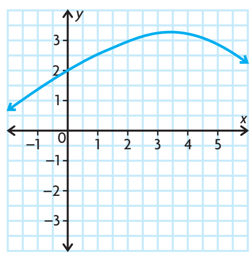


7. $a = 3, b = -9, c = -9$

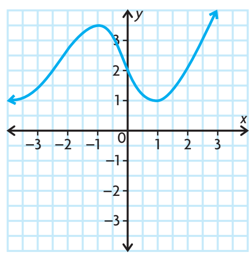
8.



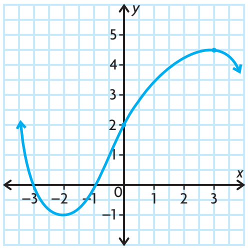
9. **a.** **i.** $x < 4$
ii. $x > 4$
iii. $x = 4$



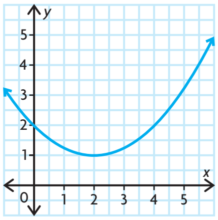
b. **i.** $x < -1, x > 1$
ii. $-1 < x < 1$
iii. $x = -1, x = 1$



c. **i.** $-2 < x < 3$
ii. $x < -2, x > 3$
iii. $x = -2, x = 3$



d. **i.** $x > 2$
ii. $x < 2$
iii. $x = 2$



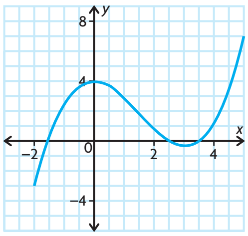
10. $f(x) = ax^2 + bx + c$
$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

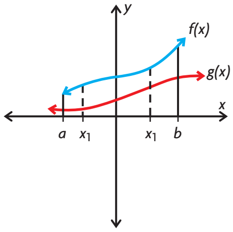
If $x < \frac{-b}{2a}, f'(x) < 0$, therefore the function is decreasing.
If $x > \frac{-b}{2a}, f'(x) > 0$, therefore the function is increasing.

11. $f'(x) = 0$ for $x = 2$,
increasing: $x > 2$,
decreasing: $x < 2$,
local minimum: $(2, -44)$

12.



13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \le x \le b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:
$f(x_2) > f(x_1)$ (1)
$g(x_2) > g(x_1)$ (2)
$yu = f(x) \cdot g(x)$
(1) × (2) results in
$f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$
The function yu or $f(x) \cdot g(x)$ is strictly increasing.

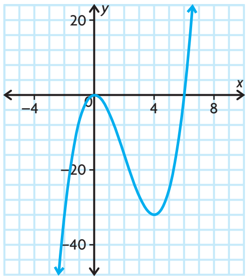


14. strictly decreasing

Section 4.2, pp. 178–180

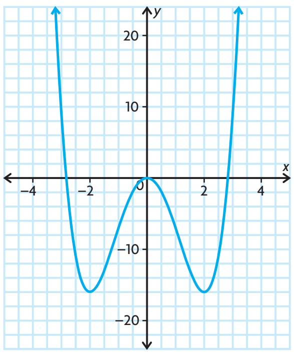
1. Determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0

2. **a.** Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.
b. $(0, 0), (4, -32)$

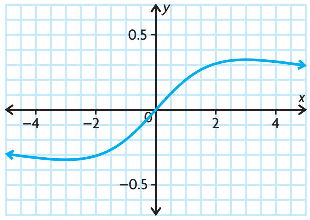


3. **a.** local minima: $(-2, -16), (2, -16)$,
local maximum: $(0, 0)$
b. local minimum: $(-3, -0.3)$,
local maximum: $(3, 0.3)$
c. local minimum: $(-2, 5)$,
local maximum: $(0, 1)$

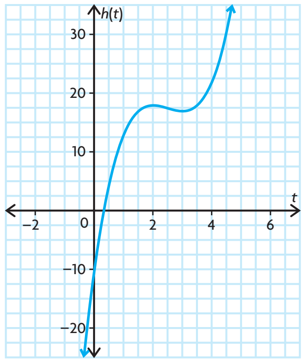
4. **a.** $(0, 0), (2\sqrt{2}, 0), (-2\sqrt{2}, 0)$



b. $(0, 0)$

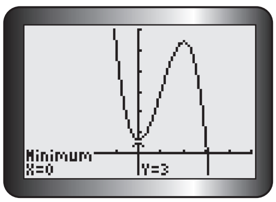


c. $(-3, 1, 0), (0, 1)$

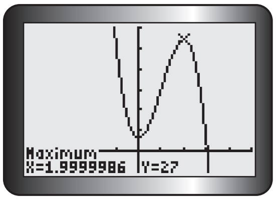


5. **a.** local minimum: $(0, 3)$,
local maximum: $(2, 27)$,
Tangent is parallel to the horizontal axis for both.
b. $(0, 0)$ neither maximum nor minimum,
Tangent is parallel to the horizontal axis.
c. $(5, 0)$; neither maximum nor minimum,
Tangent is not parallel to the horizontal axis.
d. local minimum: $(0, -1)$,
Tangent is parallel to the horizontal axis.
$(-1, 0)$ and $(1, 0)$ are neither maxima or minima.
Tangent is not parallel to the horizontal axis for either.

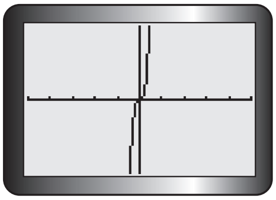
6. **a.**



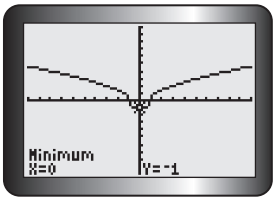
b.



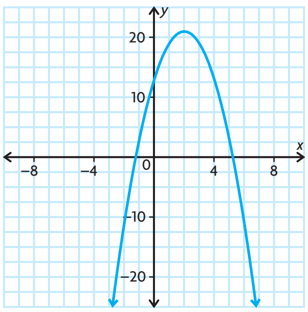
c.



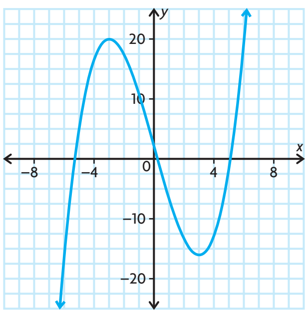
d.



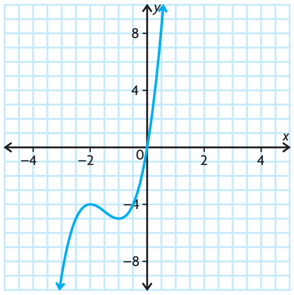
7. **a.** (2, 21) local maximum



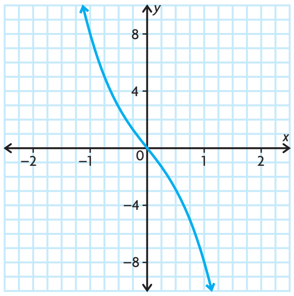
b. $(-3, 20)$ local maximum, $(3, -16)$ local minimum



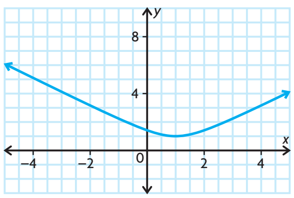
c. $(-2, -4)$ local maximum, $(-1, -5)$ local minimum



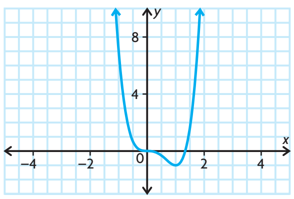
d. no critical points



e. (1, 1) local minimum

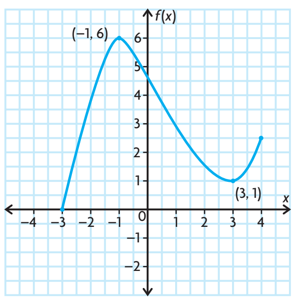


f. (0, 0) neither maximum nor minimum, $(1, -1)$ local minimum



8. local minima at $x = -6$ and $x = 2$; local maximum at $x = -1$

9.



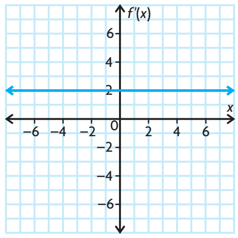
10. $a = -\frac{11}{9}, b = \frac{22}{3}, c = 1$

11. $p = -2, q = 6$
minimum; the derivative is negative to the left and positive to the right

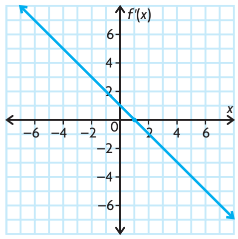
12. **a.** $k < 0$
b. $k = 0$
c. $k > 0$

13. $a = -1, b = 3, c = 0, d = 0$

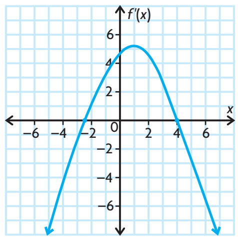
14. **a.**



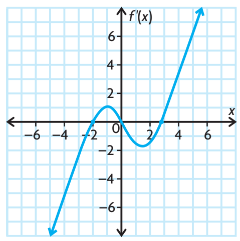
b.



c.

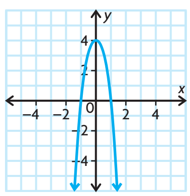


d.

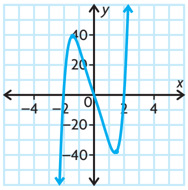


15. **a.** $a = -4, b = -36, c = 0$
b. $(3, -198)$
c. local minimum: $(-2, -73)$ and $(3, -198)$, local maximum: $(0, -9)$

16. **a.** local maximum: (0, 4)



b. local minimum: (1.41, −39.6), local maximum: (1.41, 39.6)



17. $h(x) = \frac{f(x)}{g(x)}$

Since $f(x)$ has a local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If $x < c, f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c, f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c, h'(x) > 0$ and for $x > c, h'(x) < 0$. Therefore, $h(x)$ has a local maximum at $x = c$.

Section 4.3, pp. 193–195

1. **a.** vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$
b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$

2. $f(x) = \frac{g(x)}{h(x)}$

Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim_{x \to \infty} f(x) = \infty$.

Conditions for a horizontal asymptote: $\lim_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$, or $\lim_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$.

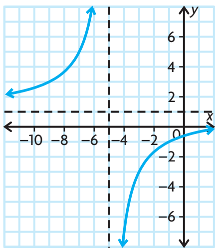
Condition for an oblique asymptote: The highest power of $g(x)$ must be one more than the highest power of $k(x)$.

3. **a.** 2
b. 5
c. $-\frac{5}{2}$
d. ∞

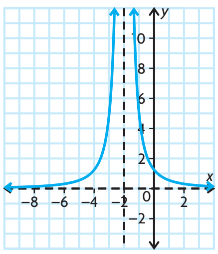
4. **a.** $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote
b. $x = 2$; large and negative to left of asymptote, large and positive to right of asymptote
c. $x = 3$; large and positive to left of asymptote, large and positive to right of asymptote
d. hole at $x = 3$, no vertical asymptote
e. $x = -3$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 1$; large and negative to left of asymptote, large and positive to right of asymptote
f. $x = -1$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 1$; large and negative to left of asymptote, large and positive to right of asymptote

5. **a.** $y = 1$; large negative: approaches from above, large positive: approaches from below
b. $y = 0$; large negative: approaches from below, large positive: approaches from above
c. $y = 3$; large negative: approaches from above, large positive: approaches from above
d. no horizontal asymptotes

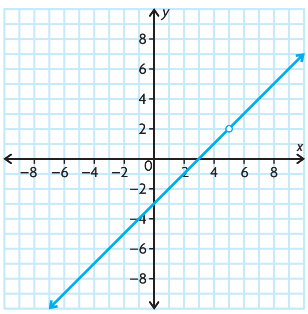
6. **a.**



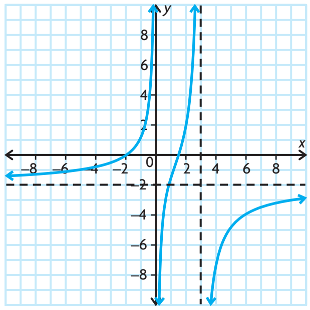
b.



c.



d.

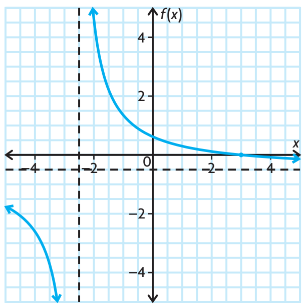


7. **a.** $y = 3x - 7$
b. $y = x + 3$
c. $y = x - 2$
d. $y = x + 3$

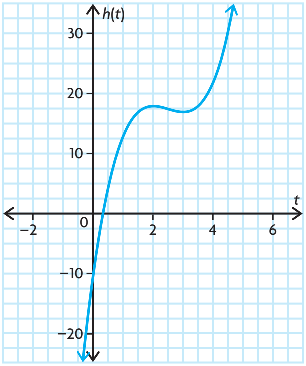
8. **a.** large negative: approaches from below, large positive: approaches from above
b. large negative: approaches from above, large positive: approaches from below

9. **a.** $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote
$y = 3$
b. $x = 1$; large and positive to left of asymptote, large and positive to right of asymptote
$y = 1$
c. $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote
$y = 1$
d. $x = 2$; large and negative to left of asymptote, large and positive to right of asymptote
$y = 1$

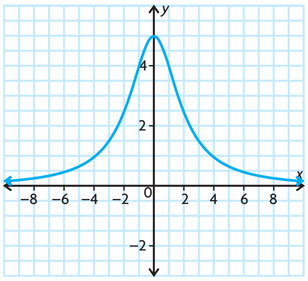
10. **a.**



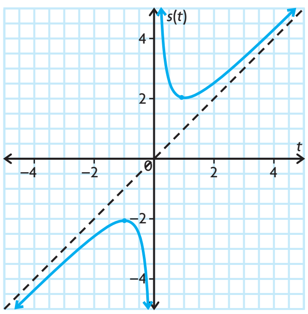
b.



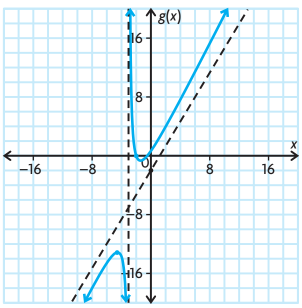
c.



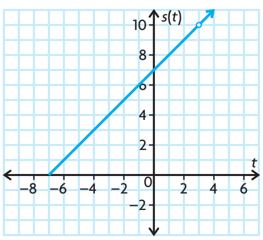
d.



e.



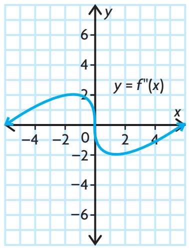
f.



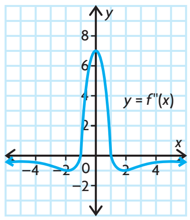
11. **a.** $y = \dfrac{a}{c}$

b. $x = -\dfrac{d}{c}$

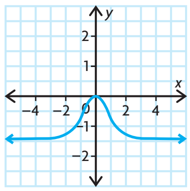
12. **a.**



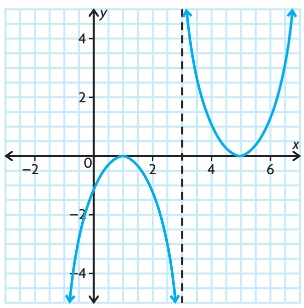
b.



13. **a.**



b.



14. **a.** $f(x)$ and $r(x)$: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} r(x)$ exist.

b. $h(x)$: the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.

c. $h(x)$: the denominator is defined for all $x \in \mathbf{R}$.

$f(x) = \dfrac{-x-3}{(x-7)(x+2)}$ has vertical Asymptotes at $x = 7$ and $x = -2$.

As $x \to -2^-, f(x) \to -\infty$.

As $x \to -2^+, f(x) \to \infty$.

As $x \to 7^-, f(x) \to \infty$.

As $x \to 7^+, f(x) \to -\infty$.

$f(x)$ has a horizontal asymptote at $y = 0$.

$g(x)$ has a vertical asymptote at $x = 3$.

As $x \to 3^-, g(x) \to \infty$.

As $x \to 3^+, g(x) \to -\infty$.

$y = x$ is an oblique asymptote for $h(x)$.

$r(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+4)}$ has vertical asymptotes at $x = -4$ and $x = 4$.

As $x \to -4^-, r(x) \to \infty$.

As $x \to -4^+, r(x) \to -\infty$.

As $x \to 4^-, r(x) \to -\infty$.

As $x \to 4^+, r(x) \to \infty$.

$r(x)$ has a horizontal asymptote at $y = 1$.

15. $a = \dfrac{9}{5}, b = \dfrac{3}{5}$

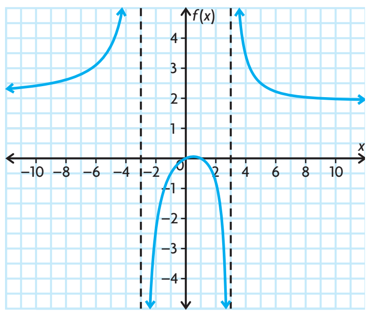
16. **a.** $\lim_{x\to\infty} \dfrac{x^2+1}{x+1} = \lim_{x\to\infty} \dfrac{x + \frac{1}{x}}{1 + \frac{1}{x}}$

$= \infty$

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x+1}$$
$$= \lim_{x\to\infty} \frac{(x+1)(x+1)}{(x+1)}$$
$$= \lim_{x\to\infty} (x+1)$$
$$= \infty$$

b. $$\lim_{x\to\infty} \left[\frac{x^2+1}{x+1} - \frac{x^2+2x+1}{x+1}\right]$$
$$= \lim_{x\to\infty} \frac{x^2+1-x^2-2x-1}{x+1}$$
$$= \lim_{x\to\infty} \frac{-2x}{x+1}$$
$$= \lim_{x\to\infty} \frac{-2}{1} = -2$$

17.



Mid-Chapter Review, pp. 196–197

1. **a.** decreasing: $(-\infty, 2)$, increasing: $(2, \infty)$

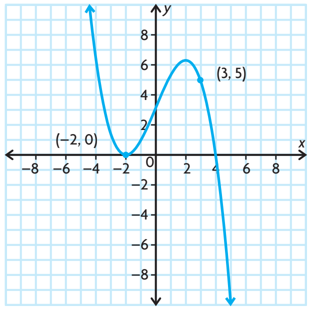
b. decreasing: $(0, 2)$, increasing: $(-\infty, 0), (2, \infty)$

c. increasing: $(-\infty, -3), (3, \infty)$

d. decreasing: $(-\infty, 0)$, increasing: $(0, \infty)$

2. increasing: $x < -1$ and $x > 2$, decreasing: $-1 < x < 2$

3.



4. a. -4
b. $0, \pm 3\sqrt{3}$
c. $0, \pm\sqrt{2}$
d. $\pm 1, \pm 2$
e. 0
f. $\pm\sqrt{2}$

5. a. $x = 1$ local maximum, $x = 2$ local minimum
b. $x = -\frac{2}{3}$ local maximum, $x = 2$ local minimum

6. ± 2

7. $x = 2$ local minimum, increasing: $x > 2$, decreasing: $x < 2$

8. a. $x = -2$; large and positive to left of asymptote, large and negative to right of asymptote
b. $x = -3$; large and negative to left of asymptote, large and positive to right of asymptote
$x = 3$; large and positive to left of asymptote, large and negative to right of asymptote
c. $x = -3$; large and negative to left of asymptote, large and positive to right of asymptote
d. $x = -\frac{2}{3}$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 5$; large and positive to left of asymptote, large and negative to right of asymptote

9. a. $y = 3$; large negative: approaches from above, large positive: approaches from below
b. $y = 1$; large negative: approaches from below, large positive: approaches from above

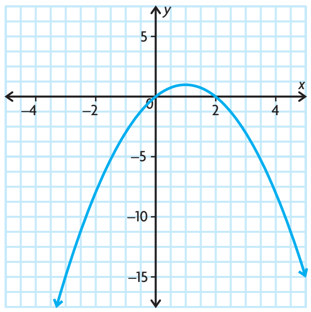
10. a. $x = 5$; large and positive to left of asymptote, large and positive to right of asymptote
b. no discontinuities
c. $x = 6 + 2\sqrt{6}$; large and negative to left of asymptote, large and positive to right of asymptote
$x = 6 - 2\sqrt{6}$; large and negative to left of asymptote, large and positive to right of asymptote

11. a. $f(x)$ is increasing.
b. $f(x)$ is decreasing.

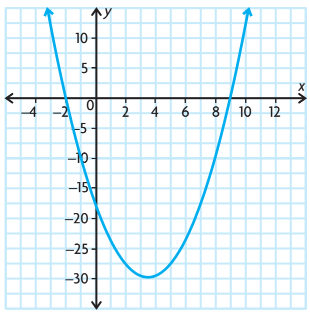
12. increasing: $0 < t < 0.97$, decreasing: $t > 0.97$

13. increasing: $t > 2.5198$, decreasing: $t < 2.5198$

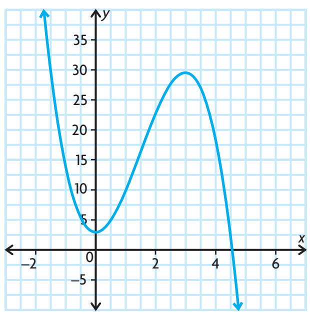
14.



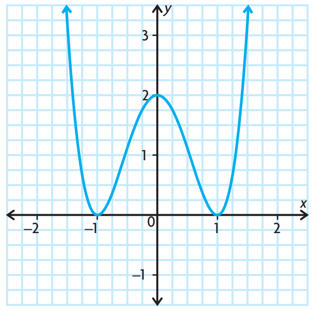
15. a. i. $x = \frac{7}{2}$
ii. increasing: $x > \frac{7}{2}$, decreasing: $x < \frac{7}{2}$
iii. local minimum at $x = \frac{7}{2}$
iv.



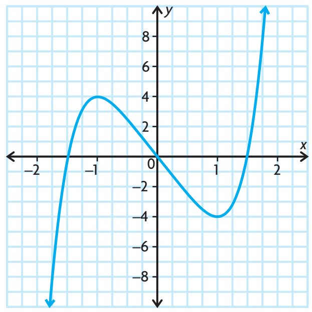
b. i. $x = 0, x = 3$
ii. increasing: $0 < x < 3$, decreasing: $x < 0, x > 3$
iii. local minimum at $x = 0$, local maximum at $x = 3$
iv.



c. i. $x = -1, x = 0, x = 1$
ii. increasing: $-1 < x < 0, x > 1$; decreasing: $x < -1, 0 < x < 1$
iii. local maximum at $x = 0$, local minimum at $x = 1, -1$
iv.



d. i. $x = -1, x = 1$
ii. increasing: $x < -1, x > 1$; decreasing: $-1 < x < 1$
iii. local maximum at $x = -1$, local minimum at $x = 1$
iv.



16. a. $x = -\frac{1}{2}$; large and positive to left of asymptote, large and negative to right of asymptote; $y = \frac{1}{2}$
b. $x = -2$; large and positive to left of asymptote, large and positive to right of asymptote; $y = 1$
c. $x = -3$; large and positive to left of asymptote, large and negative to right of asymptote; $y = -1$
d. $x = -4$; large and negative to left of asymptote, large and positive to right of asymptote; $y = 2$

17. a. $-\frac{2}{3}$
b. $\frac{1}{6}$
c. -3
d. 0
e. ∞
f. 1
g. 1
h. ∞

Section 4.4, pp. 205–206

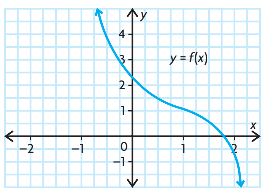
1. **a.** A: negative, B: negative, C: positive, D: positive
b. A: negative, B: negative, C: positive, D: negative

2. **a.** local minimum: $(5, -105)$, local maximum: $(-1, 20)$
b. local maximum: $\left(0, \frac{25}{48}\right)$
c. local maximum: $(-1, -2)$, local minimum: $(1, 2)$
d. $(3, 8)$ is neither a local maximum or minimum.

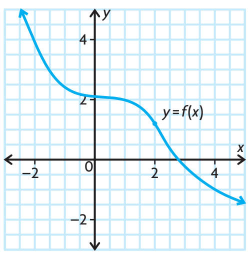
3. **a.** $\left(\frac{4}{3}, -14\frac{20}{27}\right)$
b. $\left(-4, \frac{25}{64}\right)\left(4, \frac{25}{64}\right)$
c. no points of inflection
d. $(3, 8)$

4. **a.** 24; above
b. 4; above
c. $-\frac{9}{100\sqrt{10}}$; below
d. $-\frac{2}{27}$; below

5. **a.** **i.** concave up on $x < 1$, concave down on $x > 1$
ii. $x = 1$
iii.



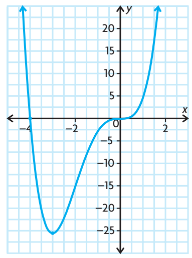
b. **i.** concave up on $x < 0$ or $x > 2$, concave down on $0 < x < 2$
ii. $x = 0$ and $x = 2$
iii.



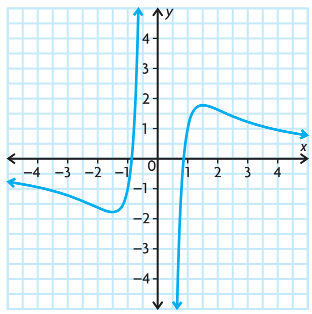
6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value. If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.

7. Use the first derivative test or the second derivative test to determine the type of critical points that may be present.

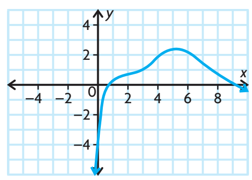
8. **a.** **i.** $(-2, -16)$, $(0, 0)$
ii.



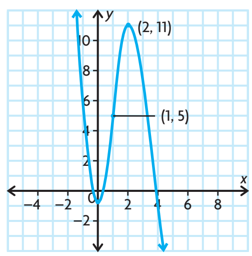
b. **i.** $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right), \left(\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9}\right)$
ii.



9.



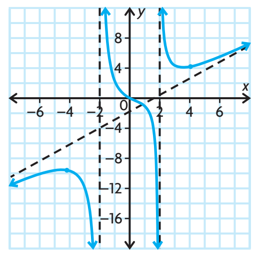
10. $a = -3$, $b = 9$, $c = -1$



11. $\frac{27}{64}$

12. $f(x) = ax^4 + bx^3$
$f'(x) = 4ax^3 + 3bx^2$
$f''(x) = 12ax^2 + 6bx$
For possible points of inflection, we solve $f''(x) = 0$:
$12ax^2 + 6bx = 0$
$6x(2ax + b) = 0$
$x = 0$ or $x = -\frac{b}{2a}$.
The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.
We know the values of $f''(x)$ have opposite signs when passing through a root. Thus, at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points.
Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$.
To find the x-intercepts, we solve $f(x) = 0$
$x^3(ax + b) = 0$
$x = 0$ or $x = -\frac{b}{a}$
The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.
The points of inflection are $(0, 0)$ and $\left(-\frac{b}{2a}, -\frac{b^4}{16a^3}\right)$.

13. **a.**



b. Answers may vary. For example, there is a section of the graph that lies between the two sections of the graph that approaches the asymptote.

14. $n = 1$, $n = 2$: no inflection points; $n = 3$, $n = 4$: inflection point at $x = c$; The graph of f has an inflection point at $x = c$ when $n \geq 3$.

Section 4.5, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

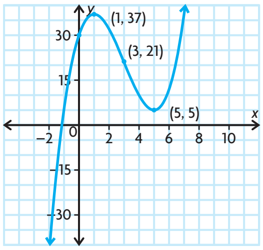
2. A polynomial of degree three has at most two local extremes. A polynomial of degree four has at most three local extremes. Since each local maximum and minimum of a function corresponds to a zero of its derivative, the number of zeros of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeros, and thus at most $n - 1$ local extremes.

3. **a.** $x = -3$ or $x = -1$

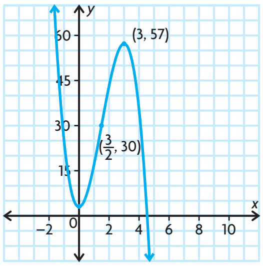
b. no vertical asymptotes

c. $x = 3$

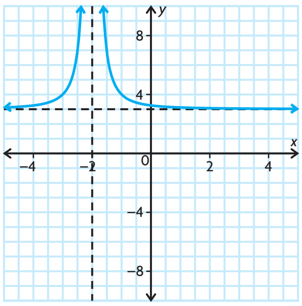
4. **a.**



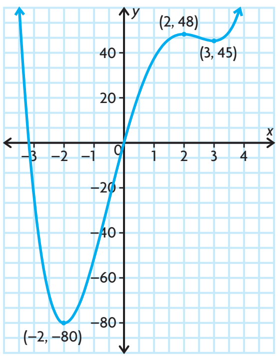
b.



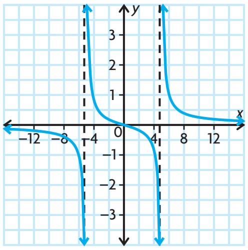
c.



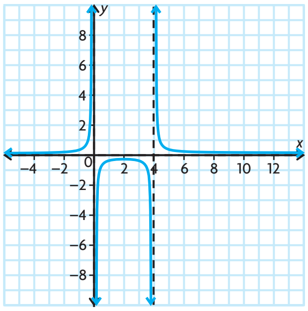
d.



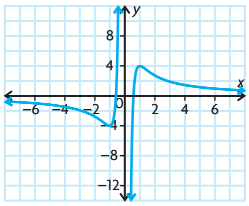
e.



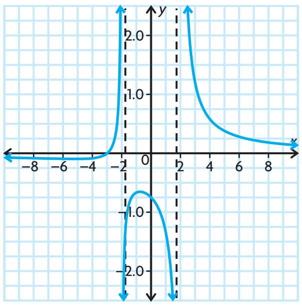
f.



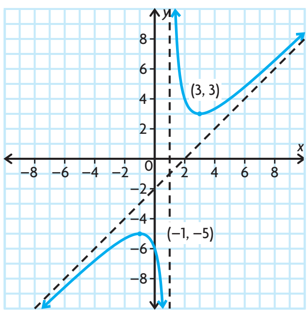
g.



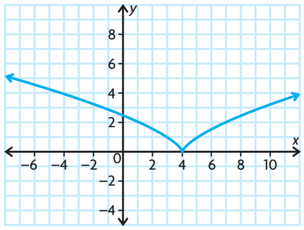
h.



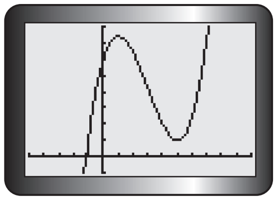
i.



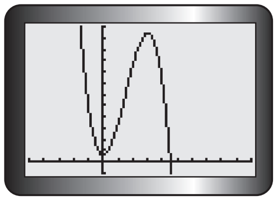
j.



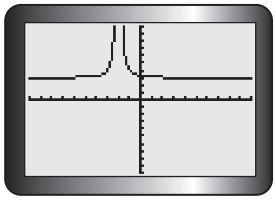
5. **a.**



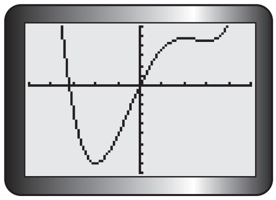
b.



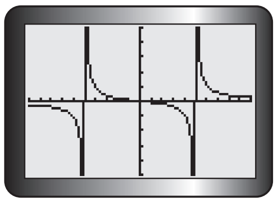
c.



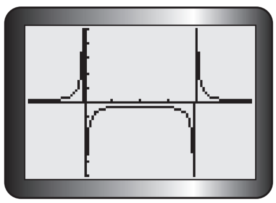
d.



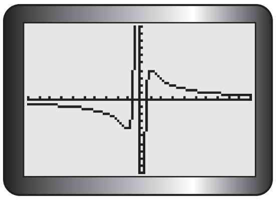
e.



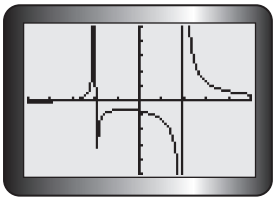
f.



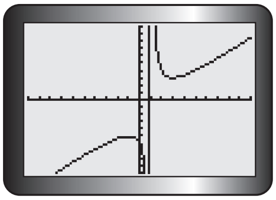
g.



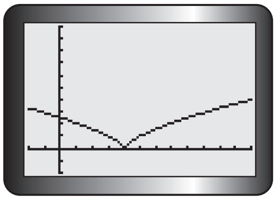
h.



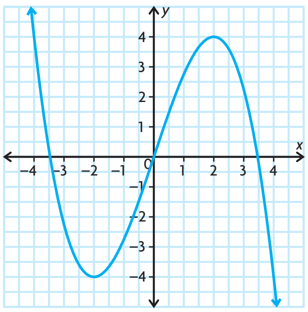
i.



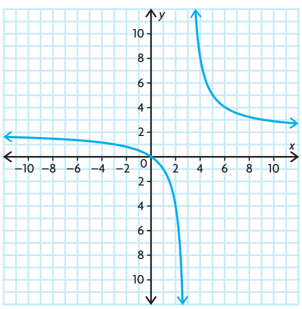
j.



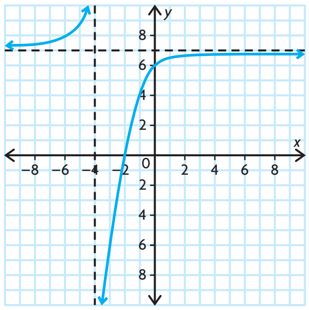
6. $a = -\frac{1}{4}, b = 0, c = 3, d = 0$



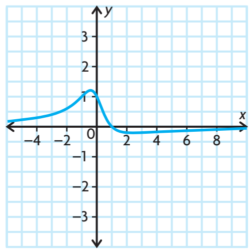
7. a. Answers may vary. For example:



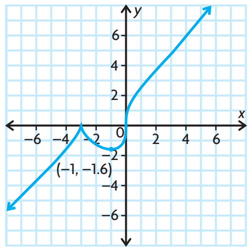
b. Answers may vary. For example:



8.



9.



10. a. $y = 1$ is a horizontal asymptote to the right-hand branch of the graph. $y = -1$ is a horizontal asymptote to the left-hand branch of the graph.

b. $y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

11. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$$x = -\frac{b}{3a}$$

The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $\frac{-b}{3a}$ to values greater than $\frac{-b}{3a}$. Thus, there is a point of inflection at $x = \frac{-b}{3a}$.

At $x = \frac{b}{3a}, \frac{dy}{dx} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c = c - \frac{b^2}{3a}$

Review Exercise, pp. 216–219

1. a. i. $x < 1$

ii. $x > 1$

iii. $(1, 20)$

b. i. $x < -3, -3 < x < 1, x > 6.5$

ii. $1 < x < 3, 3 < x < 6.5$

iii. $(1, -1), (6.5, -1)$

2. No, a counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing, yet the graph is concave down for $x < 0$ and concave up for $x > 0$.

3. a. $(0, 20)$, local minimum; tangent is horizontal

b. $(0, 6)$, local maximum; tangent is horizontal

$(3, 33)$, neither local maximum nor minimum; tangent is horizontal

c. $\left(-1, -\frac{1}{2}\right)$, local minimum;

$\left(7, \frac{1}{14}\right)$, local maximum; tangents at both points are parallel

d. $(1, 0)$, neither local maximum nor minimum; tangent is not horizontal

4. a. $a < x < b, x > e$

b. $b < x < c$

c. $x < a, d < x < e$

d. $c < x < d$

5. a. $x = 3$; large and negative to left of asymptote, large and positive to right of asymptote

b. $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote

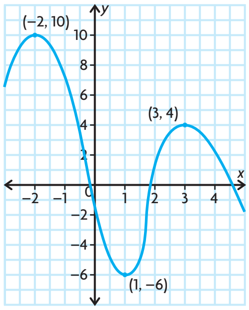
c. hole at $x = -3$

d. $x = -4$; large and positive to left of asymptote, large and negative to right of asymptote

$x = 5$; large and negative to left of asymptote, large and positive to right of asymptote

6. $(0, 5)$; Since the derivative is 0 at $x = 0$, the tangent line is parallel to the x-axis at that point. Because the derivative is always positive, the function is always increasing and, therefore, must cross the tangent line instead of just touching it.

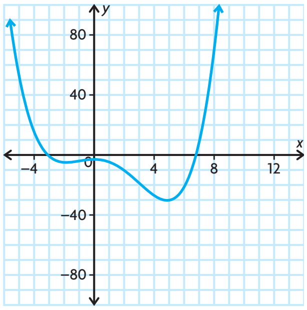
7.



8. a. i. concave up: $-1 < x < 3$; concave down: $x < -1$, $3 < x$

ii. points of inflection: $x = -1$, $x = 5$

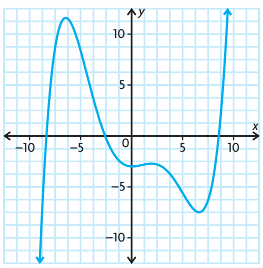
iii.



b. i. concave up: $-4.5 < x < 1$, $5 < x$ concave down: $x < -4.5$, $1 < x < 5$

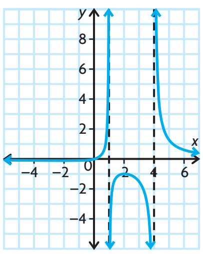
ii. points of inflection: $x = -4.5$, $x = 1$, and $x = 5$

iii.

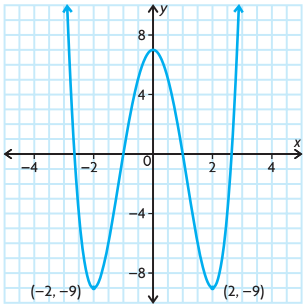


9. a. $a = 1, b = 0$

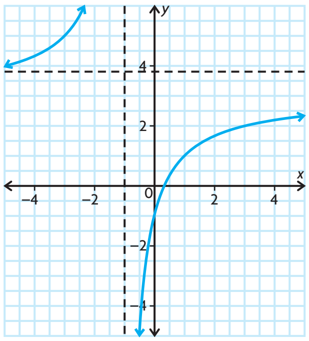
b.



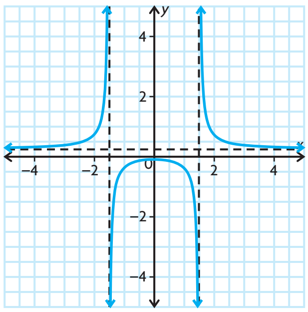
10. a.



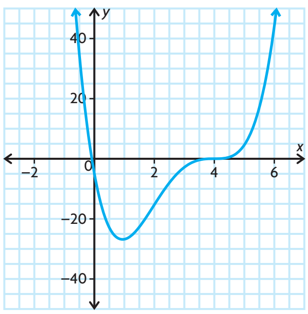
b.



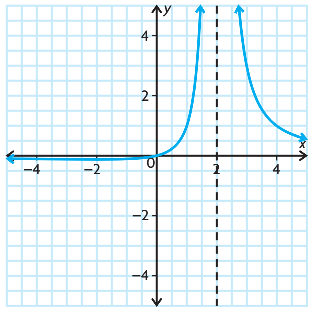
c.



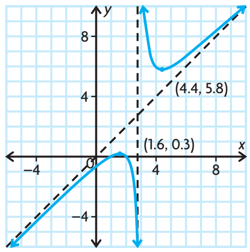
d.



e.



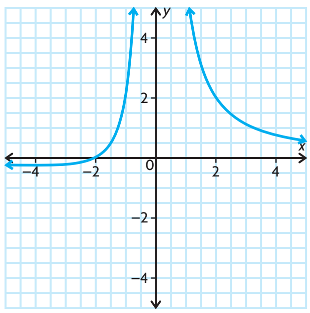
f.



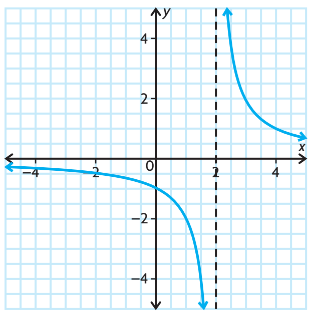
11. a. $-2 \leq k \leq 2, x \neq \pm k$

b. There are three different graphs that result for values of k chosen.

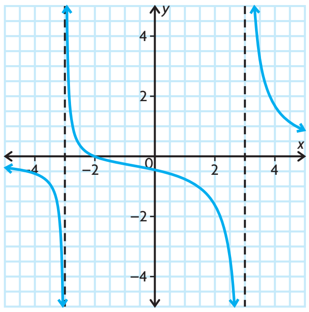
$k = 0$



$k = 2$



For all other values of k, the graph will be similar to the graph below.

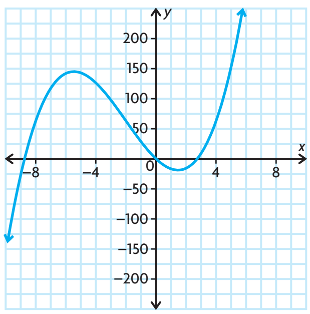


12. **a.** $y = x - 3$
b. $y = 4x + 11$

13. $x = -2, x = 0, x = 2$;
increasing: $-2 < x < 0, x > 2$;
decreasing: $x < -2, 0 < x < 2$

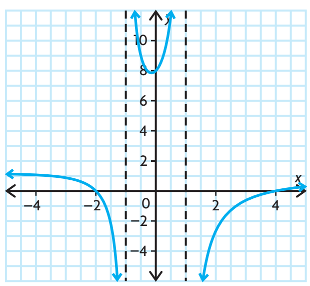
14. local maximum: $(-2.107, 17.054)$,
local minimum: $(1.107, 0.446)$,
absolute maximum: $(3, 24.5)$,
absolute minimum: $(-4, -7)$

15.

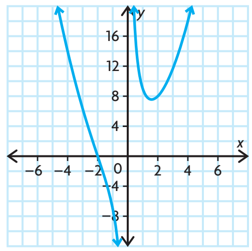


16. **a.** $p(x)$: oblique asymptote, $y = 0.75x$
$q(x)$: vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 0$
$r(x)$: vertical asymptotes at $x = -1$ and $x = 1$; horizontal asymptote at $y = 1$
$s(x)$: vertical asymptote at $y = 2$

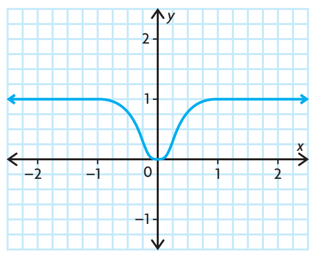
b.



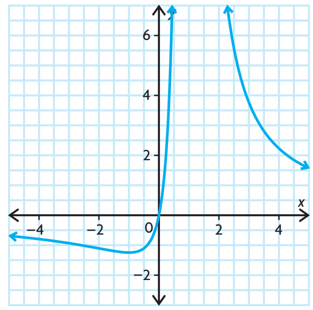
17. Domain: $\{x \in \mathbf{R} | x \neq 0\}$;
x-intercept: -2;
y-intercept: 8;
vertical asymptote: $x = 0$; large and negative to the left of the asymptote, large and positive to the right of the asymptote;
no horizontal or oblique asymptote;
increasing: $x > 1.59$;
decreasing: $x < 0, 0 < x < 1.59$;
concave up: $x < -2, x > 0$;
concave down: $-2 < x < 0$;
local minimum at $(1.59, 7.56)$;
point of inflection at $(-2, 0)$



18. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x > 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x < 0$. At a stationary point, $x = 0$. From the graph, the zero for $f'(x)$ occurs at $x = 0$. At $x = 0$, $f'(x)$ changes from negative to positve, so f has a local minimum point there. If the graph of f is concave up, then f'' is positive. From the slope of f', the graph of f is concave up for $-0.6 < x < 0.6$. If the graph of f is concave down, then f'' is negative. From the slope of f', the graph of f is concave down for $x < -0.6$ and $x > 0.6$. Graphs will vary slightly.



19. domain: $\{x \in \mathbf{R} | x \neq 1\}$;
x-intercept and y-intercept: $(0, 0)$;
vertical asymptote: $x = 1$; large and positive on either side of the asymptote;
horizontal asymptote: $y = 0$;
increasing: $-1 < x < 1$;
decreasing: $x < -1, x > 1$;
concave down: $x < -2$;
concave up: $-2 < x < 1, x > 1$;
local minimum at $(-1, -1.25)$;
point of inflection: $(-2, -1.11)$



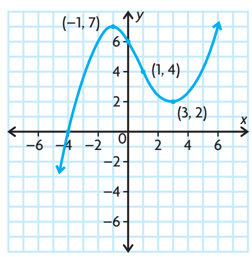
20. **a.** Graph A is f, graph C is f', and graph B is f''. We know this because when you take the derivative, the degree of the denominator increases by one. Graph A has a squared term in the denominator, graph C has a cubic term in the denominator, and graph B has a term to the power of four in the denominator.
b. Graph F is f, graph E is f' and graph D is f''. We know this because the degree of the denominator increases by one degree when the derivative is taken.

Chapter 4 Test, p. 220

1. **a.** $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$
b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$
c. $(-9, 1), (-6, -2), (0, 1), (8, -2)$
d. $x = -3, x = 4$
e. $f''(x) > 0$
f. $-3 < x < 0$ or $4 < x < 8$
g. $(-8, 0), (10, -3)$

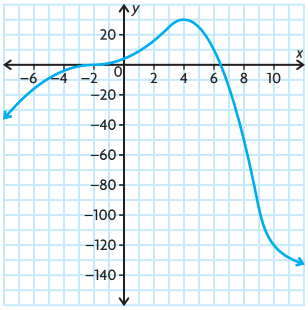
2. **a.** $x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$
b. $\left(-\frac{1}{2}, -\frac{17}{8}\right)$: local maximum
$\left(\frac{1}{2}, \frac{15}{8}\right)$: local maximum
$(3, -45)$: local minimum

3.



4. hole at $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote;
$y = 1$;
Domain: $\{x \in \mathbf{R} \mid x \neq -2, x \neq 3\}$

5.



6. There are discontinuities at $x = -3$ and $x = 3$.

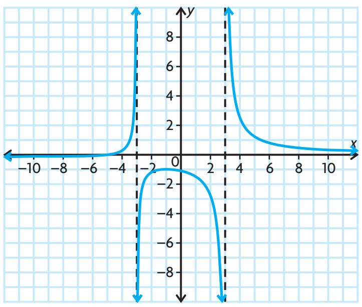
$\left.\begin{array}{l}\lim_{x \to 3^-} f(x) = \infty \\ \lim_{x \to 3^+} f(x) = -\infty\end{array}\right\} x = -3$ is a vertical asymptote.

$\left.\begin{array}{l}\lim_{x \to 3^-} f(x) = -\infty \\ \lim_{x \to 3^+} f(x) = \infty\end{array}\right\} x = 3$ is a vertical asymptote.

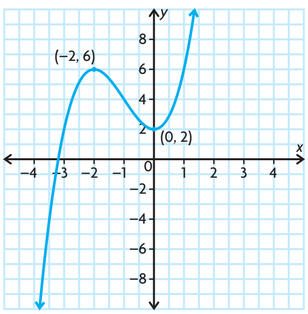
The y-intercept is $-\frac{10}{9}$ and x-intercept is -5.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$y = 0$ is a horizontal asymptote.



7. $b = 3$, $c = 2$



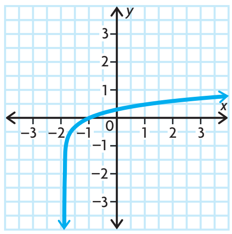
Chapter 5

Review of Prerequisite Skills, pp. 224–225

1. a. $\frac{1}{9}$ c. $\frac{1}{9}$
b. 4 d. $\frac{9}{4}$

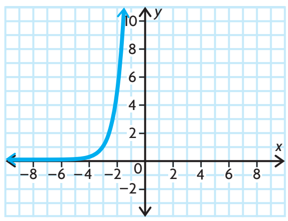
2. a. $\log_5 625 = 4$
b. $\log_4 \frac{1}{16} = -2$
c. $\log_x 3 = 3$
d. $\log_{10} 450 = w$
e. $\log_3 z = 8$
f. $\log_a T = b$

3. a.



x-intercept: $(-1, 0)$

b.



no x-intercept

4. a. $\frac{y}{r}$ b. $\frac{x}{r}$ c. $\frac{y}{x}$

5. a. 2π e. $\frac{3\pi}{2}$
b. $\frac{\pi}{4}$ f. $-\frac{2\pi}{3}$
c. $-\frac{\pi}{2}$ g. $\frac{5\pi}{4}$
d. $\frac{\pi}{6}$ h. $\frac{11\pi}{6}$

6. a. b c. a e. b
b. $\frac{b}{a}$ d. a f. $-b$

7. a. $\cos \theta = -\frac{12}{13}$,
$\tan \theta = -\frac{5}{12}$

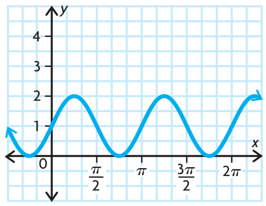
b. $\sin \theta = -\frac{\sqrt{5}}{3}$,
$\tan \theta = \frac{\sqrt{5}}{2}$

c. $\sin \theta = -\frac{2}{\sqrt{5}}$,
$\cos \theta = \frac{1}{\sqrt{5}}$

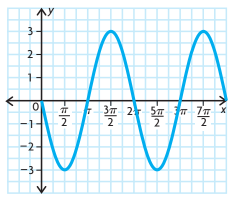
d. $\cos \frac{\pi}{2} = 0$,
$\tan \frac{\pi}{2}$ is undefined

8. a. period: π, amplitude: 1
b. period: 4π, amplitude: 2
c. period: 2, amplitude: 3
d. period: $\frac{\pi}{6}$, amplitude: $\frac{2}{7}$
e. period: 2π, amplitude: 5
f. period: 2π, amplitude: $\frac{3}{2}$

9. a.



b.



10. a. $\tan x + \cot x = \sec x \csc x$

$$
\begin{aligned}
\text{LS} &= \tan x + \cot x \\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x} \\
&= \frac{1}{\cos x + \sin x} \\
\text{RS} &= \sec x \csc x \\
&= \frac{1}{\cos x} \times \frac{1}{\sin x} \\
&= \frac{1}{\cos x \sin x}
\end{aligned}
$$

Therefore, $\tan x + \cot x = \sec x \csc x$.