Chapter 6

INTRODUCTION TO VECTORS

Have you ever tried to swim across a river with a strong current or run into a head wind? Have you ever tried sailing across a windy lake? If your answer is yes, then you have experienced the effect of vector quantities. Vectors were developed in the late nineteenth century as mathematical tools for studying physics. In the following century, vectors became an essential tool for anyone using mathematics including social sciences. In order to navigate, pilots need to know what effect a crosswind will have on the direction in which they intend to fly. In order to build bridges, engineers need to know what load a particular design will support. In this chapter, you will learn more about vectors and how they represent quantities possessing both magnitude and direction.

CHAPTER EXPECTATIONS

In this chapter, you will

- represent vectors as directed line segments, **Section 6.1**
- recognize a vector as a quantity with both magnitude and direction, **Section 6.1**
- perform mathematical operations on geometric vectors, **Sections 6.2, 6.3**
- determine some properties of the operations performed on vectors, **Section 6.4**
- determine the Cartesian representation of a vector in two- and three-dimensional space, **Sections 6.5, 6.6, 6.7, 6.8**
- perform mathematical operations on algebraic vectors in two- and three-dimensional space, **Sections 6.6, 6.7, 6.8**

Review of Prerequisite Skills

In this chapter, you will be introduced to the concept of a vector, a mathematical entity having both magnitude and direction. You will examine geometric and algebraic representations of vectors in two- and three-dimensional space. Before beginning this introduction to vectors, you may wish to review some basic facts of trigonometry.

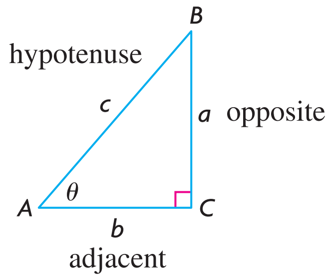
TRIGONOMETRIC RATIOS

In a right-angled triangle, as shown, $a^2 + b^2 = c^2$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

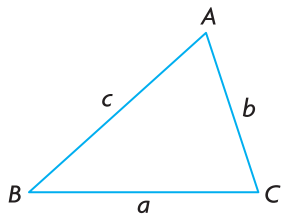
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$



THE SINE LAW

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



THE COSINE LAW

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

SOLVING A TRIANGLE

- To solve a triangle means to find the measures of the sides and angles whose values are not given.
- Solving a triangle may require the use of trigonometric ratios, the Pythagorean theorem, the sine law, and/or the cosine law.

Exercise

1. State the exact value of each of the following:

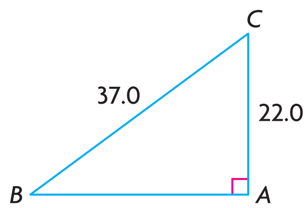
a. sin 60° c. cos 60° e. sin 135°

b. tan 120° d. cos 30° f. tan 45°

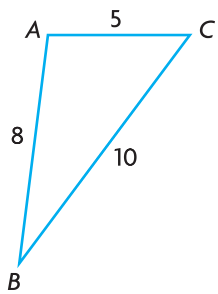
2. In $\triangle ABC$, $AB = 6$, $\angle B = 90°$, and $AC = 10$. State the exact value of tan A.

3. Solve $\triangle ABC$, to one decimal place.

a.



b.



4. In $\triangle XYZ$, $XY = 6$, $\angle X = 60°$, and $\angle Y = 70°$. Determine the values of XZ, YZ, and $\angle Z$, to two digit accuracy.

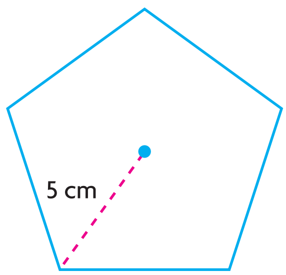
5. In $\triangle RST$, $RS = 4$, $RT = 7$, and $ST = 5$. Determine the measures of the angles to the nearest degree.

6. An aircraft control tower, T, is tracking two airplanes at points A, 3.5 km from T, and B, 6 km from T. If $\angle ATB = 70°$, determine the distance between the two airplanes to two decimal places.

7. Three ships are at points P, Q, and R such that $PQ = 2$ km, $PR = 7$ km, and $\angle QPR = 142°$. What is the distance between Q and R, to two decimal places.

8. Two roads intersect at an angle of 48º. A car and truck collide at the intersection, and then leave the scene of the accident. The car travels at 100 km/h down one road, while the truck goes 80 km/h down the other road. Fifteen minutes after the accident, a police helicopter locates the car and pulls it over. Twenty minutes after the accident, a police cruiser pulls over the truck. How far apart are the car and the truck at this time?

9. A regular pentagon has all sides equal and all central angles equal. Calculate, to the nearest tenth, the area of the pentagon shown.



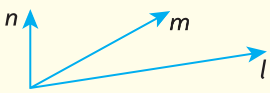
CHAPTER 6: FIGURE SKATING

Figure skaters are exceptional athletes and artists. Their motion while skating is also an illustration of the use of vectors. The ice they skate on is a nearly frictionless surface—so any force applied by the skater has a direct impact on speed, momentum, and direction. Vectors can be used to describe a figure skater's path on the ice. When the skater starts moving in a direction, she will continue moving in that direction and at that speed until she applies a force to change or stop her motion. This is more apparent with pairs figure skaters. To stay together, each skater must skate with close to the same speed as their partner in the same direction. If one skater uses less force or applies the force in a different direction, the skaters will either bump into each other or separate and fly away from each other. If they don't let go of each other, the opposing forces may cause them to spin.

Case Study—Throwing a Triple Salchow

The Triple Salchow throw is one of the more difficult moves in pairs figure skating. Both partners skate together in one direction with a lot of speed. Next, the male skater plants his feet to throw his partner and add his momentum to that of the female skater. She applies force with one skate to jump into the air. In order to make herself spin, she applies force at an angle to the initial direction and spins three times in the air before landing. There are three main vectors at work here. These vectors are the initial thrust of both skaters, the force the male skater applies to the female skater, and the vertical force of the jump.

Vector	Magnitude (size of the force)
Both skaters' initial thrust (*l*)	60
Female skater's change in direction to cause spin (*m*)	40
Female skater's vertical leap (*n*)	20



DISCUSSION QUESTIONS

1. What operation on vectors *l* and *m* do you think should be done to find the resulting thrust vector (*a*) for the female skater?
2. If we performed the same operation as in problem 1 to the vectors *a* and *n*, what would the resulting vector represent? You may assume that the angle between *a* and *n* is 90°.
3. Can you think of a three-dimensional figure that would represent all of the vectors *l*, *m*, and *n* at the same time, as well as the vectors found in the previous two problems? Give as complete a description as possible for this figure, including any properties you notice. For example, is it constructed from any familiar two-dimensional objects?

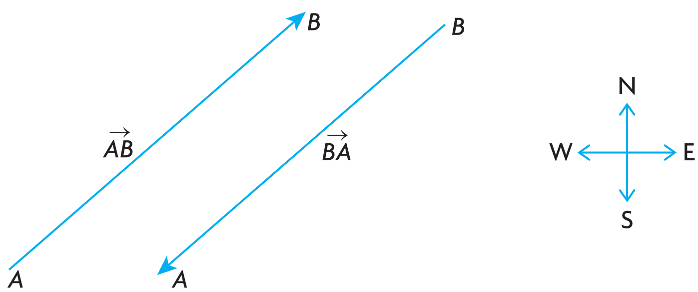
Section 6.1—An Introduction to Vectors

In mathematics and science, you often come in contact with different quantities. Some of these quantities, those whose **magnitude** (or size) can be completely specified by just one number, are called **scalars**. Some examples of scalars are age, volume, area, speed, mass, and temperature. On the other hand, some quantities (such as weight, velocity, or friction) require both a magnitude and a direction for a complete description and are called **vectors**.

Defining the Characteristics of Vectors

A vector can be represented by a directed line segment. A directed line segment has a length, called its magnitude, and a direction indicated by an arrowhead.

The diagram below can help to make the distinction between a vector and a scalar. If an airplane is travelling at a speed of 500 km/h, this description is useful, but for navigation and computational purposes, incomplete. If we add the fact that the airplane is travelling in a northeasterly direction, we now have a description of its velocity because we have specified both its speed and direction. This defines velocity as a vector quantity. If we refer to the speed of the airplane, we are describing it with just a single number, which defines speed as a scalar quantity.



Scale: 1 cm is equivalent to 100 km/h

In the diagram, $\overrightarrow{AB}$ is an example of a vector. In this case, it is a line segment running from A to B with its tail at A and head at B. Its actual size, or magnitude, is denoted by $|\overrightarrow{AB}|$. The magnitude of a vector is always non-negative. The vector $\overrightarrow{AB}$ could be used to represent the velocity of any airplane heading in a northeasterly direction at 500 km/h (using a scale of 1 cm to 100 km/h, i.e., $|\overrightarrow{AB}| = 5$ cm $= 500$ km/h). The direction of the "arrow" represents the direction of the airplane, and its length represents its speed.

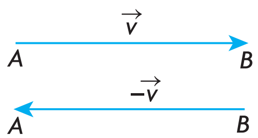
What is a Vector?

A vector is a mathematical quantity having both magnitude and direction.

In the diagram on the previous page, $\overrightarrow{BA}$ is a vector pointing from B to A. The vector $\overrightarrow{BA}$ represents an airplane travelling in a southwesterly direction at 500 km/h. Note that the magnitudes of the two vectors are equal, i.e., $|\overrightarrow{AB}| = |\overrightarrow{BA}|$, but that the vectors themselves are not equal because they point in opposite directions. For this reason, we describe these as **opposite vectors**.

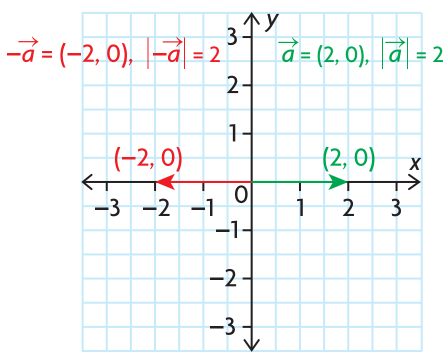
Opposite Vectors

Two vectors that are opposites have the same magnitude but point in opposite directions.



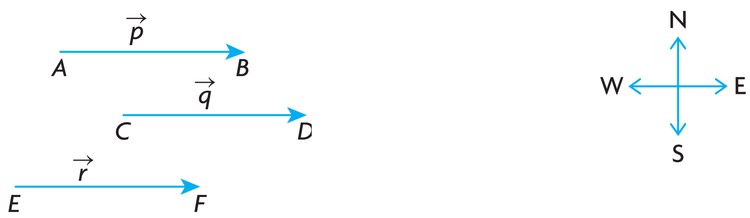
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposites, and $\overrightarrow{AB} = -\overrightarrow{BA}$. In this case, $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and the vectors are parallel but point in opposite directions. Vectors can also be represented with lower-case letters. In the diagram above, vectors $\vec{v}$ and $-\vec{v}$ have the same magnitude, i.e., $|\vec{v}| = |-\vec{v}|$, but point in opposite directions, so $\vec{v}$ and $-\vec{v}$ are also opposites.

No mention has yet been made of using coordinate systems to represent vectors. In the diagram below, it is helpful to note that $\vec{a} = (2, 0)$ is a vector having its tail at the origin and head at $(2, 0)$; this vector has magnitude 2, i.e., $|\vec{a}| = 2$. Also, observe that $-\vec{a} = (-2, 0)$ and $|-\vec{a}| = 2$. The vectors $\vec{a}$ and $-\vec{a}$ are opposites.



It is not always appropriate or necessary to describe a quantity by both a magnitude and direction. For example, the description of the area of a square or rectangle does not require a direction. In referring to a person's age, it is clear what is meant by just the number. By their nature, quantities of this type do not have a direction associated with them and, thus, are not vectors.

Vectors are equal, or equivalent, if they have the same direction and the same magnitude. This means that the velocity vector for an airplane travelling in an easterly direction at 400 km/h could be represented by any of the three vectors in the following diagram.

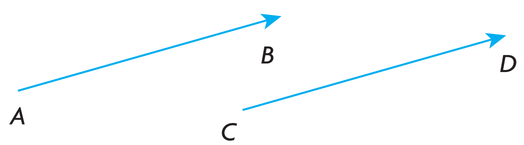


Scale: 1 cm is equivalent to 100 km/h

Notice that any one of these vectors could be translated to be **coincident** with either of the other two. (When vectors are translated, it means they are picked up and moved without changing either their direction or size.) This implies that the velocity vector of an airplane travelling at 400 km/h in an easterly direction from Calgary is identical to that of an airplane travelling at 400 km/h in an easterly direction from Toronto.

Note that, in the diagram above, we have also used lower-case letters to represent the three vectors. It is convenient to write the vector $\overrightarrow{AB}$ as $\vec{p}$, for example, and in this case $\vec{p} = \vec{q} = \vec{r}$.

Equal Vectors



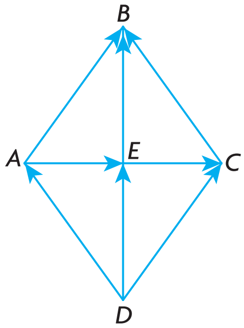
Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal (or equivalent) if and only if

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to each other, and the direction from A to B is the same as the direction from C to D
2. the magnitude of $\overrightarrow{AB}$ equals the magnitude of $\overrightarrow{CD}$, i.e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

EXAMPLE 1 Connecting vectors to two-dimensional figures

Rhombus $ABCD$ is drawn and its two diagonals AC and BD are drawn as shown. Name vectors equal to each of the following.

a. $\overrightarrow{AB}$ b. $\overrightarrow{DA}$ c. $\overrightarrow{EB}$ d. $\overrightarrow{AE}$



Solution

A rhombus is a parallelogram with its opposite sides parallel and the four sides equal in length. Thus, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{DA} = \overrightarrow{CB}$ and $|\overrightarrow{AB}| = |\overrightarrow{DC}| = |\overrightarrow{DA}| = |\overrightarrow{CB}|$. Note that $\overrightarrow{AB} \neq \overrightarrow{DA}$ because these vectors have different directions, even though they have equal magnitudes, i.e., $|\overrightarrow{AB}| = |\overrightarrow{DA}|$.

Since the diagonals in a rhombus bisect each other, $\overrightarrow{AE} = \overrightarrow{EC}$ and $\overrightarrow{EB} = \overrightarrow{DE}$. Note also that, if the arrow had been drawn from C to D instead of from D to C, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ would be opposites and would not be equal, even though they are of the same length. If these vectors are opposites, then the relationship between them can be expressed as $\overrightarrow{AB} = -\overrightarrow{CD}$. This implies that these vectors have the same magnitude but opposite directions.

In summary: a. $\overrightarrow{AB} = \overrightarrow{DC}$ b. $\overrightarrow{DA} = \overrightarrow{CB}$ c. $\overrightarrow{EB} = \overrightarrow{DE}$ d. $\overrightarrow{AE} = \overrightarrow{EC}$

In our discussion of vectors thus far, we have illustrated our ideas with **geometric vectors**. Geometric vectors are those that are considered without reference to coordinate axes. The ability to use vectors in applications usually requires us to place them on a coordinate plane. These are referred to as **algebraic vectors**; they will be introduced in the exercises and examined in detail in Section 6.5. Algebraic vectors will become increasingly important in our work.

IN SUMMARY

Key Ideas

- A vector is a mathematical quantity having both magnitude and direction, for example velocity.
- A scalar is a mathematical quantity having only magnitude, for example, speed.

Need to Know

- $\overrightarrow{AB}$ represents a vector running from A to B, with its tail at A and head at B.
- $|\overrightarrow{AB}|$ represents the magnitude of a vector and is always non-negative.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposite if they are parallel and have the same magnitude but opposite directions. It follows that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and $\overrightarrow{AB} = -\overrightarrow{BA}$.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if they are parallel and have the same magnitude and the same direction. It follows that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and $\overrightarrow{AB} = \overrightarrow{CD}$.

Exercise 6.1

PART A

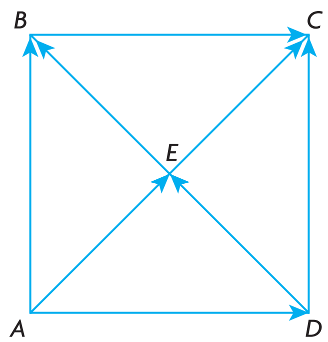
1. State whether each statement is true or false. Justify your decision.
   a. If two vectors have the same magnitude, then they are equal.
   b. If two vectors are equal, then they have the same magnitude.
   c. If two vectors are parallel, then they are either equal or opposite vectors.
   d. If two vectors have the same magnitude, then they are either equal or opposite vectors.

2. For each of the following, state whether the quantity is a scalar or a vector and give a brief explanation why: height, temperature, weight, mass, area, volume, distance, displacement, speed, force, and velocity.

3. Friction is considered to be a vector because friction can be described as the force of resistance between two surfaces in contact. Give two examples of friction from everyday life, and explain why they can be described as vectors.

PART B

4. Square *ABCD* is drawn as shown below with the diagonals intersecting at *E*.



   a. State four pairs of equivalent vectors.
   b. State four pairs of opposite vectors.
   c. State two pairs of vectors whose magnitudes are equal but whose directions are perpendicular to each other.

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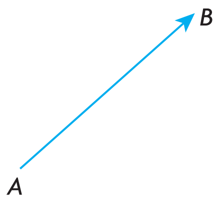
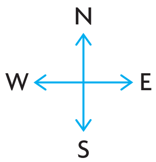
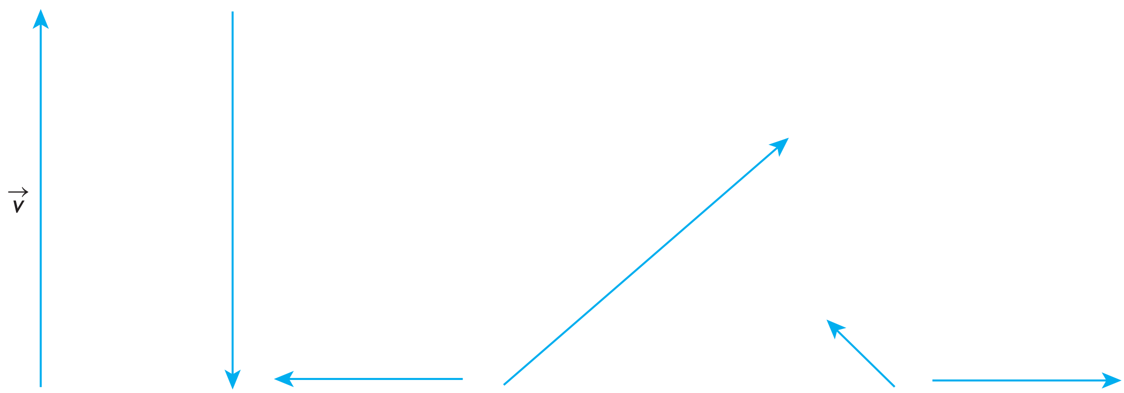


5. Given the vector $\overrightarrow{AB}$ as shown, draw a vector
   a. equal to $\overrightarrow{AB}$
   b. opposite to $\overrightarrow{AB}$
   c. whose magnitude equals $|\overrightarrow{AB}|$ but is not equal to $\overrightarrow{AB}$
   d. whose magnitude is twice that of $\overrightarrow{AB}$ and in the same direction
   e. whose magnitude is half that of $\overrightarrow{AB}$ and in the opposite direction



6. Using a scale of 1 cm to represent 10 km/h, draw a velocity vector to represent each of the following:
   a. a bicyclist heading due north at 40 km/h
   b. a car heading in a southwesterly direction at 60 km/h
   c. a car travelling in a northeasterly direction at 100 km/h
   d. a boy running in a northwesterly direction at 30 km/h
   e. a girl running around a circular track travelling at 15 km/h heading due east

7. The vector shown, $\vec{v}$, represents the velocity of a car heading due north at 100 km/h. Give possible interpretations for each of the other vectors shown.



a. b. c. d. e.

C

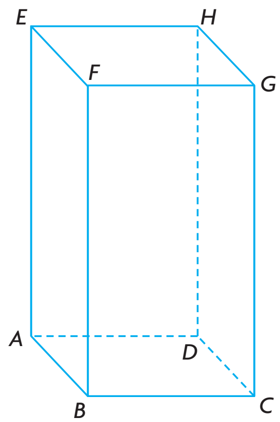
8. For each of the following vectors, describe the opposite vector.
   a. an airplane flies due north at 400 km/h
   b. a car travels in a northeasterly direction at 70 km/h
   c. a bicyclist pedals in a northwesterly direction at 30 km/h
   d. a boat travels due west at 25 km/h

T

9. a. Given the square-based prism shown where $AB = 3$ cm and $AE = 8$ cm, state whether each statement is true or false. Explain.

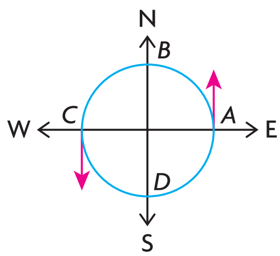
i) $\overrightarrow{AB} = \overrightarrow{GH}$ ii) $|\overrightarrow{EA}| = |\overrightarrow{CG}|$ iii) $|\overrightarrow{AD}| = |\overrightarrow{DC}|$ iv) $\overrightarrow{AH} = \overrightarrow{BG}$

b. Calculate the magnitude of $\overrightarrow{BD}$, $\overrightarrow{BE}$, and $\overrightarrow{BH}$.



A

10. James is running around a circular track with a circumference of 1 km at a constant speed of 15 km/h. His velocity vector is represented by a vector tangent to the circle. Velocity vectors are drawn at points A and C as shown. As James changes his position on the track, his velocity vector changes.



a. Explain why James's velocity can be represented by a vector tangent to the circle.

b. What does the length of the vector represent?

c. As he completes a lap running at a constant speed, explain why James's velocity is different at every point on the circle.

d. Determine the point on the circle where James is heading due south.

e. In running his first lap, there is a point at which James is travelling in a northeasterly direction. If he starts at point A how long would it have taken him to get to this point?

f. At the point he has travelled $\frac{3}{8}$ of a lap, in what direction would James be heading? Assume he starts at point A.

PART C

11. $\overrightarrow{AB}$ is a vector whose tail is at $(-4, 2)$ and whose head is at $(-1, 3)$.

a. Calculate the magnitude of $\overrightarrow{AB}$.

b. Determine the coordinates of point D on vector $\overrightarrow{CD}$, if $C(-6, 0)$ and $\overrightarrow{CD} = \overrightarrow{AB}$.

c. Determine the coordinates of point E on vector $\overrightarrow{EF}$, if $F(3, -2)$ and $\overrightarrow{EF} = \overrightarrow{AB}$.

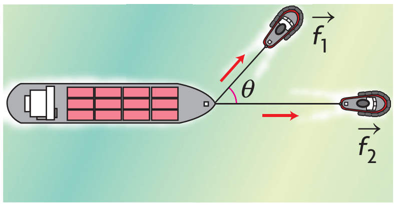
d. Determine the coordinates of point G on vector $\overrightarrow{GH}$, if $G(3, 1)$ and $\overrightarrow{GH} = -\overrightarrow{AB}$.

Section 6.2—Vector Addition

In this section, we will examine ways that vectors can be used in different physical situations. We will consider a variety of contexts and use them to help develop rules for the application of vectors.

Examining Vector Addition

Suppose that a cargo ship has a mechanical problem and must be towed into port by two tugboats. This situation is represented in the following diagram.



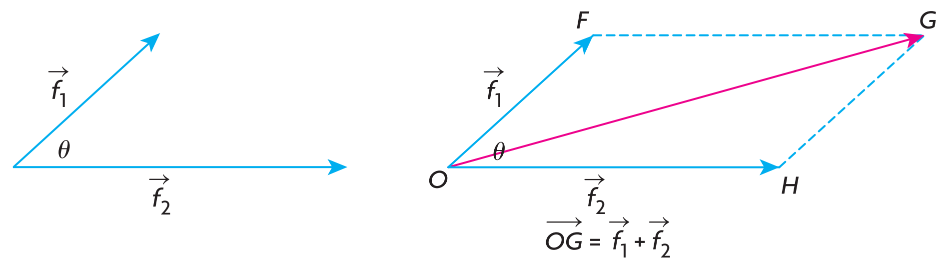
The force exerted by the first tugboat is denoted by $\vec{f_1}$ and that of the second tugboat as $\vec{f_2}$. They are denoted as vectors because these forces have both magnitude and direction. θ is the angle between the two forces shown in the diagram, where the vectors are placed tail to tail.

In considering this situation, a number of assumptions have been made:

1. The direction of the force exerted by each of the tugboats is indicated by the direction of the arrows.
2. The magnitude of the force exerted by each of the two tugboats is proportional to the length of the corresponding force vector. This means that the longer the force vector, the greater the exerted force.
3. The forces that have been exerted have been applied at a common point on the ship.

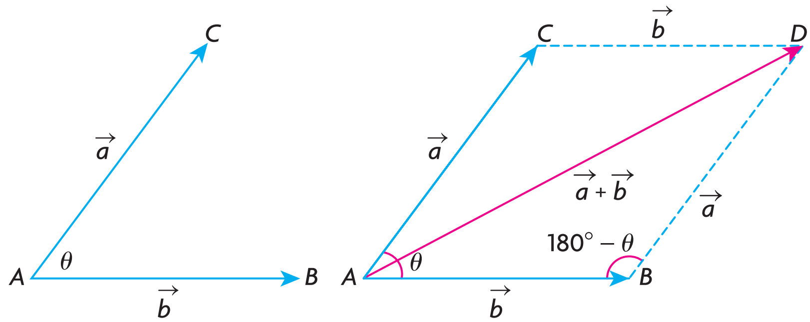
What we want to know is whether we can predict the direction the ship will move and with what force. Intuitively, we know that the ship will move in a direction somewhere between the direction of the forces, but because $|\vec{f_2}| > |\vec{f_1}|$ (the magnitude of the second force is greater than that of the first force), the boat should move closer to the direction of $\vec{f_2}$ rather than $\vec{f_1}$. The combined magnitude of the two forces should be greater than either of $|\vec{f_1}|$ or $|\vec{f_2}|$ but not equal to their sum, because they are pulling at an angle of θ to each other, i.e. they are not pulling in exactly the same direction.
There are several other observations to be made in this situation. The actions of the two tug boats are going to pull the ship in a way that combines the force vectors. The ship is going to be towed in a constant direction with a certain force, which, in effect, means the two smaller force vectors can be replaced with just

one vector. To find this single vector to replace $\vec{f_1}$ and $\vec{f_2}$, the parallelogram determined by these vectors is constructed. The main diagonal of the parallelogram is called the **resultant** or sum of these two vectors and represents the combined effect of the two vectors. The resultant of $\vec{f_1}$ and $\vec{f_2}$ has been shown in the following diagram as the diagonal, $\overrightarrow{OG}$, of the parallelogram.



The length (or magnitude) of each vector representing a force is proportional to the actual force exerted. After the tugboats exert their forces, the ship will head in the direction of $\overrightarrow{OG}$ with a force proportional to the length of $\overrightarrow{OG}$.

The Parallelogram Law for Adding Two Vectors

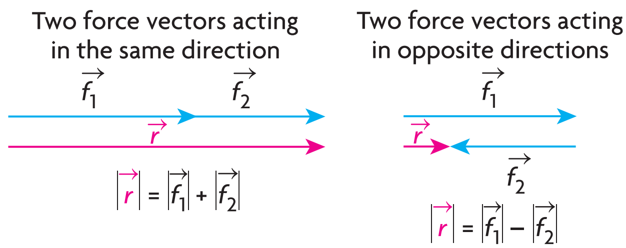


To determine the sum of the two vectors $\vec{a}$ and $\vec{b}$, complete the parallelogram formed by these two vectors when placed tail to tail. Their sum is the vector $\overrightarrow{AD}$, the diagonal of the constructed parallelogram,

$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$.

Consider the triangle formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$. It is important to note that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$. This means that the magnitude of the sum $\vec{a} + \vec{b}$ is less than or equal to the combined magnitudes of $\vec{a}$ and $\vec{b}$. The magnitude of $\vec{a} + \vec{b}$ is equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$ only when these three vectors lie in the same direction.

In the tugboat example, this means the overall effect of the two tugboats is less than the sum of their individual efforts. If the tugs pulled in the same direction, the overall magnitude would be equal to the sum of their individual magnitudes. If they pulled in opposite directions, the overall magnitude would be their difference.

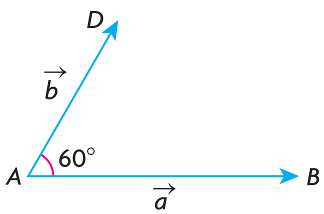


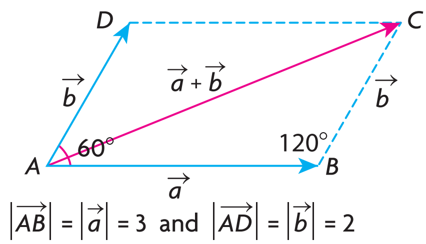
EXAMPLE 1 Selecting a strategy to determine the magnitude of a resultant vector

Given vectors $\vec{a}$ and $\vec{b}$ such that the angle between the two vectors is 60°, $|\vec{a}| = 3$, and $|\vec{b}| = 2$, determine $|\vec{a} + \vec{b}|$.

Solution

If it is stated that the angle between the vectors is θ, this means that the vectors are placed tail to tail and the angle between the vectors is θ. In this problem, the angle between the vectors is given to be 60°, so the vectors are placed tail to tail as shown.





To calculate the value of $|\vec{a} + \vec{b}|$, draw the diagonal of the related parallelogram. From the diagram, $\vec{AB} + \vec{BC} = \vec{a} + \vec{b} = \vec{AC}$. Note that the angle between $\vec{AB}$ and $\vec{BC}$ is 120°, the supplement of 60°.

$$\text{Now, } |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\angle ABC) \qquad \text{(Cosine law)}$$

$$|\vec{a} + \vec{b}|^2 = 3^2 + 2^2 - 2(3)(2)\cos 120° \qquad \text{(Substitution)}$$

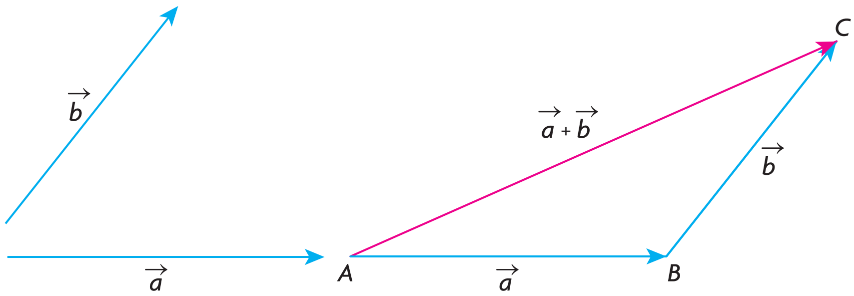
$$|\vec{a} + \vec{b}|^2 = 13 - 2(3)(2)\left(\frac{-1}{2}\right)$$

$$|\vec{a} + \vec{b}|^2 = 19$$

Therefore, $|\vec{a} + \vec{b}| = \sqrt{19} \doteq 4.36$.

When finding the sum of two or more vectors, it is not necessary to draw a parallelogram each time. In the following, we show how to add vectors using the triangle law of addition.

Triangle Law of Addition

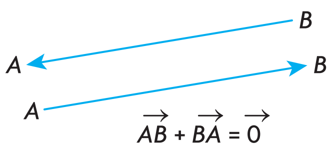


In the diagram, the sum of the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b}$, is found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. This could also have been done by translating $\vec{a}$ so that its tail was at the head of $\vec{b}$. In either case, the sum of the vectors $\vec{a}$ and $\vec{b}$ is $\overrightarrow{AC}$.

A way of thinking about the sum of two vectors is to imagine that, if you start at point A and walk to point B and then to C, you end up in the exact location as if you walked directly from point A to C. Thus, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The Zero Vector

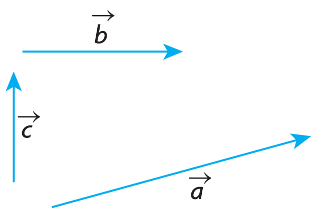
An observation that comes directly from the triangle law of addition is that when two opposite vectors are added, the resultant is the zero vector. This means that the combined effect of a vector and its opposite is the zero vector. In symbols, $\overrightarrow{AB} + \overrightarrow{BA} = \vec{0}$.



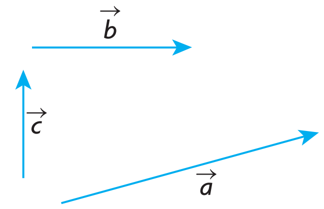
The zero vector has a magnitude of 0, i.e., $|\vec{0}| = 0$, and no defined direction.

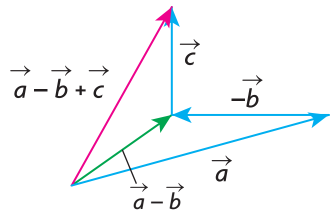
EXAMPLE 2 **Representing a combination of three vectors using the triangle law of addition**

Suppose you are given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as shown below. Using these three vectors, sketch $\vec{a} - \vec{b} + \vec{c}$.



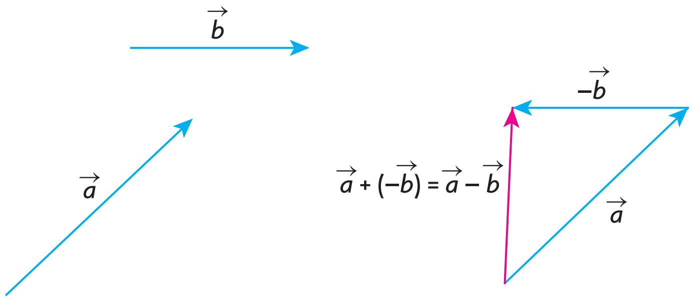
Solution





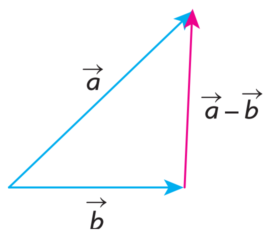
To draw the required vector, first draw $-\vec{b}$, the opposite of $\vec{b}$, and then place the vectors head to tail as shown. It should be emphasized that $\vec{a} - \vec{b}$ actually means $\vec{a} + (-\vec{b})$. Note that the required resultant vector $\vec{a} - \vec{b} + \vec{c}$ is also the resultant vector of $(\vec{a} - \vec{b}) + \vec{c}$ by the triangle law of addition.

The Difference of Two Vectors, $\vec{a} - \vec{b}$



In the diagram above, the difference between vectors $\vec{a}$ and $\vec{b}$ is found by adding the opposite vector $\vec{b}$ to $\vec{a}$ using the triangle law of addition.

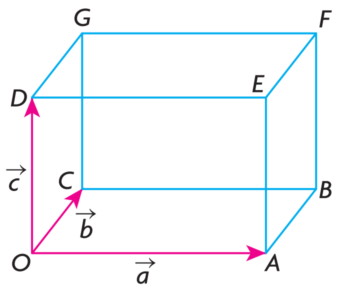
Another way to think about $\vec{a} - \vec{b}$ is to arrange the vectors tail to tail. In this case, $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is illustrated in the following diagram. Using the vectors above, the difference vector is the same as the one produced by adding the opposite.



The concept of addition and subtraction is applied in Example 3.

EXAMPLE 3 Representing a single vector as a combination of vectors

In the rectangular box shown below, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{b}$, and $\overrightarrow{OD} = \vec{c}$.



Express each of the following vectors in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

a. $\overrightarrow{BC}$ b. $\overrightarrow{GF}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{AC}$ e. $\overrightarrow{BG}$ f. $\overrightarrow{OF}$

Solution

a. $\overrightarrow{BC}$ is the opposite of $\vec{a}$, so $\overrightarrow{BC} = -\vec{a}$.

b. $\overrightarrow{GF}$ is the same as $\vec{a}$, so $\overrightarrow{GF} = \vec{a}$.

c. In rectangle $OABC$, $\overrightarrow{OB}$ is the diagonal of the rectangle, so $\overrightarrow{OB} = \vec{a} + \vec{b}$.

d. Since $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{AO} = -\vec{a}$, $\overrightarrow{AC} = -\vec{a} + \vec{b}$ or $\overrightarrow{AC} = \vec{b} - \vec{a}$.

e. Since $\overrightarrow{BG} = \overrightarrow{BC} + \overrightarrow{CG}$, $\overrightarrow{BC} = -\vec{a}$, and $\overrightarrow{CG} = \vec{c}$, $\overrightarrow{BG} = -\vec{a} + \vec{c}$ or $\overrightarrow{BG} = \vec{c} - \vec{a}$.

f. Since $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF}$, $\overrightarrow{OB} = \vec{a} + \vec{b}$, and $\overrightarrow{BF} = \vec{c}$, $\overrightarrow{OF} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + \vec{b} + \vec{c}$.

In the next example, we demonstrate how vectors might be used in a situation involving velocity.

EXAMPLE 4 Solving a problem using vectors

An airplane heads due south at a speed of 300 km/h and meets a wind from the west at 100 km/h. What is the resultant velocity of the airplane (relative to the ground)?

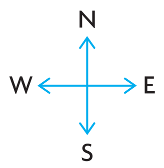
Solution

Let $\vec{v}$ represent the air speed of the airplane (velocity of the airplane without the wind).

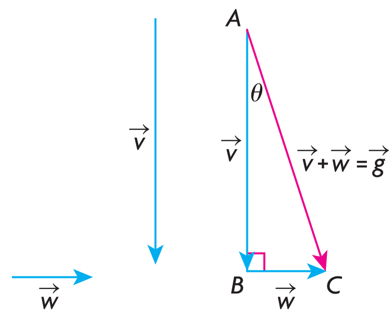
Let $\vec{w}$ represent the velocity of the wind.

Let $\vec{g}$ represent the ground speed of the airplane (the resultant velocity of the airplane with the wind taken into account relative to a fixed point on the ground).

The vectors are drawn so that their lengths are proportionate to their speed. That is to say, $|\vec{v}| = 3|\vec{w}|$.



In order to calculate the resultant ground velocity, the vectors are arranged head-to-tail as shown. The resultant ground velocity, $\vec{v} + \vec{w}$, is the hypotenuse of a right triangle.

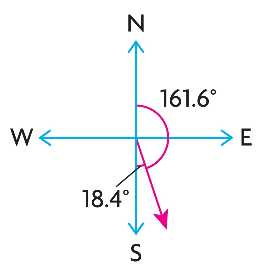


$$\text{So, } |\vec{g}|^2 = |\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$$
$$|\vec{v} + \vec{w}|^2 = 300^2 + 100^2$$
$$|\vec{v} + \vec{w}|^2 = 100\,000$$
$$|\vec{v} + \vec{w}| = \sqrt{100\,000}$$
$$\doteq 316.23$$

Since we are calculating the resultant ground velocity, we must also determine the new direction of the airplane. To do so, we must determine θ.

Thus, $\tan\theta = \dfrac{|\vec{w}|}{|\vec{v}|} = \dfrac{100}{300} = \dfrac{1}{3}$ and $\theta = \tan^{-1}\left(\dfrac{1}{3}\right) \doteq 18.4°$.

This means that the airplane is heading S18.4°E at a speed of 316.23 km/h. The wind has not only thrown the airplane off course, but it has also caused it to speed up. When we say the new direction of the airplane is S18.4°E, this means that the airplane is travelling in a south direction, 18.4° toward the east. This is illustrated in the following diagram.



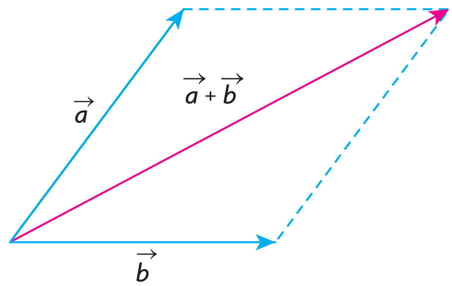
Other ways of stating this would be E71.6°S or a **bearing** of 161.6° (i.e., 161.6° rotated clockwise from due North).

In calculating the velocity of an object, such as an airplane, the velocity must always be calculated relative to some fixed object or some frame of reference. For example, if you are walking forward in an airplane at 5 km/h, your velocity relative to the airplane is 5 km/h in the same direction as the airplane, but relative to the ground, your velocity is 5 km/h in the same direction as the airplane plus the velocity of the airplane relative to the ground. If the airplane is travelling forward at 800 km/h, then your velocity relative to the ground is 805 km/h in the same direction as the airplane. In our example, the velocities given are measured relative to the ground, as is the final velocity. This is often referred to as the ground velocity.

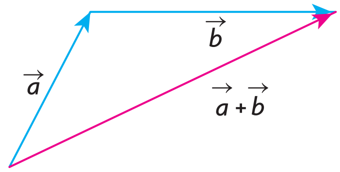
IN SUMMARY

Key Ideas

- To determine the sum of any two vectors $\vec{a}$ and $\vec{b}$, arranged tail-to-tail, complete the parallelogram formed by the two vectors. Their sum is the vector that is the diagonal of the constructed parallelogram.

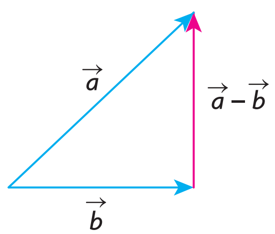


- The sum of the vectors $\vec{a}$ and $\vec{b}$ is also found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. The resultant is the vector from the tail of $\vec{a}$ to the head of $\vec{b}$.



Need to Know

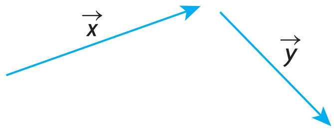
- When two opposite vectors are added, the resultant is the zero vector.
- The zero vector has a magnitude of 0 and no defined direction.
- To think about $\vec{a} - \vec{b}$, arrange the vectors tail to tail. $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is the vector from the head of $\vec{b}$ to the head of $\vec{a}$. This vector is also equivalent to $\vec{a} + \left(-\vec{b}\right)$.



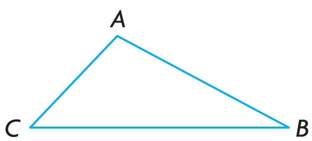
Exercise 6.2

PART A

1. The vectors $\vec{x}$ and $\vec{y}$ are drawn as shown below. Draw a vector equivalent to each of the following.



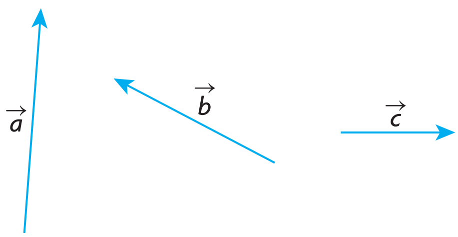
a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{y} - \vec{x}$ d. $(-\vec{y}) + (-\vec{x})$



2. Given $\triangle ABC$, draw appropriate arrows on the sides of the triangle and give a single vector equivalent for each of the following:

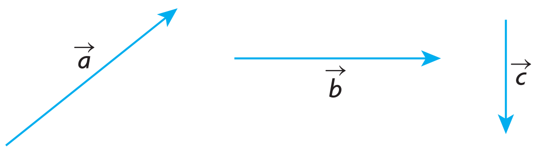
a. $\overrightarrow{BC} + \overrightarrow{CA}$ b. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ c. $\overrightarrow{AB} - \overrightarrow{AC}$ d. $-\overrightarrow{BC} + \overrightarrow{BA}$

3. Given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, construct vectors equivalent to each of the following.



a. $\vec{a} + \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} - \vec{c}$ c. $\vec{a} - \vec{b} - \vec{c}$ d. $\vec{a} - \vec{b} + \vec{c}$

4. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



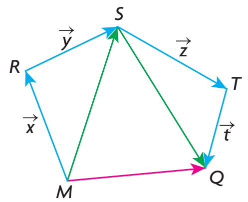
a. Construct $\vec{a} + \left(\vec{b} + \vec{c}\right)$.

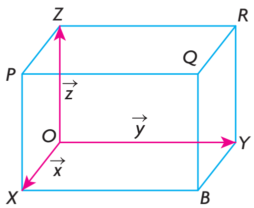
b. Construct $\left(\vec{a} + \vec{b}\right) + \vec{c}$.

c. Compare your results from parts a. and b.

5. Each of the following vector expressions can be simplified and written as a single vector. Write the single vector corresponding to each expression and illustrate your answer with a sketch.
   a. $\overrightarrow{PQ} - \overrightarrow{RQ} + \overrightarrow{RS}$
   b. $\overrightarrow{PS} + \overrightarrow{RQ} - \overrightarrow{RS} - \overrightarrow{PQ}$

C 6. Explain why $(\vec{x} + \vec{y}) + (\vec{z} + \vec{t})$ equals $\overrightarrow{MQ}$ in the following diagram.

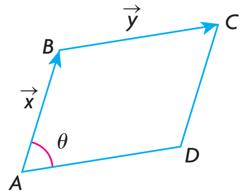




7. The rectangular box shown is labelled with $\overrightarrow{OX} = \vec{x}$, $\overrightarrow{OY} = \vec{y}$, and $\overrightarrow{OZ} = \vec{z}$.

   Express each of the following vectors in terms of $\vec{x}$, $\vec{y}$, and $\vec{z}$.
   a. $\overrightarrow{BY}$ b. $\overrightarrow{XB}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{XY}$ e. $\overrightarrow{OQ}$ f. $\overrightarrow{QZ}$ g. $\overrightarrow{XR}$ h. $\overrightarrow{PO}$

8. In the diagram, $\vec{x}$ and $\vec{y}$ represent adjacent sides of a parallelogram.



   a. Draw vectors that are equivalent to $\vec{x} - \vec{y}$ and $\vec{y} - \vec{x}$.
   b. To calculate $|\vec{x} - \vec{y}|$, the formula $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\theta$ is used. Show, by drawing the vector $\vec{y} - \vec{x}$, that the formula for calculating $|\vec{y} - \vec{x}|$ is the same.

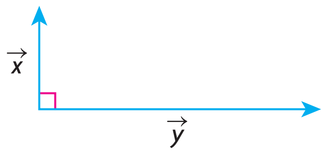
PART B

9. In still water, Maria can paddle at the rate of 7 km/h. The current in which she paddles has a speed of 4 km/h.
   a. At what velocity does she travel downstream?
   b. Using vectors, draw a diagram that illustrates her velocity going downstream.
   c. If Maria changes her direction and heads upstream instead, what is her speed? Using vectors, draw a diagram that illustrates her velocity going upstream.

10. a. In the example involving a ship being towed by the two tugboats, draw $\vec{f_1}$, $\vec{f_2}$, θ, and $\vec{f_1} + \vec{f_2}$.
    b. Show that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}| + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.

A 11. A small airplane is flying due north at 150 km/h when it encounters a wind of 80 km/h from the east. What is the resultant ground velocity of the airplane?

K 12. $|\vec{x}| = 7$ and $|\vec{y}| = 24$. If the angle between these vectors is 90°, determine $|\vec{x} + \vec{y}|$ and calculate the angle between $\vec{x}$ and $\vec{x} + \vec{y}$.

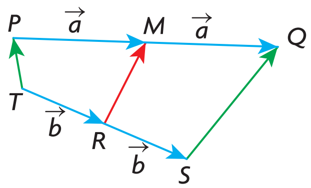


13. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two unit vectors (vectors with magnitude 1) with an angle of 150° between them. Calculate $|\overrightarrow{AB} + \overrightarrow{AC}|$.

T 14. $ABCD$ is a parallelogram whose diagonals BD and AC meet at the point E. Prove that $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \vec{0}$.

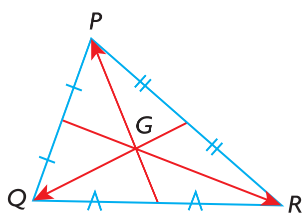
PART C

15. M is the midpoint of line segment PQ, and R is the midpoint of TS. If $\overrightarrow{PM} = \overrightarrow{MQ} = \vec{a}$ and $\overrightarrow{TR} = \overrightarrow{RS} = \vec{b}$, as shown, prove that $2\overrightarrow{RM} = \overrightarrow{TP} + \overrightarrow{SQ}$.



16. Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are such that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. Show that $\vec{a}$ and $\vec{b}$ must represent the sides of a rectangle.

17. The three medians of $\triangle PQR$ meet at a common point G. The point G divides each median in a 2:1 ratio. Prove that $\overrightarrow{GP} + \overrightarrow{GQ} + \overrightarrow{GR} = \vec{0}$.



Section 6.3—Multiplication of a Vector by a Scalar

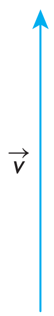
In this section, we will demonstrate the effect of multiplying a vector, $\vec{a}$, by a number k to produce a new vector, $k\vec{a}$. The number k used for multiplication is called a scalar and can be any real number. Previously, the distinction was made between scalars and vectors by saying that scalars have magnitude and not direction, whereas vectors have both. In this section, we are giving a more general meaning to the word *scalar* so that it means any real number. Since real numbers have magnitude (size) but not direction, this meaning is consistent with our earlier understanding.

Examining Scalar Multiplication

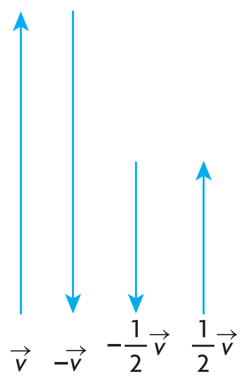
Multiplying $\vec{a}$ by different values of k can affect the direction and magnitude of a vector, depending on the values of k that are chosen. The following example demonstrates the effect on a velocity vector when it is multiplied by different scalars.

EXAMPLE 1 **Reasoning about the meaning of scalar multiplication**

An airplane is heading due north at 1000 km/h. The airplane's velocity is represented by $\vec{v}$. Draw the vectors $-\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{1}{2}\vec{v}$ and give an interpretation for each.



Scale: 1 cm is equivalent to 250 km/h



Solution

We interpret the vectors in the following way:

$\vec{v}$: the velocity vector for an airplane heading due north at 1000 km/h

$-\vec{v}$: the velocity vector for an airplane heading due south at 1000 km/h

$\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due north at 500 km/h

$-\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due south at 500 km/h

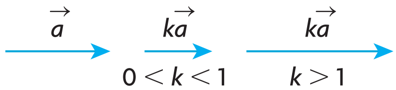
The previous example illustrates how multiplication of a vector by different values of a scalar k can change the magnitude and direction of a vector. The effect of multiplying a vector by a scalar is summarized as follows.

Multiplication of a Vector by a Scalar

For the vector $k\vec{a}$, where k is a scalar and $\vec{a}$ is a nonzero vector:

1. If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.

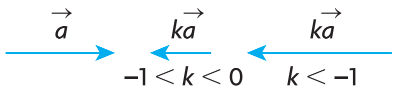
   For $k > 0$, two different possibilities will be considered and are illustrated in the following diagram:

   For $0 < k < 1$, the vector is shortened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{1}{2}\vec{a}$ is half the length of the original vector and in the same direction, i.e., $\left|\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$.

   For $k > 1$, the vector is lengthened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ and in the same direction, i.e., $\left|\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

2. If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$. Again, two situations will be considered for $k < 0$.

   For $-1 < k < 0$, the vector is shortened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{1}{2}\vec{a}$ is half the length of the original vector $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$. In the situation where $k < -1$, the vector is lengthened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

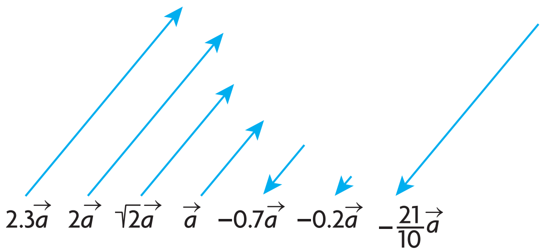
Collinear Vectors

A separate comment should be made about the cases $k = 0$ and $k = -1$.

If we multiply any vector $\vec{a}$ by the scalar 0, the result is always the zero vector, i.e., $0\vec{a} = \vec{0}$. Note that the right side of this equation is a vector, not a scalar.

When we multiply a vector by -1, i.e., $(-1)\vec{a}$, we normally write this as $-\vec{a}$. When any vector is multiplied by -1, its magnitude is unchanged but the direction changes to the opposite. For example, the vectors $-4\vec{a}$ and $4\vec{a}$ have the same magnitude (length) but are opposite.

The effect of multiplying a vector, $\vec{a}$, by different scalars is shown below.



When two vectors are parallel or lie on the same straight line, these vectors are described as being **collinear**. They are described as being collinear because they can be translated so that they lie in the same straight line. Vectors that are not collinear are not parallel. All of the vectors shown above are scalar multiples of $\vec{a}$ and are collinear. When discussing vectors, the terms *parallel* and *collinear* are used interchangeably.

Two vectors u and v are collinear if and only if it is possible to find a nonzero scalar k such that $\vec{u} = k\vec{v}$.

In the following example, we combine concepts learned in the previous section with those introduced in this section.

EXAMPLE 2 Selecting a strategy to determine the magnitude and direction of a vector

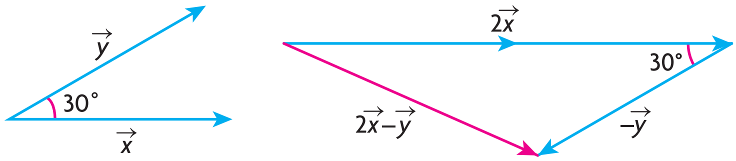
The vectors $\vec{x}$ and $\vec{y}$ are unit vectors (vectors with magnitude 1) that make an angle of 30° with each other.

a. Calculate the value of $|2\vec{x} - \vec{y}|$.

b. Determine the direction of $2\vec{x} - \vec{y}$.

Solution

a.

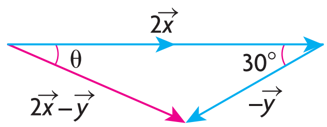


To calculate the value of $|2\vec{x} - \vec{y}|$, construct $2\vec{x} - \vec{y}$ by drawing $2\vec{x}$ and $-\vec{y}$ head-to-tail and then adding them.

Using the cosine law, $|2\vec{x} - \vec{y}|^2 = |2\vec{x}|^2 + |-\vec{y}|^2 - 2|2\vec{x}||-\vec{y}|\cos 30°$

$$|2\vec{x} - \vec{y}|^2 = 2^2 + 1^2 - 2(2)(1)\frac{\sqrt{3}}{2}$$
$$|2\vec{x} - \vec{y}|^2 = 5 - 2\sqrt{3}$$
$$|2\vec{x} - \vec{y}| = \sqrt{5 - 2\sqrt{3}}$$
$$\text{Therefore, } |2\vec{x} - \vec{y}| \doteq 1.24.$$

b. To determine the direction of $2\vec{x} - \vec{y}$, we will calculate θ using the sine law and describe the direction relative to the
the
direction of $\vec{x}$.



$$\frac{\sin\theta}{|-\vec{y}|} = \frac{\sin 30°}{|2\vec{x} - \vec{y}|}$$

$$\frac{\sin\theta}{1} \doteq \frac{\sin 30°}{1.24}$$

$$\theta = \sin^{-1}\left(\frac{\sin 30°}{1.24}\right)$$

$$\theta \doteq 23.8°$$

Therefore, $2\vec{x} - \vec{y}$ has a direction of 23.8° rotated clockwise relative to $\vec{x}$.

In many practical situations that involve velocities, we use specialized notation to describe direction. In the following example, we use this notation along with scalar multiplication to help illustrate its meaning.

EXAMPLE 3 Representing velocity using vectors

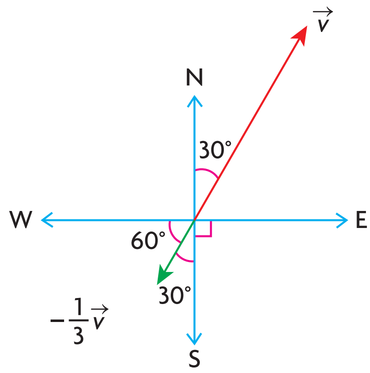
An airplane is flying in the direction $N30°E$ at an airspeed of 240 km/h. The velocity vector for this airplane is represented by $\vec{v}$.

a. Draw a sketch of $-\frac{1}{3}\vec{v}$ and state the direction of this vector.

b. For the vector $\frac{3}{2}\vec{v}$, state its direction and magnitude.

Solution

a.

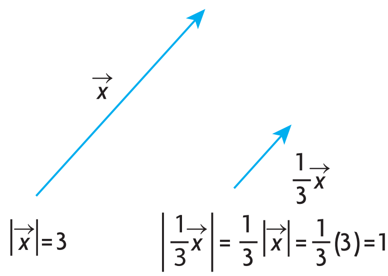


Scale: 1 cm is equivalent to 40 km/h

The vector $-\frac{1}{3}\vec{v}$ represents a speed of $\frac{1}{3}(240\text{ km/h}) = 80\text{ km/h}$ and points in the opposite direction as $\vec{v}$. The direction for this vector can be described as W60°S, S30°W, or a bearing of 210°.

b. The velocity vector $\frac{3}{2}\vec{v}$ represents a speed of $\frac{3}{2}(240\text{ km/h}) = 360\text{ km/h}$ in the same direction as $\vec{v}$.

It is sometimes useful to multiply the nonzero vector $\vec{x}$ by the scalar $\frac{1}{|\vec{x}|}$. When we multiply $\vec{x}$ by $\frac{1}{|\vec{x}|}$, we get the vector $\frac{1}{|\vec{x}|}\vec{x}$. This vector of length one and called a unit vector, which points in the same direction as $\vec{x}$.



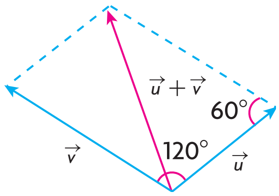
The concept of unit vector will prove to be very useful when we discuss applications of vectors.

EXAMPLE 4 Using a scalar to create a unit vector

Given that $|\vec{u}| = 4$ and $|\vec{v}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°, determine the unit vector in the same direction as $\vec{u} + \vec{v}$.

Solution

Draw a sketch and determine $|\vec{u} + \vec{v}|$.



Using the cosine law,

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$
$$|\vec{u} + \vec{v}|^2 = 4^2 + 5^2 - 2(4)(5)\cos 60°$$
$$|\vec{u} + \vec{v}|^2 = 21$$
$$|\vec{u} + \vec{v}| = \sqrt{21}$$

To create a unit vector in the same direction as $\vec{u} + \vec{v}$, multiply by the scalar equal to $\frac{1}{|\vec{u} + \vec{v}|}$. In this case, the unit vector is $\frac{1}{\sqrt{21}}(\vec{u} + \vec{v}) = \frac{1}{\sqrt{21}}\vec{u} + \frac{1}{\sqrt{21}}\vec{v}$.

IN SUMMARY

Key Idea

- For the vector $k\vec{a}$ where k is a scalar and $\vec{a}$ is a nonzero vector:
  - If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.
  - If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$.

Need to Know

- If two or more vectors are nonzero scalar multiples of the same vector, then all these vectors are collinear.
- $\frac{1}{|\vec{x}|}\vec{x}$ is a vector of length one, called a unit vector, in the direction of the nonzero vector $\vec{x}$.
- $-\frac{1}{|\vec{x}|}\vec{x}$ is a unit vector in the opposite direction of the nonzero vector $\vec{x}$.

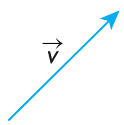
Exercise 6.3

PART A

1. Explain why the statement $\vec{a} = 2|\vec{b}|$ is not meaningful.

2. An airplane is flying at an airspeed of 300 km/h. Using a scale of 1 cm equivalent to 50 km/h, draw a velocity vector to represent each of the following:
   a. a speed of 150 km/h heading in the direction N45°E
   b. a speed of 450 km/h heading in the direction E15°S
   c. a speed of 100 km/h heading in an easterly direction
   d. a speed of 300 km/h heading on a bearing of 345°

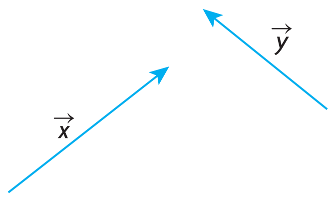
3. An airplane's direction is E25°N. Explain why this is the same as N65°E or a bearing of 65°.



4. The vector $\vec{v}$ has magnitude 2, i.e., $|\vec{v}| = 2$. Draw the following vectors and express each of them as a scalar multiple of $\vec{v}$.
   a. a vector in the same direction as $\vec{v}$ with twice its magnitude
   b. a vector in the same direction as $\vec{v}$ with one-half its magnitude
   c. a vector in the opposite direction as $\vec{v}$ with two-thirds its magnitude
   d. a vector in the opposite direction as $\vec{v}$ with twice its magnitude
   e. a unit vector in the same direction as $\vec{v}$

PART B

K 5. The vectors $\vec{x}$ and $\vec{y}$ are shown below. Draw a diagram for each of the following.



   a. $\vec{x} + 3\vec{y}$ b. $\vec{x} - 3\vec{y}$ c. $-2\vec{x} + \vec{y}$ d. $-2\vec{x} - \vec{y}$

6. Draw two vectors, $\vec{a}$ and $\vec{b}$, that do not have the same magnitude and are noncollinear. Using the vectors you drew, construct the following:
   a. $2\vec{a}$ b. $3\vec{b}$ c. $-3\vec{b}$ d. $2\vec{a} + 3\vec{b}$ e. $2\vec{a} - 3\vec{b}$

7. Three collinear vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, are such that $\vec{a} = \frac{2}{3}\vec{b}$ and $\vec{a} = \frac{1}{2}\vec{c}$.
   a. Determine integer values for m and n such that $m\vec{c} + n\vec{b} = \vec{0}$. How many values are possible for m and n to make this statement true?
   b. Determine integer values for d, e, and f such that $d\vec{a} + e\vec{b} + f\vec{c} = \vec{0}$. Are these values unique?

8. The two vectors $\vec{a}$ and $\vec{b}$ are collinear and are chosen such that $|\vec{a}| = |\vec{b}|$. Draw a diagram showing different possible configurations for these two vectors.

9. The vectors $\vec{a}$ and $\vec{b}$ are perpendicular. Are the vectors $4\vec{a}$ and $-2\vec{b}$ also perpendicular? Illustrate your answer with a sketch.

10. If the vectors $\vec{a}$ and $\vec{b}$ are noncollinear, determine which of the following pairs of vectors are collinear and which are not.

    a. $2\vec{a}, -3\vec{a}$ b. $2\vec{a}, 3\vec{b}$ c. $5\vec{a}, -\frac{3}{2}\vec{b}$ d. $-\vec{b}, 2\vec{b}$

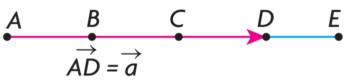
C 11. In the discussion, we defined $\frac{1}{|\vec{x}|}\vec{x}$. Using your own scale, draw your own vector to represent $\vec{x}$.

    a. Sketch $\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

    b. Sketch $-\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

12. Two vectors, $\vec{a}$ and $\vec{b}$, are such that $2\vec{a} = -3\vec{b}$. Draw a possible sketch of these two vectors. What is the value of m, if $|\vec{b}| = m|\vec{a}|$?

A 13. The points B, C, and D are drawn on line segment AE dividing it into four equal lengths. If $\overrightarrow{AD} = \vec{a}$, write each of the following in terms of $\vec{a}$ and $|\vec{a}|$.



    a. $\overrightarrow{EC}$ b. $\overrightarrow{BC}$ c. $|\overrightarrow{ED}|$ d. $|\overrightarrow{AC}|$ e. $\overrightarrow{AE}$

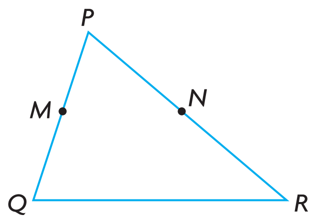
14. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 90° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

15. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 30° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

16. Prove that $\frac{1}{|\vec{a}|}\vec{a}$ is a unit vector pointing in the same direction as $\vec{a}$.
    (*Hint*: Let $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$ and then find the magnitude of each side of this equation.)

T 17. In $\triangle ABC$, a median is drawn from vertex A to the midpoint of BC, which is labelled D. If $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$, prove that $\overrightarrow{AD} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$.

18. Let PQR be a triangle in which M is the midpoint of PQ and N is the midpoint of PR. If $\overrightarrow{PM} = \vec{a}$ and $\overrightarrow{PN} = \vec{b}$, find vector expressions for $\overrightarrow{MN}$ and $\overrightarrow{QR}$ in terms of $\vec{a}$ and $\vec{b}$. What conclusions can be drawn about MN and QR? Explain.



19. Draw rhombus $ABCD$ where $AB = 3$ cm. For each of the following, name two vectors $\vec{u}$ and $\vec{v}$ in your diagram such that

a. $\vec{u} = \vec{v}$

b. $\vec{u} = 2\vec{v}$

c. $\vec{u} = -\vec{v}$

d. $\vec{u} = 0.5\vec{v}$

PART C

20. Two vectors, $\vec{x}$ and $\vec{y}$, are drawn such that $|\vec{x}| = 3|\vec{y}|$. Considering $m\vec{x} + n\vec{y} = \vec{0}$, determine all possible values for m and n such that

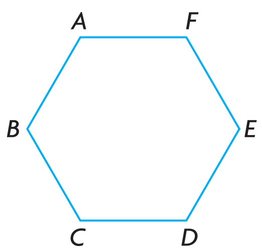
a. $\vec{x}$ and $\vec{y}$ are collinear

b. $\vec{x}$ and $\vec{y}$ are noncollinear

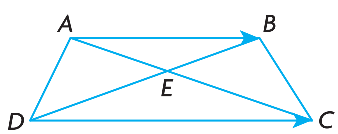
21. $ABCDEF$ is a regular hexagon such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

a. Express $\overrightarrow{CD}$ in terms of $\vec{a}$ and $\vec{b}$.

b. Prove that BE is parallel to CD and that $|\overrightarrow{BE}| = 2|\overrightarrow{CD}|$.



22. $ABCD$ is a trapezoid whose diagonals AC and BD intersect at the point E. If $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$, prove that $\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AD}$.



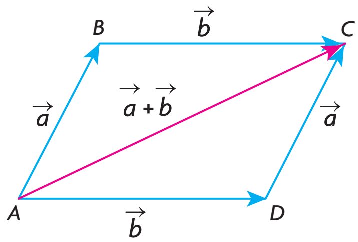
Section 6.4—Properties of Vectors

In previous sections, we developed procedures for adding and subtracting vectors and for multiplying a vector by a scalar. In carrying out these computations, certain assumptions were made about how to combine vectors without these rules being made explicit. Although these rules will seem apparent, they are important for understanding the basic structure underlying vectors, and for their use in computation. Initially, three specific rules for dealing with vectors will be discussed, and we will show that these rules are similar to those used in dealing with numbers and basic algebra. Later, we demonstrate an additional three rules.

Properties of Vector Addition

1. Commutative Property of Addition: When we are dealing with numbers, the order in which they are added does not affect the final answer. For example, if we wish to add 2 and 3, the answer is the same if it is written as $2 + 3$ or as $3 + 2$. In either case, the answer is 5. This property of being able to add numbers, in any chosen order, is called the commutative property of addition for real numbers. This property also works for algebra, because algebraic expressions are themselves numerical in nature. We make this assumption when simplifying in the following example: $2x + 3y + 3x = 2x + 3x + 3y = 5x + 3y$. Being able to switch the order like this allows us to carry out addition without concern for the order of the terms being added.

   This property that we have identified also holds for vectors, as can be seen in the following diagram:

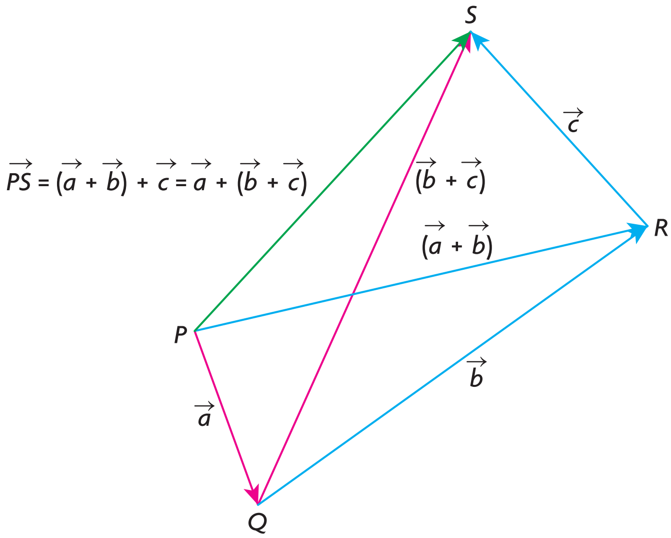
   From triangle ABC, using the vector addition rules,
   $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$
   From triangle ADC,
   $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$
   So, $\overrightarrow{AC} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$.

   Although vector addition is commutative, certain types of vector operations are not always commutative. We will see this when dealing with cross products in Chapter 7.

2. Associative Property of Addition: When adding numbers, the associative property is used routinely. If we wish to add 3, 5 and 8, for example, we can do this as $(3 + 5) + 8$ or as $3 + (5 + 8)$. Doing it either way, we get the answer 16. In doing this calculation, we are free to associate the numbers however we choose. This property also holds when adding algebraic expressions, such as $2x + 3x - 7x = (2x + 3x) - 7x = 2x + (3x - 7x) = -2x$.

   In adding vectors, we are free to associate them in exactly the same way as we do for numbers or algebraic expressions. For vectors, this property is stated as $\vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   We will use the following diagram and addition of vectors to demonstrate the associative property.

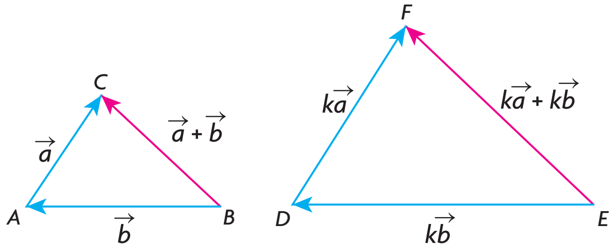


   In the diagram, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, and $\overrightarrow{RS} = \vec{c}$. From triangle PRQ, $\overrightarrow{PR} = \vec{a} + \vec{b}$, and then from triangle PSR, $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = (\vec{a} + \vec{b}) + \vec{c}$. Similarly, from triangle SQR, $\overrightarrow{QS} = \vec{b} + \vec{c}$ and then from triangle PQS, $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = \vec{a} + (\vec{b} + \vec{c})$. So $\overrightarrow{PS} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   It is interesting to note, just as we did with the commutative property, that the associative property holds for the addition of vectors but does not hold for certain kinds of multiplication.

3. Distributive Property of Addition: The distributive property is something we have used implicitly from the first day we thought about numbers or algebra. In calculating the perimeter of a rectangle with width w and length l, we write the perimeter as $P = 2(w + l) = 2w + 2l$. In this case, the 2 has been distributed across the brackets to give $2w$ and $2l$.

Demonstrating the distributive law for vectors depends on being able to multiply vectors by scalars and on the addition law for vectors.



In this diagram, we started with $\vec{a}$ and $\vec{b}$ and then multiplied each of them by k, a positive scalar, to give the vectors $k\vec{a}$ and $k\vec{b}$, respectively. In $\triangle ABC$, $\overrightarrow{BC} = \vec{a} + \vec{b}$, and in $\triangle DEF$, $\overrightarrow{EF} = k\vec{a} + k\vec{b}$. However, the two triangles are similar, so $\overrightarrow{EF} = k(\vec{a} + \vec{b})$. Since $\overrightarrow{EF} = k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, we have shown that the distributive law is true for $k > 0$ and any pair of vectors.

Although we chose k to be a positive number, we could have chosen any real number for k.

Properties of Vector Addition

1. Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
3. Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$

EXAMPLE 1 Selecting appropriate vector properties to determine an equivalent vector

Simplify the following expression: $3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$.

Solution

$3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$

$= 6\vec{a} + 3\vec{b} + 3\vec{c} - \vec{a} - 3\vec{b} + 2\vec{c}$ (Distributive property)

$= 6\vec{a} - \vec{a} + 3\vec{b} - 3\vec{b} + 3\vec{c} + 2\vec{c}$ (Commutative property)

$= (6 - 1)\vec{a} + (3 - 3)\vec{b} + (3 + 2)\vec{c}$ (Distributive property for scalars)

$= 5\vec{a} + \vec{0} + 5\vec{c}$

$= 5\vec{a} + 5\vec{c}$

In doing the calculation in Example 1, assumptions were made that are implicit but that should be stated.

Further Laws of Vector Addition and Scalar Multiplication

1. Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
2. Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
3. Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

It is important to be aware of all these properties when calculating, but the properties can be assumed without having to refer to them for each simplification.

EXAMPLE 2 Selecting appropriate vector properties to create new vectors

If $\vec{x} = 3\vec{i} - 4\vec{j} + \vec{k}$, $\vec{y} = \vec{j} - 5\vec{k}$, and $\vec{z} = -\vec{i} - \vec{j} + 4\vec{k}$, determine each of the following in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{x} - 2\vec{y} + 3\vec{z}$

Solution

a. $$\begin{aligned}\vec{x} + \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) + (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{j} + \vec{k} - 5\vec{k}\\ &= 3\vec{i} - 3\vec{j} - 4\vec{k}\end{aligned}$$

b. $$\begin{aligned}\vec{x} - \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) - (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} - \vec{j} + \vec{k} + 5\vec{k}\\ &= 3\vec{i} - 5\vec{j} + 6\vec{k}\end{aligned}$$

c. $$\begin{aligned}\vec{x} - 2\vec{y} + 3\vec{z} &= (3\vec{i} - 4\vec{j} + \vec{k}) - 2(\vec{j} - 5\vec{k}) + 3(-\vec{i} - \vec{j} + 4\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{k} - 2\vec{j} + 10\vec{k} - 3\vec{i} - 3\vec{j} + 12\vec{k}\\ &= -9\vec{j} + 23\vec{k}\end{aligned}$$

As stated previously, it is not necessary to state the rules as we simplify, and furthermore, it is better to try to simplify without writing in every step.

The rules that were developed in this section will prove useful as we move ahead. They are necessary for our understanding of linear combinations, which will be dealt with later in this chapter.

IN SUMMARY

Key Idea

- Properties used to evaluate numerical expressions and simplify algebraic expressions also apply to vector addition and scalar multiplication.

Need to Know

- Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$
- Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
- Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
- Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

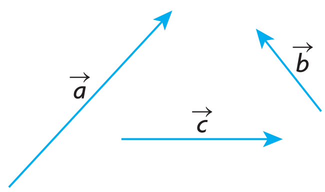
Exercise 6.4

PART A

1. If $*$ is an operation on a set, S, the element x, such that $a * x = a$, is called the identity element for the operation $*$.
   a. For the addition of numbers, what is the identity element?
   b. For the multiplication of numbers, what is the identity element?
   c. For the addition of vectors, what is the identity element?
   d. For scalar multiplication, what is the identity element?

2. Illustrate the commutative law for two vectors that are perpendicular.

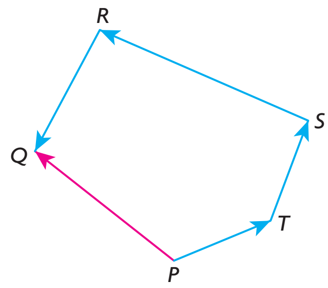
C 3. Redraw the following three vectors and illustrate the associative law.



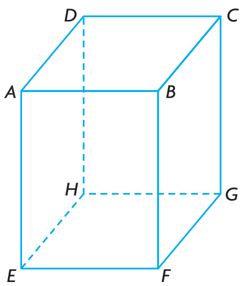
4. With the use of a diagram, show that the distributive law, $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, holds where $k < 0$, $k \in \mathbf{R}$.

PART B

K 5. Using the given diagram, show that the following is true.



$$\begin{aligned}\overrightarrow{PQ} &= (\overrightarrow{RQ} + \overrightarrow{SR}) + \overrightarrow{TS} + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + (\overrightarrow{SR} + \overrightarrow{TS}) + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + \overrightarrow{SR} + (\overrightarrow{TS} + \overrightarrow{PT})\end{aligned}$$



6. $ABCDEFGH$ is a rectangular prism.
   a. Write a single vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GH} + \overrightarrow{HD} + \overrightarrow{DC}$.
   b. Write a vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GD} + \overrightarrow{DE}$.
   c. Is it true that $|\overrightarrow{HB}| = |\overrightarrow{GA}|$? Explain.

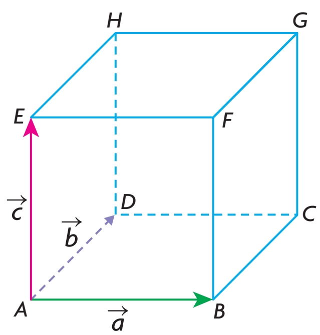
7. Write the following vector in simplified form:
   $3(\vec{a} - 2\vec{b} - 5\vec{c}) - 3(2\vec{a} - 4\vec{b} + 2\vec{c}) - (\vec{a} - 3\vec{b} + 3\vec{c})$

8. If $\vec{a} = 3\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + 3\vec{j} - \vec{k}$, express each of the following in terms of $\vec{i}, \vec{j}$, and $\vec{k}$.
   a. $2\vec{a} - 3\vec{b}$
   b. $\vec{a} + 5\vec{b}$
   c. $2(\vec{a} - 3\vec{b}) - 3(-2\vec{a} - 7\vec{b})$

T 9. If $2\vec{x} + 3\vec{y} = \vec{a}$ and $-\vec{x} + 5\vec{y} = 6\vec{b}$, express $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

10. If $\vec{x} = \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z}$, $\vec{x} - \vec{y} = \vec{a}$, and $\vec{y} - \vec{z} = \vec{b}$, show that $\vec{a} = -\frac{1}{3}\vec{b}$.

A 11. A cube is constructed from the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, as shown below.



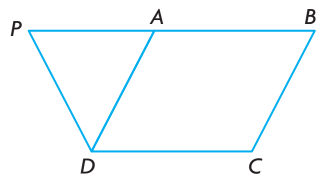
   a. Express each of the diagonals $\overrightarrow{AG}$, $\overrightarrow{BH}$, $\overrightarrow{CE}$, and $\overrightarrow{DF}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   b. Is $|\overrightarrow{AG}| = |\overrightarrow{BH}|$? Explain.

PART C

12. In the trapezoid $TXYZ$, $\overrightarrow{TX} = 2\overrightarrow{ZY}$. If the diagonals meet at O, find an expression for $\overrightarrow{TO}$ in terms of $\overrightarrow{TX}$ and $\overrightarrow{TZ}$.

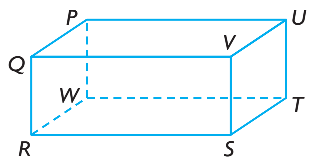
Mid-Chapter Review

1. $ABCD$ is a parallelogram, and $|\overrightarrow{PD}| = |\overrightarrow{DA}|$.



   a. Determine which vectors (if any) are equal to $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AD}$, $\overrightarrow{CB}$, and $\overrightarrow{AP}$.
   b. Explain why $|\overrightarrow{PD}| = |\overrightarrow{BC}|$.

2. The diagram below represents a rectangular prism. State a single vector equal to each of the following.



   a. $\overrightarrow{RQ} + \overrightarrow{RS}$
   b. $\overrightarrow{RQ} + \overrightarrow{QV}$
   c. $\overrightarrow{PW} + \overrightarrow{WS}$
   d. $(\overrightarrow{RQ} + \overrightarrow{RS}) + \overrightarrow{VU}$
   e. $\overrightarrow{PW} - \overrightarrow{VP}$
   f. $\overrightarrow{PW} + \overrightarrow{WR} + \overrightarrow{RQ}$

3. Two vectors, $\vec{a}$ and $\vec{b}$, have a common starting point with an angle of 120° between them. The vectors are such that $|\vec{a}| = 3$ and $|\vec{b}| = 4$.
   a. Calculate $|\vec{a} + \vec{b}|$.
   b. Calculate the angle between $\vec{a}$ and $\vec{a} + \vec{b}$.

4. Determine all possible values for t if the length of the vector $\vec{x} = t\vec{y}$ is $4|\vec{y}|$.

5. $PQRS$ is a quadrilateral where A, B, C, and D are the midpoints of SP, PQ, QR, and RS, respectively. Prove, using vector methods, that $ABCD$ is a parallelogram.

6. Given that $|\vec{u}| = 8$ and $|\vec{v}| = 10$ and the angle between vectors $\vec{u}$ and $\vec{v}$ is 60° determine:
   a. $|\vec{u} - \vec{v}|$
   b. the direction of $\vec{u} - \vec{v}$ relative to $\vec{u}$
   c. the unit vector in the direction of $\vec{u} + \vec{v}$
   d. $|5\vec{u} + 2\vec{v}|$

7. The vectors $\vec{p}$ and $\vec{q}$ are distinct unit vectors that are placed in a tail-to-tail position. If these two vectors have an angle of 60° between them, determine $|2\vec{p} - \vec{q}|$.

8. The vector $\vec{m}$ is collinear (parallel) to $\vec{n}$ but in the opposite direction. Express the magnitude of $\vec{m} + \vec{n}$ in terms of the magnitudes of $\vec{m}$ and $\vec{n}$.

9. $ABCD$ is a parallelogram. If $\overrightarrow{AB} = \vec{x}$ and $\overrightarrow{DA} = \vec{y}$, express $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{BD}$, and $\overrightarrow{AC}$ in terms of $\vec{x}$ and $\vec{y}$.

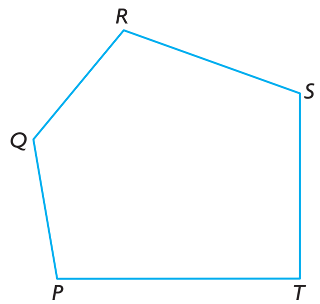
10. If A, B, and C are three collinear points with B at the midpoint of AC, and O is any point not on the line AC, prove that $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OB}$. (*Hint*: $\overrightarrow{AB} = \overrightarrow{BC}$.)

11. $ABCD$ is a quadrilateral with $\overrightarrow{AB} = \vec{x}$, $\overrightarrow{CD} = 2\vec{y}$, and $\overrightarrow{AC} = 3\vec{x} - \vec{y}$. Express $\overrightarrow{BD}$ and $\overrightarrow{BC}$ in terms of $\vec{x}$ and $\vec{y}$.

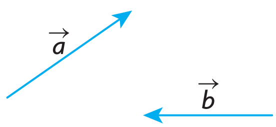
12. An airplane is heading due south at a speed of 500 km/h when it encounters a head wind from the south at 40 km/h. What is the resultant ground velocity of the airplane?

13. $PQRST$ is a pentagon. State a single vector that is equivalent to each of the following:

    a. $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RT}$ b. $\overrightarrow{PQ} + \overrightarrow{QR} - \overrightarrow{TR}$ c. $\overrightarrow{PR} - (\overrightarrow{PT} - \overrightarrow{ST})$



14. The vectors $\vec{a}$ and $\vec{b}$ are given below. Use these vectors to sketch each of the following.



    a. $\frac{1}{3}\vec{a} + \vec{b}$ b. $\frac{3}{2}\vec{a} - 2\vec{b}$ c. $-\vec{b} + \vec{a}$ d. $\frac{\vec{b} + \vec{a}}{2}$

15. $PQRS$ is a quadrilateral with $\overrightarrow{PQ} = 2\vec{a}$, $\overrightarrow{QR} = 3\vec{b}$, and $\overrightarrow{QS} = 3\vec{b} - 3\vec{a}$. Express $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\vec{a}$ and $\vec{b}$.

Section 6.5—Vectors in R^2 and R^3

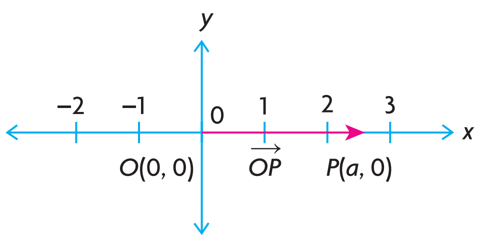
In the introduction to this chapter, we said vectors are important because of their application to a variety of different areas of study. In these areas, the value of using vectors is derived primarily from being able to consider them in coordinate form, or algebraic form, as it is sometimes described. Our experience with coordinate systems in mathematics thus far has been restricted to the xy-plane, but we will soon begin to see how ideas in two dimensions can be extended to higher dimensions and how this results in a greater range of applicability.

Introduction to Algebraic Vectors

Mathematicians started using coordinates to analyze physical situations in about the fourteenth century. However, a great deal of the credit for developing the methods used with coordinate systems should be given to the French mathematician Rene Descartes (1596–1650). Descartes was the first to realize that using a coordinate system would allow for the use of algebra in geometry. Since then, this idea has become important in the development of mathematical ideas in many areas. For our purposes, using algebra in this way leads us to the consideration of ideas involving vectors that otherwise would not be possible.

At the beginning of our study of algebraic vectors, there are a number of ideas that must be introduced and that form the foundation for what we are doing. After we start to work with vectors, these ideas are used implicitly without having to be restated each time.

One of the most important ideas that we must consider is that of the **unique** representation of vectors in the xy-plane. The unique representation of the vector $\overrightarrow{OP}$ is a matter of showing the unique representation of the point P because $\overrightarrow{OP}$ is determined by this point. The uniqueness of vector representation will be first considered for the **position vector** $\overrightarrow{OP}$, which has its head at the point $P(a, 0)$ and its tail at the origin $O(0, 0)$ shown on the x-axis below. The x-axis is the set of real numbers, $\mathbf{R}$, which is made up of rational and irrational numbers.

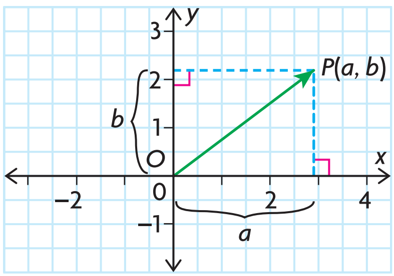


The point P is a distance of a units away from the origin and occupies exactly one position on the x-axis. Since each point P has a unique position on this axis, this implies that $\overrightarrow{OP}$ is also unique because this vector is determined by P.

The xy-plane is often referred to as R^2, which means that each of the x- and y-coordinates for any point on the plane is a real number. In technical terms, we would say that $R^2 = \{(x, y)$, where x and y are real numbers$\}$.

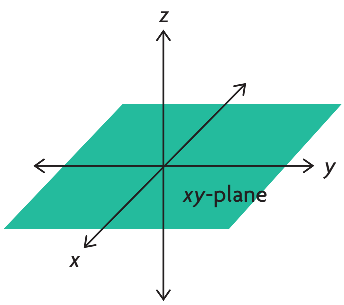
Points and Vectors in R^2

In the following diagram, $\overrightarrow{OP}$ can also be represented in component form by the vector defined as (a, b). This is a vector with its tail at $O(0, 0)$ and its head at $P(a, b)$. Perpendicular lines have been drawn from P to the two axes to help show the meaning of (a, b) in relation to $\overrightarrow{OP}$. a is called the x-component and b is the y-component of $\overrightarrow{OP}$. Again, each of the coordinates of this point is a unique real number and, because of this, the associated vector, $\overrightarrow{OP}$, has a unique location in the xy-plane.



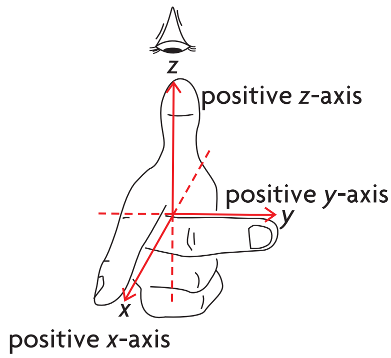
Points and Vectors in R^3

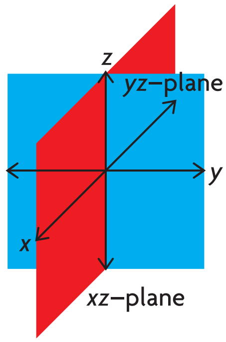
All planes in R^2 are flat surfaces that extend infinitely far in all directions and are said to be two-dimensional because each point is located using an x and a y or two coordinates. It is also useful to be able to represent points and vectors in three dimensions. The designation R^3 is used for three dimensions because each of the coordinates of a point $P(a, b, c)$, and its associated vector $\overrightarrow{OP} = (a, b, c)$, is a real number. Here, $O(0, 0, 0)$ is the origin in three dimensions. As in R^2, each point has a unique location in R^3, which again implies that each position vector $\overrightarrow{OP}$ is unique in R^3.



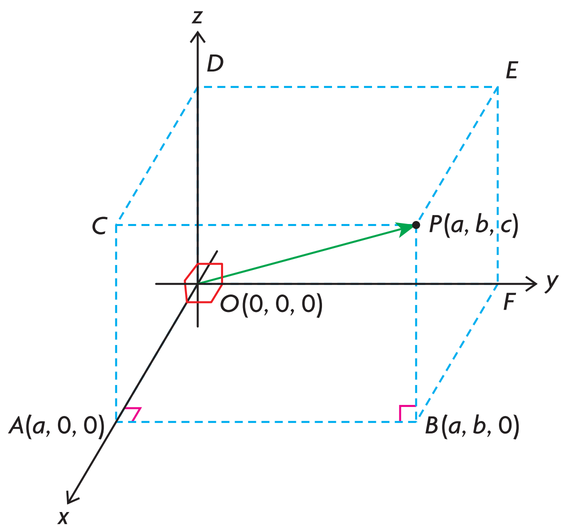
In placing points in R^3, we choose three axes called the x-, y-, and z-axis. Each pair of axes is perpendicular, and each axis is a copy of the real number line. There are several ways to choose the orientation of the positive axes, but we will use what is called a right-handed system. If we imagine ourselves looking down the positive z-axis onto the xy-plane so that, when the positive x-axis is rotated 90° *counterclockwise* it becomes coincident with the positive y-axis, then this is called a right-handed system. A right-handed system is normally what is used to represent R^3, and we will use this convention in this book.

Right-Handed System of Coordinates





Each pair of axes determines a plane. The xz-plane is determined by the x- and z-axes, and the yz-plane is determined by the y- and z-axes. Notice that, when we are discussing, for example, the xy-plane in R^3, this plane extends infinitely far in both the positive and negative directions. One way to visualize a right-handed system is to think of the y- and z-axes as lying in the plane of a book, determining the yz-plane, with the positive x-axis being perpendicular to the plane of the book and pointing directly toward you.



Each point $P(a, b, c)$ in R^3 has its location determined by an ordered triple. In the diagram above, the positive x-, y-, and z-axes are shown such that each pair of axes is perpendicular to the other and each axis represents a real number line. If we wish to locate $P(a, b, c)$, we move along the x-axis to $A(a, 0, 0)$, then in a direction perpendicular to the xz-plane, and parallel to the y-axis, to the point $B(a, b, 0)$. From there, we move in a direction perpendicular to the xy-plane and parallel to the z-axis to the point $P(a, b, c)$. This point is a vertex of a right rectangular prism.

Notice that the coordinates are signed, and so, for example, if we are locating the point $A(-2, 0, 0)$ we would proceed along the *negative* x-axis.

A source of confusion might be the meaning of $P(a, b, c)$ because it may be confused as either being a point or a vector. When referring to a vector, it will be stated explicitly that we are dealing with a vector and will be written as $\overrightarrow{OP} = (a, b, c)$, where a, b, and c are the x-, y-, and z-components respectively of the vector. In the diagram, this position vector is formed by joining the origin $O(0, 0, 0)$ to $P(a, b, c)$. When dealing with points, $P(a, b, c)$ will be named specifically as a point. In most situations, the distinction between the two should be evident from the context.

EXAMPLE 1 Reasoning about the coordinates of points in R^3

In the diagram on the previous page, determine the coordinates of C, D, E, and F.

Solution

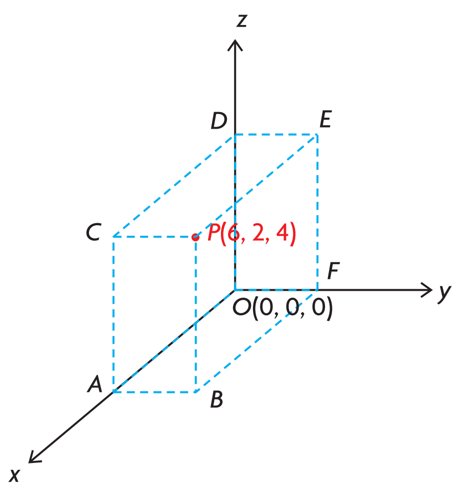
C is on the xz-plane and has coordinates $(a, 0, c)$, D is on the z-axis and has coordinates $(0, 0, c)$, E is on yz-plane and has coordinates $(0, b, c)$, and F is on the y-axis and has coordinates $(0, b, 0)$.

In the following example, we show how to locate points with the use of a rectangular box (prism) and line segments. It is useful, when we first start labelling points in R^3, to draw the box to gain familiarity with the coordinate system.

EXAMPLE 2 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point $P(6, 2, 4)$ is located in R^3. What are the coordinates of A, B, C, D, E, and F?

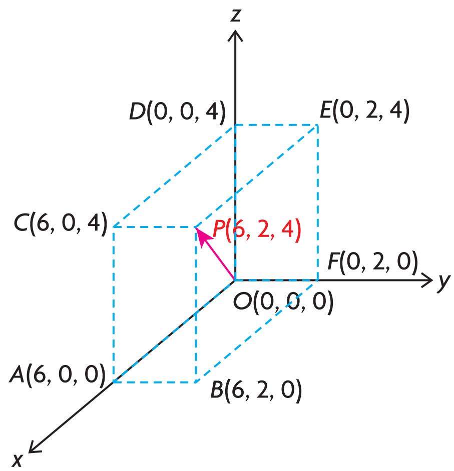
b. Draw the vector $\overrightarrow{OP}$.



Solution

a. $A(6, 0, 0)$ is a point on the positive x-axis, $B(6, 2, 0)$ is a point on the xy-plane, $C(6, 0, 4)$ is a point on the xz-plane, $D(0, 0, 4)$ is a point on the positive z-axis, $E(0, 2, 4)$ is a point on the yz-plane, and $F(0, 2, 0)$ is a point on the positive y-axis.

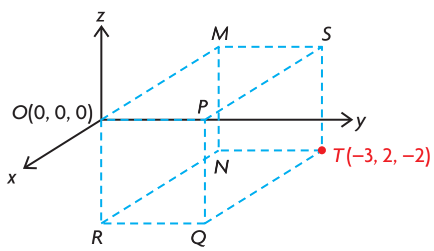
b. The vector $\overrightarrow{OP}$ is the vector associated with the point $P(a, b, c)$. It is the vector with its tail at the origin and its head at $P(6, 2, 4)$ and is named $\overrightarrow{OP} = (6, 2, 4)$.



EXAMPLE 3 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point T is located in R^3. What are the coordinates of P, Q, R, M, N, and S?

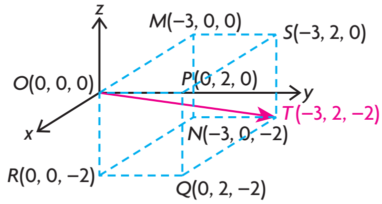
b. Draw the vector $\overrightarrow{OT}$.



Solution

a. The point $P(0, 2, 0)$ is a point on the positive y-axis. The point $Q(0, 2, -2)$ is on the yz-plane. The point $R(0, 0, -2)$ is on the negative z-axis. The point

$M(-3, 0, 0)$ is on the negative x-axis. The point $N(-3, 0, -2)$ is on the xz-plane. The point $S(-3, 2, 0)$ is on the xy-plane.



b. The vector $\overrightarrow{OT}$ is the vector associated with the point $T(-3, 2, -2)$ and is a vector with O as its tail and T as its head and is named $\overrightarrow{OT} = (-3, 2, -2)$.

When working with coordinate systems in R^3, it is possible to label planes using equations, which is demonstrated in the following example.

EXAMPLE 4 Representing planes in R^3 with equations

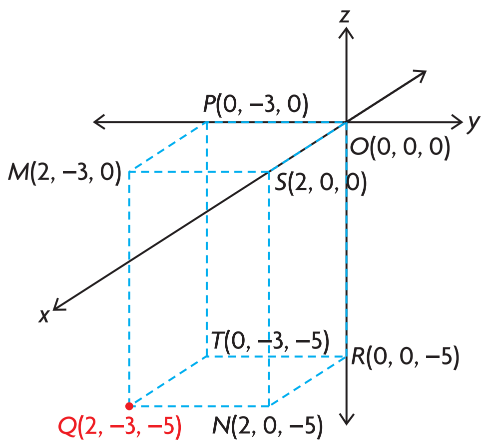
The point $Q(2, -3, -5)$ is shown in R^3.

a. Write an equation for the xy-plane.

b. Write an equation for the plane containing the points P, M, Q, and T.

c. Write a mathematical description of the set of points in rectangle $PMQT$.

d. What is the equation of the plane parallel to the xy-plane passing through $R(0, 0, -5)$?



Solution

a. Every point on the xy-plane has a z-component of 0, with every point on the plane having the form $(x, y, 0)$, where x and y are real numbers. The equation is $z = 0$.

b. Every point on this plane has a y-component equal to -3, with every point on the plane having the form $(x, -3, z)$, where x and z are real numbers. The equation is $y = -3$.

c. Every point in the rectangle has a y-component equal to -3, with every point in the rectangle having the form $(x, -3, z)$, where x and z are real numbers such that $0 \leq x \leq 2$ and $-5 \leq z \leq 0$.

d. Every point on this plane has a z-component equal to -5, with every point on the plane having the form $(x, y, -5)$, where x and y are real numbers. The equation is $z = -5$.

There is one further observation that should be made about placing points on coordinate axes. When using R^2 to describe the plane, which is two-dimensional, the exponent, n, in R^n is 2. Similarly, in three dimensions, the exponent is 3. The exponent in R^n corresponds to the number of dimensions of the coordinate system.

IN SUMMARY

Key Idea

- In R^2 or R^3, the location of every point is unique. As a result, every vector drawn with its tail at the origin and its head at a point is also unique. This type of vector is called a position vector.

Need to Know

- In R^2, $P(a, b)$ is a point that is a units from $O(0, 0)$ along the x-axis and b units parallel to the y-axis.
- The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0)$ and its head at $P(a, b)$. $\overrightarrow{OP} = (a, b)$
- In R^3, $P(a, b, c)$ is a point that is a units from $O(0, 0, 0)$ along the x-axis, b units parallel to the y-axis, and c units parallel to the z-axis. The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0, 0)$ and its head at $P(a, b, c)$. $\overrightarrow{OP} = (a, b, c)$
- In R^3, the three mutually perpendicular axes form a *right-handed* system.

Exercise 6.5

PART A

1. In R^3, is it possible to locate the point $P\left(\frac{1}{2}, \sqrt{-1}, 3\right)$? Explain.

2. a. Describe in your own words what it means for a point and its associated vector to be uniquely represented in R^3.

   b. Suppose that $\overrightarrow{OP} = (a, -3, c)$ and $\overrightarrow{OP} = (-4, b, -8)$. What are the corresponding values for a, b, and c? Why are we able to be certain that the determined values are correct?

3. a. The points $A(5, b, c)$ and $B(a, -3, 8)$ are located at the same point in R^3. What are the values of a, b, and c?

   b. Write the vector corresponding to $\overrightarrow{OA}$.

4. In R^3, each of the components for each point or vector is a real number. If we use the notation I^3, where I represents the set of integers, explain why $\overrightarrow{OP} = (-2, 4, -\sqrt{3})$ would not be an acceptable vector in I^3. Why is $\overrightarrow{OP}$ an acceptable vector in R^3?

5. Locate the points $A(4, -4, -2)$, $B(-4, 4, 2)$, and $C(4, 4, -2)$ using coordinate axes that you construct yourself. Draw the corresponding rectangular box (prism) for each, and label the coordinates of its vertices.

6. a. On what axis is $A(0, -1, 0)$ located? Name three other points on this axis.
   b. Name the vector $\overrightarrow{OA}$ associated with point A.

7. a. Name three vectors with their tails at the origin and their heads on the z-axis.
   b. Are the vectors you named in part a. collinear? Explain.
   c. How would you represent a general vector with its head on the z-axis and its tail at the origin?

K 8. Draw a set of x-, y-, and z-axes and plot the following points:

a. $A(1, 0, 0)$	c. $C(0, 0, -3)$	e. $E(2, 0, 3)$
b. $B(0, -2, 0)$	d. $D(2, 3, 0)$	f. $F(0, 2, 3)$

PART B

9. a. Draw a set of x-, y-, and z-axes and plot the following points: $A(3, 2, -4)$, $B(1, 1, -4)$, and $C(0, 1, -4)$.
   b. Determine the equation of the plane containing the points A, B, and C.

10. Plot the following points in R^3, using a rectangular prism to illustrate each coordinate.

a. $A(1, 2, 3)$	c. $C(1, -2, 1)$	e. $E(1, -1, 1)$
b. $B(-2, 1, 1)$	d. $D(1, 1, 1)$	f. $F(1, -1, -1)$

11. Name the vector associated with each point in question 10, express it in component form, and show the vectors associated with each of the points in the diagrams.

12. $P(2, a - c, a)$ and $Q(2, 6, 11)$ represent the same point in R^3.
    a. What are the values of a and c?
    b. Does $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$? Explain.

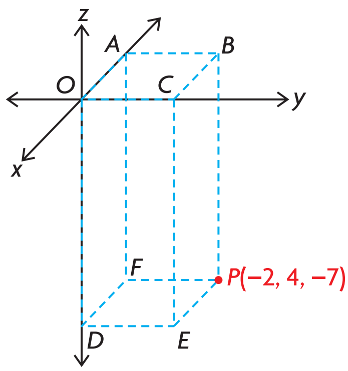
13. Each of the points $P(x, y, 0)$, $Q(x, 0, z)$, and $R(0, y, z)$ represent general points on three different planes. Name the three planes to which each corresponds.

C

14. a. What is the equation of the plane that contains the points $M(1, 0, 3)$, $N(4, 0, 6)$, and $P(7, 0, 9)$? Explain your answer.
    b. Explain why the plane that contains the points M, N, and P also contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

A

15. The point $P(-2, 4, -7)$ is located in R^3 as shown on the coordinate axes below.



    a. Determine the coordinates of points A, B, C, D, E, and F.
    b. What are the vectors associated with each of the points in part a.?
    c. How far below the xy-plane is the rectangle *DEPF?*
    d. What is the equation of the plane containing the points B, C, E, and P?
    e. Describe mathematically the set of points contained in rectangle *BCEP*.

16. Draw a diagram on the appropriate coordinate system for each of the following vectors:

a. $\overrightarrow{OP} = (4, -2)$	c. $\overrightarrow{OC} = (2, 4, 5)$	e. $\overrightarrow{OF} = (0, 0, 5)$
b. $\overrightarrow{OD} = (-3, 4)$	d. $\overrightarrow{OM} = (-1, 3, -2)$	f. $\overrightarrow{OJ} = (-2, -2, 0)$

PART C

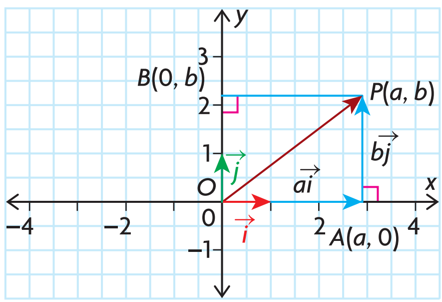
T

17. Draw a diagram illustrating the set of points $\{(x, y, z) \in R^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.
18. Show that if $\overrightarrow{OP} = (5, -10, -10)$, then $|\overrightarrow{OP}| = 15$.
19. If $\overrightarrow{OP} = (-2, 3, 6)$ and $B(4, -2, 8)$, determine the coordinates of point A such that $\overrightarrow{OP} = \overrightarrow{AB}$.

Section 6.6—Operations with Algebraic Vectors in R^2

In the previous section, we showed how to locate points and vectors in both two and three dimensions and then showed their connection to algebraic vectors. In R^2, we showed that $\overrightarrow{OP} = (a, b)$ was the vector formed when we joined the origin, $O(0, 0)$, to the point $P(a, b)$. We showed that the same meaning could be given to $\overrightarrow{OP} = (a, b, c)$, where the point $P(a, b, c)$ was in R^3 and $O(0, 0, 0)$ is the origin. In this section, we will deal with vectors in R^2 and show how a different representation of $\overrightarrow{OP} = (a, b)$ leads to many useful results.

Defining a Vector in R^2 in Terms of Unit Vectors



A second way of writing $\overrightarrow{OP} = (a, b)$ is with the use of the unit vectors $\vec{i}$ and $\vec{j}$.

The vectors $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ have magnitude 1 and lie along the positive x- and y-axes, respectively, as shown on the graph.

Our objective is to show how $\overrightarrow{OP}$ can be written in terms of $\vec{i}$ and $\vec{j}$. In the diagram, $\overrightarrow{OA} = (a, 0)$ and, since $\overrightarrow{OA}$ is just a scalar multiple of $\vec{i}$, we can write $\overrightarrow{OA} = a\vec{i}$. In a similar way, $\overrightarrow{OB} = b\vec{j}$. Using the triangle law of addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = a\vec{i} + b\vec{j}$. Since $\overrightarrow{OP} = (a, b)$, it follows that $(a, b) = a\vec{i} + b\vec{j}$.

This means that $\overrightarrow{OP} = (-3, 8)$ can also be written as $\overrightarrow{OP} = -3\vec{i} + 8\vec{j}$. Notice that this result allows us to write *all* vectors in the plane in terms of $\vec{i}$ and $\vec{j}$ and, just as before, their representation is unique.

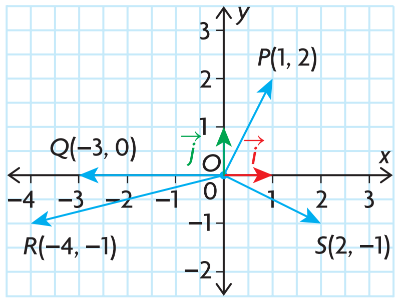
Representation of Vectors in R^2

The position vector $\overrightarrow{OP}$ can be represented as either $\overrightarrow{OP} = (a, b)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j}$, where $O(0, 0)$ is the origin, $P(a, b)$ is any point on the plane, and $\vec{i}$ and $\vec{j}$ are the standard unit vectors for R^2. Standard unit vectors, $\vec{i}$ and $\vec{j}$, are unit vectors that lie along the x- and y-axes, respectively, so $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Every vector in R^2, given in terms of its components, can also be written uniquely in terms of $\vec{i}$ and $\vec{j}$. For this reason, vectors $\vec{i}$ and $\vec{j}$ are also called the standard basis vectors in R^2.

EXAMPLE 1 Representing vectors in R^2 in two equivalent forms

a. Four position vectors, $\overrightarrow{OP} = (1, 2)$, $\overrightarrow{OQ} = (-3, 0)$, $\overrightarrow{OR} = (-4, -1)$, and $\overrightarrow{OS} = (2, -1)$, are shown. Write each of these vectors using the unit vectors $\vec{i}$ and $\vec{j}$.

b. The vectors $\overrightarrow{OA} = -\vec{i}$, $\overrightarrow{OB} = \vec{i} + 5\vec{j}$, $\overrightarrow{OC} = -5\vec{i} + 2\vec{j}$, and $\overrightarrow{OD} = \sqrt{2}\vec{i} - 4\vec{j}$ have been written using the unit vectors $\vec{i}$ and $\vec{j}$. Write them in component form (a, b).



Solution

a. $\overrightarrow{OP} = \vec{i} + 2\vec{j}$, $\overrightarrow{OQ} = -3\vec{i}$, $\overrightarrow{OR} = -4\vec{i} - \vec{j}$, $\overrightarrow{OS} = 2\vec{i} - \vec{j}$

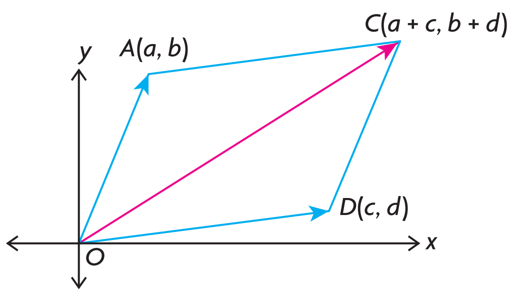
b. $\overrightarrow{OA} = (-1, 0)$, $\overrightarrow{OB} = (1, 5)$, $\overrightarrow{OC} = (-5, 2)$, and $\overrightarrow{OD} = (\sqrt{2}, -4)$

The ability to write vectors using $\vec{i}$ and $\vec{j}$ allows us to develop many of the same results with algebraic vectors that we developed with geometric vectors.

Addition of Two Vectors Using Component Form

We start by drawing the position vectors, $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OD} = (c, d)$, where A and D are any two points in R^2. For convenience, we choose these two points in the first quadrant. We rewrite each of the two position vectors, $\overrightarrow{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\overrightarrow{OD} = (c, d) = c\vec{i} + d\vec{j}$.

Adding these vectors gives
$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{OD} &= a\vec{i} + b\vec{j} + c\vec{i} + d\vec{j}\\ &= a\vec{i} + c\vec{i} + b\vec{j} + d\vec{j}\\ &= (a + c)\vec{i} + (b + d)\vec{j}\\ &= (a + c, b + d)\\ &= \overrightarrow{OC}\end{aligned}$$



To find $\overrightarrow{OC}$, it was necessary to use the commutative and distributive properties of vector addition, along with the ability to write vectors in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

To determine the sum of two vectors, $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OD} = (c, d)$, add their corresponding x- and y-components.
So, $\overrightarrow{OA} + \overrightarrow{OD} = (a, b) + (c, d) = (a + c, b + d) = \overrightarrow{OC}$

The process is similar for subtraction.
$\overrightarrow{OA} - \overrightarrow{OD} = (a, b) - (c, d) = (a - c, b - d)$

Scalar Multiplication of Vectors Using Components

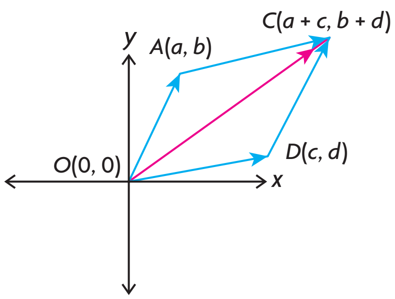
When dealing with geometric vectors, the meaning of multiplying a vector by a scalar was shown. The multiplication of a vector by a scalar in component form has the same meaning. In essence, if $\overrightarrow{OP} = (a, b)$, we wish to know how the coordinates of $m\overrightarrow{OP}$ are determined, where m is a real number. This can be determined by using various distributive properties for scalar multiplication of vectors along with the $\vec{i}$, $\vec{j}$ representation of a vector.

In algebraic form,
$$\begin{aligned} m\overrightarrow{OP} &= m(a, b) \\ &= m(a\vec{i} + b\vec{j}) \\ &= (ma)\vec{i} + (mb)\vec{j} \\ &= (ma, mb) \end{aligned}$$

To multiply an algebraic vector by a scalar, each component of the algebraic vector is multiplied by that scalar.

Adding Two Vectors in R^2

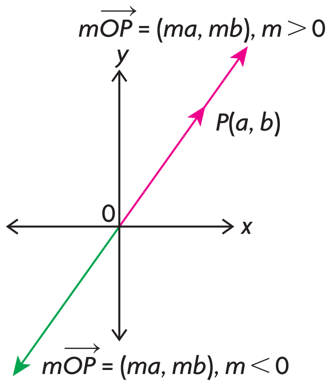
To determine the sum of two algebraic vectors, add their corresponding x- and y-components.



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OD} = (a + c, b + d)$

Multiplying a Vector by a Scalar in R^2

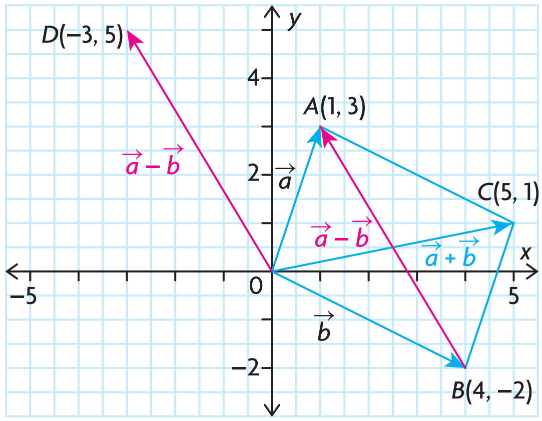
To multiply an algebraic vector by a scalar, multiply both x- and y-components by the scalar.



EXAMPLE 2 Representing the sum and difference of two algebraic vectors in R^2

Given $\vec{a} = \overrightarrow{OA} = (1, 3)$ and $\overrightarrow{OB} = \vec{b} = (4, -2)$, determine the components of $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, and illustrate each of these vectors on the graph.

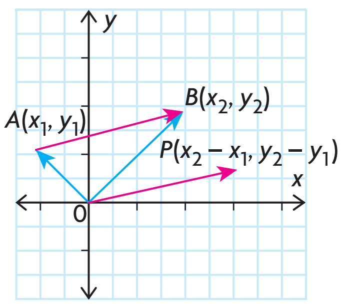
Solution



$$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = (1, 3) + (4, -2) = (1 + 4, 3 + (-2)) = (5, 1) = \overrightarrow{OC}$$
$$\vec{a} - \vec{b} = \overrightarrow{OA} - \overrightarrow{OB} = (1, 3) - (4, -2) = (1 - 4, 3 + 2) = (-3, 5) = \overrightarrow{OD}$$

From the diagram, we can see that $\vec{a} + \vec{b}$ and $\overrightarrow{BA}$ represent the diagonals of the parallelogram. It should be noted that the position vector, $\overrightarrow{OD}$, is a vector that is equivalent to diagonal $\overrightarrow{BA}$. The vector $\overrightarrow{OD} = \vec{a} - \vec{b}$ is described as a position vector because it has its tail at the origin and is equivalent to $\overrightarrow{BA}$, since their magnitudes are the same and they have the same direction.

Vectors in R^2 Defined by Two Points



$\overrightarrow{OP}$ is the position vector for $\overrightarrow{AB}$.

In considering the vector $\overrightarrow{AB}$, determined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$, an important consideration is to be able to find its related position vector and to calculate $|\overrightarrow{AB}|$. In order to do this, we use the triangle law of addition. From the diagram on the left, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$. Thus, the components of the algebraic vector are found by subtracting the coordinates of its tail from the coordinates of its head.

To determine $|\overrightarrow{AB}|$, use the Pythagorean theorem.

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The formula for determining $|\overrightarrow{AB}|$ is the same as the formula for finding the distance between two points.

Position Vectors and Magnitudes in R^2

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the vector $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ is its related position vector $\overrightarrow{OP}$, and $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

EXAMPLE 3

Using algebraic vectors to solve a problem

$A(-3, 7)$, $B(5, 22)$, and $C(8, 18)$ are three points in R^2.

a. Calculate the value of $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$, the perimeter of triangle ABC.

b. Calculate the value of $|\overrightarrow{AB} + \overrightarrow{BC}|$.

Solution

a. Calculate a position vector for each of the three sides.

$\overrightarrow{AB} = (5 - (-3), 22 - 7) = (8, 15)$, $\overrightarrow{BC} = (8 - 5, 18 - 22) = (3, -4)$, and $\overrightarrow{CA} = (-3 - 8, 7 - 18) = (-11, -11)$

$|\overrightarrow{AB}| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$, $|\overrightarrow{BC}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$, and $|\overrightarrow{CA}| = \sqrt{(-11)^2 + (-11)^2} = \sqrt{121 + 121} = \sqrt{242} \doteq 15.56$

The perimeter of the triangle is approximately 37.56.

b. Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, $\overrightarrow{AC} = -\overrightarrow{CA} = (11, 11)$, and $|\overrightarrow{AC}| = \sqrt{11^2 + 11^2} = \sqrt{242}$, then $|\overrightarrow{AC}| \doteq 15.56$. Note that $|\overrightarrow{AC}| = |\overrightarrow{CA}| \doteq 15.56$.

EXAMPLE 4

Selecting a strategy to combine two vectors

For the vectors $\vec{x} = 2\vec{i} - 3\vec{j}$ and $\vec{y} = -4\vec{i} - 3\vec{j}$, determine $|\vec{x} + \vec{y}|$ and $|\vec{x} - \vec{y}|$.

Solution

Method 1: (Component Form)

Since $\vec{x} = 2\vec{i} - 3\vec{j}$, $\vec{x} = (2, -3)$. Similarly, $\vec{y} = (-4, -3)$.

The sum is $\vec{x} + \vec{y} = (2, -3) + (-4, -3) = (-2, -6)$.

The difference is $\vec{x} - \vec{y} = (2, -3) - (-4, -3) = (6, 0)$.

Method 2: (Standard Unit Vectors)

The sum is
$\vec{x} + \vec{y} = (2\vec{i} - 3\vec{j}) + (-4\vec{i} - 3\vec{j}) = (2 - 4)\vec{i} + (-3 - 3)\vec{j} = -2\vec{i} - 6\vec{j}$.

The difference is
$\vec{x} - \vec{y} = (2\vec{i} - 3\vec{j}) - (-4\vec{i} - 3\vec{j}) = (2 + 4)\vec{i} + (-3 + 3)\vec{j} = 6\vec{i}$.

Thus, $|\vec{x} + \vec{y}| = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} \doteq 6.32$ and
$|\vec{x} - \vec{y}| = \sqrt{6^2} = \sqrt{36} = 6$.

EXAMPLE 5

Calculating the magnitude of a vector in R^2

If $\vec{a} = (5, -6)$, $\vec{b} = (-7, 3)$, and $\vec{c} = (2, 8)$, calculate $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right|$.

Solution

$$\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c} = (5, -6) - 3(-7, 3) - \frac{1}{2}(2, 8)$$
$$= (5, -6) + (21, -9) + (-1, -4) = (25, -19)$$

Thus, $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right| = \sqrt{25^2 + (-19)^2} = \sqrt{625 + 361} = \sqrt{986} \doteq 31.40$

IN SUMMARY

Key Ideas

- In R^2, $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Both are unit vectors on the x- and y-axes, respectively.
- $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$
- The vector between two points with its tail at $A(x_1, y_1)$ and head at $B(x_2, y_2)$ is determined as follows:

  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same: $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Need to Know

- If $\overrightarrow{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\overrightarrow{OD} = (c, d) = c\vec{i} + d\vec{j}$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + c, b + d)$.
- $m\overrightarrow{OP} = m(a, b) = (ma, mb)$

Exercise 6.6

PART A

1. For $A(-1, 3)$ and $B(2, 5)$, draw a coordinate plane and place the points on the graph.
   a. Draw vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$, and give vectors in component form equivalent to each of these vectors.
   b. Determine $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$.
   c. Calculate $|\overrightarrow{AB}|$ and state the value of $|\overrightarrow{BA}|$.

2. Draw the vector $\overrightarrow{OA}$ on a graph, where point A has coordinates $(6, 10)$.
   a. Draw the vectors $m\overrightarrow{OA}$, where $m = \frac{1}{2}, \frac{-1}{2}, 2$, and -2.
   b. Which of these vectors have the same magnitude?

3. For the vector $\overrightarrow{OA} = 3\vec{i} - 4\vec{j}$, calculate $|\overrightarrow{OA}|$.

4. a. If $a\vec{i} + 5\vec{j} = (-3, b)$, determine the values of a and b.
   b. Calculate $|(-3, b)|$ after finding b.

5. If $\vec{a} = (-60, 11)$ and $\vec{b} = (-40, -9)$, calculate each of the following:
   a. $|\vec{a}|$ and $|\vec{b}|$
   b. $|\vec{a} + \vec{b}|$ and $|\vec{a} - \vec{b}|$

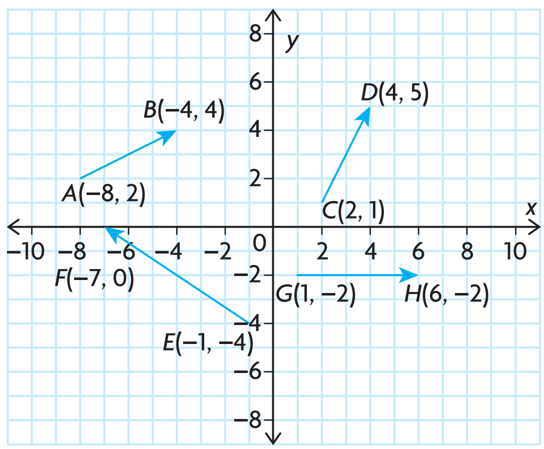
PART B

6. Find a single vector equivalent to each of the following:
   a. $2(-2, 3) + (2, 1)$
   b. $-3(4, -9) - 9(2, 3)$
   c. $\frac{-1}{2}(6, -2) + \frac{2}{3}(6, 15)$

K 7. Given $\vec{x} = 2\vec{i} - \vec{j}$ and $\vec{y} = -\vec{i} + 5\vec{j}$, find a vector equivalent to each of the following:
   a. $3\vec{x} - \vec{y}$
   b. $-(\vec{x} + 2\vec{y}) + 3(-\vec{x} - 3\vec{y})$
   c. $2(\vec{x} + 3\vec{y}) - 3(\vec{y} + 5\vec{x})$

8. Using $\vec{x}$ and $\vec{y}$ given in question 7, determine each of the following:
   a. $|\vec{x} + \vec{y}|$
   b. $|\vec{x} - \vec{y}|$
   c. $|2\vec{x} - 3\vec{y}|$
   d. $|3\vec{y} - 2\vec{x}|$

9. a. For each of the vectors shown below, determine the components of the related position vector.
   b. Determine the magnitude of each vector.



A 10. Parallelogram $OBCA$ is determined by the vectors $\overrightarrow{OA} = (6, 3)$ and $\overrightarrow{OB} = (11, -6)$.

a. Determine $\overrightarrow{OC}$, $\overrightarrow{BA}$, and $\overrightarrow{BC}$.

b. Verify that $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.

11. $\triangle ABC$ has vertices at $A(2, 3)$, $B(6, 6)$, and $C(-4, 11)$.

a. Sketch and label each of the points on a graph.

b. Calculate each of the lengths $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{CB}|$.

c. Verify that triangle ABC is a right triangle.

12. A parallelogram has three of its vertices at $A(-1, 2)$, $B(7, -2)$, and $C(2, 8)$.

a. Draw a grid and locate each of these points.

b. On your grid, draw the three locations for a fourth point that would make a parallelogram with points A, B, and C.

c. Determine all possible coordinates for the point described in part b.

13. Determine the value of x and y in each of the following:

a. $3(x, 1) - 5(2, 3y) = (11, 33)$

b. $-2(x, x + y) - 3(6, y) = (6, 4)$

C 14. Rectangle $ABCD$ has vertices at $A(2, 3)$, $B(-6, 9)$, $C(x, y)$, and $D(8, 11)$.

a. Draw a sketch of the points A, B, and D, and locate point C on your graph.

b. Explain how you can determine the coordinates of point C.

T 15. $A(5, 0)$ and $B(0, 2)$ are points on the x- and y-axes, respectively.

a. Find the coordinates of point $P(a, 0)$ on the x-axis such that $|\overrightarrow{PA}| = |\overrightarrow{PB}|$.

b. Find the coordinates of a point on the y-axis such that $|\overrightarrow{QB}| = |\overrightarrow{QA}|$.

PART C

16. Find the components of the unit vector in the direction opposite to $\overrightarrow{PQ}$, where $\overrightarrow{OP} = (11, 19)$ and $\overrightarrow{OQ} = (2, -21)$.

17. Parallelogram $OPQR$ is such that $\overrightarrow{OP} = (-7, 24)$ and $\overrightarrow{OR} = (-8, -1)$.

a. Determine the angle between the vectors $\overrightarrow{OR}$ and $\overrightarrow{OP}$.

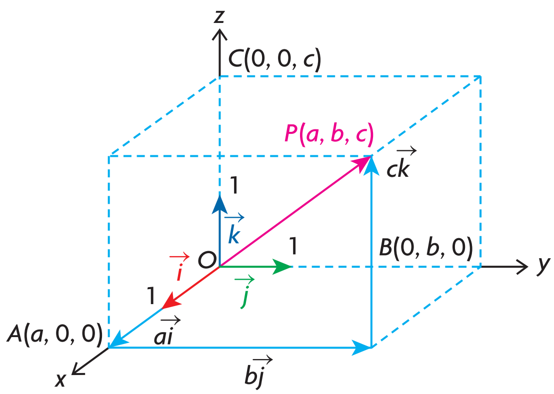
b. Determine the acute angle between the diagonals $\overrightarrow{OQ}$ and $\overrightarrow{RP}$.

Section 6.7—Operations With Vectors in R^3

The most important applications of vectors occur in R^3. In this section, results will be developed that will allow us to begin to apply ideas in R^3.

Defining a Vector in R^3 in Terms of Unit Vectors

In R^2, the vectors $\vec{i}$ and $\vec{j}$ were chosen as basis vectors. In R^3, the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ were chosen as basis vectors. These are vectors that each have magnitude 1, but now we introduce $\vec{k}$ as a vector that lies along the positive z-axis. If we use the same reasoning applied for two dimensions, then it can be seen that each vector $\overrightarrow{OP} = (a, b, c)$ can be written as $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Each of the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are shown below, as well as $\overrightarrow{OP} = (a, b, c)$.



From the diagram, $\overrightarrow{OA} = a\vec{i}$, $\overrightarrow{OB} = b\vec{j}$, and $\overrightarrow{OC} = c\vec{k}$. Using the triangle law of addition, $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Since $\overrightarrow{OP} = (a, b, c)$, we conclude that $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k} = (a, b, c)$. This result is analogous to the result derived for R^2.

Representation of Vectors in R^3

The position vector, $\overrightarrow{OP}$, whose tail is at the origin and whose head is located at point P, can be represented as either $\overrightarrow{OP} = (a, b, c)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$, where $O(0, 0, 0)$ is the origin, $P(a, b, c)$ is a point in R^3, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the standard unit vectors along the x-, y- and z- axes, respectively. This means that $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. Every vector in R^3 can be expressed uniquely in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

EXAMPLE 1 Representing vectors in R^3 in two equivalent forms

a. Write each of the vectors $\vec{OP} = (2, 1, -3)$, $\vec{OQ} = (-3, 1, -5)$, $\vec{OR} = (0, -2, 0)$, and $\vec{OS} = (3, 0, 0)$ using the standard unit vectors.

b. Express each of the following vectors in component form: $\vec{OP} = \vec{i} - 2\vec{j} - \vec{k}$, $\vec{OS} = 3\vec{k}$, $\vec{OM} = 2\vec{i} - 6\vec{k}$, and $\vec{ON} = \vec{i} - \vec{j} - 7\vec{k}$.

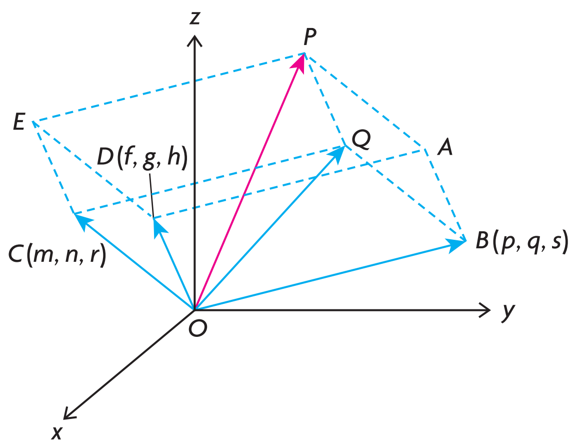
Solution

a. $\vec{OP} = 2\vec{i} + \vec{j} - 3\vec{k}$, $\vec{OQ} = -3\vec{i} + \vec{j} - 5\vec{k}$, $\vec{OR} = -2\vec{j}$, and $\vec{OS} = 3\vec{i}$

b. $\vec{OP} = (1, -2, -1)$, $\vec{OS} = (0, 0, 3)$, $\vec{OM} = (2, 0, -6)$, and $\vec{ON} = (1, -1, -7)$

In R^2, we showed how to add two algebraic vectors. The result in R^3 is analogous to this result.

Addition of Three Vectors in R^3



Writing each of the three given position vectors in terms of the standard basis vectors, $\vec{OB} = p\vec{i} + q\vec{j} + s\vec{k}$, $\vec{OC} = m\vec{i} + n\vec{j} + r\vec{k}$, and $\vec{OD} = f\vec{i} + g\vec{j} + h\vec{k}$.

Using the parallelogram law of addition, $\vec{OP} = \vec{OD} + \vec{OQ}$ and $\vec{OQ} = \vec{OB} + \vec{OC}$.

Substituting, $\vec{OP} = \vec{OD} + (\vec{OB} + \vec{OC})$.

Therefore,

$$\begin{aligned}\vec{OP} &= (f\vec{i} + g\vec{j} + h\vec{k}) + ((p\vec{i} + q\vec{j} + s\vec{k}) + (m\vec{i} + n\vec{j} + r\vec{k})) \\ &= (f\vec{i} + p\vec{i} + m\vec{i}) + (g\vec{j} + q\vec{j} + n\vec{j}) + (h\vec{k} + s\vec{k} + r\vec{k})\end{aligned}$$

(Commutative and associative properties of vector addition)

$$\begin{aligned}&= (f + p + m)\vec{i} + (g + q + n)\vec{j} + (h + s + r)\vec{k} \\ &= (f + p + m, g + q + n, h + s + r)\end{aligned}$$

(Distributive property of scalars)

This result demonstrates that the method for adding algebraic vectors in R^3 is the same as in R^2. Adding two vectors means adding their respective components. It should also be noted that the result for the subtraction of vectors in R^3 is analogous to the result in R^2. If $\overrightarrow{OA} = (a_1, a_2, a_3)$ and $\overrightarrow{OB} = (b_1, b_2, b_3)$, then $\overrightarrow{OA} - \overrightarrow{OB} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

In R^3, the shape that was used to generate the result for the addition of three vectors was not a parallelogram but a *parallelepiped*, which is a box-like shape with pairs of opposite faces being identical parallelograms. From our diagram, it can be seen that parallelograms *ODAB* and *CEPQ* are copies of each other. It is also interesting to note that the parallelepiped is completely determined by the components of the three position vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, and $\overrightarrow{OD}$. That is to say, the coordinates of *all* the vertices of the parallelepiped can be determined by the repeated application of the Triangle Law of Addition.

For vectors in R^2, we showed that the multiplication of an algebraic vector by a scalar was produced by multiplying each component of the vector by the scalar. In R^3, this result also holds, i.e., $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc), m \in \mathbf{R}$.

EXAMPLE 2 Selecting a strategy to determine a combination of vectors in R^3

Given $\vec{a} = -\vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$, determine each of the following:

a. $2\vec{a} - \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} + \vec{c}$

Solution

a. *Method 1* (Standard Unit Vectors)

$$\begin{aligned}
2\vec{a} - \vec{b} + \vec{c} &= 2(-\vec{i} + 2\vec{j} + \vec{k}) - (2\vec{j} - 3\vec{k}) + (\vec{i} - 3\vec{j} + 2\vec{k}) \\
&= -2\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{j} + 3\vec{k} + \vec{i} - 3\vec{j} + 2\vec{k} \\
&= -2\vec{i} + \vec{i} + 4\vec{j} - 2\vec{j} - 3\vec{j} + 2\vec{k} + 3\vec{k} + 2\vec{k} \\
&= -\vec{i} - \vec{j} + 7\vec{k} \\
&= (-1, -1, 7)
\end{aligned}$$

Method 2 (Components)

Converting to component form, we have $\vec{a} = (-1, 2, 1)$, $\vec{b} = (0, 2, -3)$, and $\vec{c} = (1, -3, 2)$.

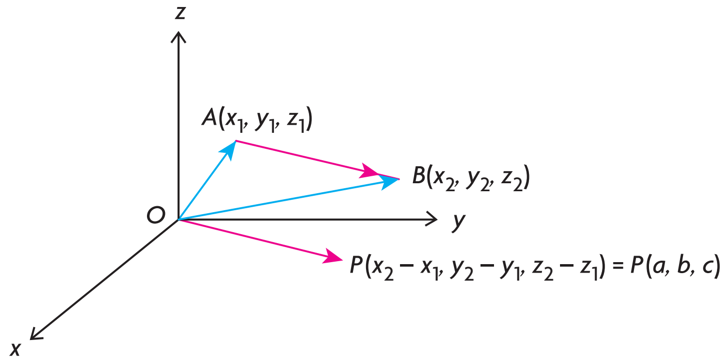
$$\begin{aligned}
\text{Therefore, } 2\vec{a} - \vec{b} + \vec{c} &= 2(-1, 2, 1) - (0, 2, -3) + (1, -3, 2) \\
&= (-2, 4, 2) + (0, -2, 3) + (1, -3, 2) \\
&= (-2 + 0 + 1, 4 - 2 - 3, 2 + 3 + 2) \\
&= (-1, -1, 7) \\
&= -\vec{i} - \vec{j} + 7\vec{k}
\end{aligned}$$

b. Using components, $\vec{a} + \vec{b} + \vec{c} = (-1, 2, 1) + (0, 2, -3) + (1, -3, 2)$

$$= (-1 + 0 + 1, 2 + 2 + (-3), 1 + (-3) + 2)$$
$$= (0, 1, 0) = \vec{j}$$

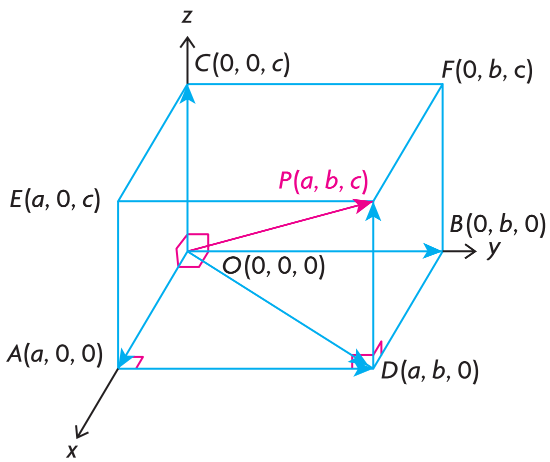
Vectors in R^3 Defined by Two Points

Position vectors and their magnitude in R^3 are calculated in a manner similar to R^2.



To determine the components of $\overrightarrow{AB}$, the same method is used in R^3 as was used in R^2, i.e., $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, which implies that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, or, in component form, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Thus, the components of the algebraic vector $\overrightarrow{AB}$ can be found by subtracting the coordinates of point A from the coordinates of point B.

If P has coordinates (a, b, c), we can calculate the magnitude of $\overrightarrow{OP}$.



From the diagram, we first note that $|\overrightarrow{OA}| = |a|$, $|\overrightarrow{OB}| = |b|$, and $|\overrightarrow{OC}| = |c|$. We also observe, using the Pythagorean theorem, that $|\overrightarrow{OP}|^2 = |\overrightarrow{OD}|^2 + |\overrightarrow{OC}|^2$ and, since $|\overrightarrow{OD}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$, substitution gives $|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{OC}|^2$.

Writing this expression in its more familiar coordinate form, we get $\left|\overrightarrow{OP}\right|^2 = |a|^2 + |b|^2 + |c|^2$ or $\left|\overrightarrow{OP}\right| = \sqrt{|a|^2 + |b|^2 + |c|^2}$. The use of the absolute value signs in the formula guarantees that the components are positive before they are squared. Because squaring components guarantees the result will be positive, it would have been just as easy to write the formula as $\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2 + c^2}$, which gives an identical result.

Position Vectors and Magnitude in R^3

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the vector $\left|\overrightarrow{AB}\right| = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (a, b, c)$ is equivalent to the related position vector, $\overrightarrow{OP}$, and

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2 + c^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

EXAMPLE 3 Connecting vectors in R^3 with their components

If $A(7, -11, 13)$ and $B(4, -7, 25)$ are two points in R^3, determine each of the following:

a. $|\overrightarrow{OA}|$ b. $|\overrightarrow{OB}|$ c. $\overrightarrow{AB}$ d. $|\overrightarrow{AB}|$

Solution

a. $\left|\overrightarrow{OA}\right| = \sqrt{7^2 + (-11)^2 + 13^2} = \sqrt{339} \doteq 18.41$

b. $\left|\overrightarrow{OB}\right| = \sqrt{4^2 + (-7)^2 + 25^2} = \sqrt{690} \doteq 26.27$

c. $\overrightarrow{AB} = (4 - 7, -7 - (-11), 25 - 13) = (-3, 4, 12)$

d. $\left|\overrightarrow{AB}\right| = \sqrt{(-3)^2 + 4^2 + 12^2} = \sqrt{169} = 13$

In this section, we developed further properties of algebraic vectors. In the next section, we will demonstrate how these properties can be used to understand the geometry of R^3.

IN SUMMARY

Key Ideas

- In R^3, $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$. All are unit vectors along the x-, y- and z-axes, respectively.
- $\overrightarrow{OP} = (a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$
- The vector between two points with its tail at $A(x_1, y_1, z_1)$ and head at $B(x_1, y_1, z_1)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same.
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Need to Know

- If $\overrightarrow{OA} = (a, b, c)$ and $\overrightarrow{OD} = (d, e, f)$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + d, b + e, c + f)$.
- $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc)$, $m \in \mathbf{R}$

Exercises 6.7

PART A

1. a. Write the vector $\overrightarrow{OA} = (-1, 2, 4)$ using the standard unit vectors.
   b. Determine $|\overrightarrow{OA}|$.
2. Write the vector $\overrightarrow{OB} = 3\vec{i} + 4\vec{j} - 4\vec{k}$ in component form and calculate its magnitude.
3. If $\vec{a} = (1, 3, -3)$, $\vec{b} = (-3, 6, 12)$, and $\vec{c} = (0, 8, 1)$, determine $\left|\vec{a} + \frac{1}{3}\vec{b} - \vec{c}\right|$.
4. For the vectors $\overrightarrow{OA} = (-3, 4, 12)$ and $\overrightarrow{OB} = (2, 2, -1)$, determine the following:
   a. the components of vector $\overrightarrow{OP}$, where $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$
   b. $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OP}|$
   c. $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. What does $\overrightarrow{AB}$ represent?

PART B

K 5. Given $\vec{x} = (1, 4, -1)$, $\vec{y} = (1, 3, -2)$, and $\vec{z} = (-2, 1, 0)$, determine a vector equivalent to each of the following:

a. $\vec{x} - 2\vec{y} - \vec{z}$

b. $-2\vec{x} - 3\vec{y} + \vec{z}$

c. $\frac{1}{2}\vec{x} - \vec{y} + 3\vec{z}$

d. $3\vec{x} + 5\vec{y} + 3\vec{z}$

6. Given $\vec{p} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{q} = -\vec{i} - \vec{j} + \vec{k}$, determine the following in terms of the standard unit vectors.

a. $\vec{p} + \vec{q}$

b. $\vec{p} - \vec{q}$

c. $2\vec{p} - 5\vec{q}$

d. $-2\vec{p} + 5\vec{q}$

7. If $\vec{m} = 2\vec{i} - \vec{k}$ and $\vec{n} = -2\vec{i} + \vec{j} + 2\vec{k}$, calculate each of the following:

a. $|\vec{m} - \vec{n}|$

b. $|\vec{m} + \vec{n}|$

c. $|2\vec{m} + 3\vec{n}|$

d. $|-5\vec{m}|$

8. Given $\vec{x} + \vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{x} - \vec{y} = 3\vec{i} + 6\vec{j} - 7\vec{k}$, determine $\vec{x}$ and $\vec{y}$.

C 9. Three vectors, $\overrightarrow{OA} = (a, b, 0)$, $\overrightarrow{OB} = (a, 0, c)$, and $\overrightarrow{OC} = (0, b, c)$, are given.

a. In a sentence, describe what each vector represents.

b. Write each of the given vectors using the standard unit vectors.

c. Determine a formula for each of $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OC}|$.

d. Determine $\overrightarrow{AB}$. What does $\overrightarrow{AB}$ represent?

10. Given the points $A(-2, -6, 3)$ and $B(3, -4, 12)$, determine each of the following:

a. $|\overrightarrow{OA}|$

b. $|\overrightarrow{OB}|$

c. $\overrightarrow{AB}$

d. $|\overrightarrow{AB}|$

e. $\overrightarrow{BA}$

f. $|\overrightarrow{BA}|$

11. The vertices of quadrilateral $ABCD$ are given as $A(0, 3, 5)$, $B(3, -1, 17)$, $C(7, -3, 15)$, and $D(4, 1, 3)$. Prove that $ABCD$ is a parallelogram.

12. Given $2\vec{x} + \vec{y} - 2\vec{z} = \vec{0}$, $\vec{x} = (-1, b, c)$, $\vec{y} = (a, -2, c)$, and $\vec{z} = (-a, 6, c)$, determine the value of the unknowns.

A 13. A parallelepiped is determined by the vectors $\overrightarrow{OA} = (-2, 2, 5)$, $\overrightarrow{OB} = (0, 4, 1)$, and $\overrightarrow{OC} = (0, 5, -1)$.

a. Draw a sketch of the parallelepiped formed by these vectors.

b. Determine the coordinates of all of the vertices for the parallelepiped.

T 14. Given the points $A(-2, 1, 3)$ and $B(4, -1, 3)$, determine the coordinates of the point on the x-axis that is equidistant from these two points.

PART C

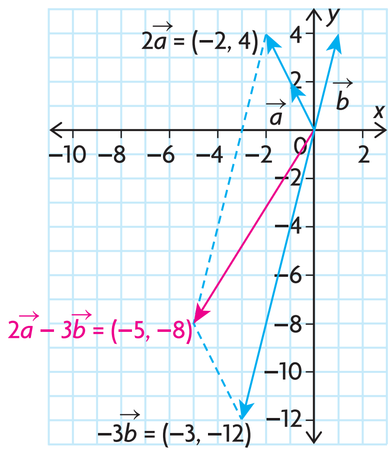
15. Given $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{a} + \vec{b}| = 7$, determine $|\vec{a} - \vec{b}|$.

Section 6.8—Linear Combinations and Spanning Sets

We have discussed concepts involving geometric and algebraic vectors in some detail. In this section, we are going to use these ideas as a basis for understanding the notion of a **linear combination**, an important idea for understanding the geometry of three dimensions.

Examining Linear Combinations of Vectors in R^2

We'll begin by considering linear combinations in R^2. If we consider the vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$ and write $2(-1, 2) - 3(1, 4) = (-5, -8)$, then the expression on the left side of this equation is called a linear combination. In this case, the linear combination produces the vector $(-5, -8)$. Whenever vectors are multiplied by scalars and then added, the result is a new vector that is a linear combination of the vectors. If we take the two vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$, then $2\vec{a} - 3\vec{b}$ is a vector on the xy-plane and is the diagonal of the parallelogram formed by the vectors $2\vec{a}$ and $-3\vec{b}$, as shown in the diagram.



Linear Combination of Vectors

For noncollinear vectors, $\vec{u}$ and $\vec{v}$, a linear combination of these vectors is $a\vec{u} + b\vec{v}$, where a and b are scalars (real numbers). The vector $a\vec{u} + b\vec{v}$ is the diagonal of the parallelogram formed by the vectors $a\vec{u}$ and $b\vec{v}$.

It was shown that every vector in the xy-plane can be written uniquely in terms of the unit vectors $\vec{i}$ and $\vec{j}$. $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. This can be done in only one way. Writing $\overrightarrow{OP}$ in this way is really just writing this vector as a linear combination of $\vec{i}$ and $\vec{j}$. Because every vector in R^2 can be written as a linear combination of these two vectors, we say that $\vec{i}$ and $\vec{j}$ span R^2. Another way of stating this is to say that the set of vectors $\{\vec{i}, \vec{j}\}$ forms a **spanning set** for R^2.

Spanning Set for R^2

The set of vectors $\{\vec{i}, \vec{j}\}$ is said to form a spanning set for R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.

What is interesting about spanning sets is that there is not just one set of vectors that can be used to span R^2. There is an infinite number of sets, each set containing a minimum of exactly two vectors, that would serve the same purpose. The concepts of span and spanning set will prove significant for the geometry of planes studied in Chapter 8.

EXAMPLE 1 Representing a vector as a linear combination of two other vectors

Show that $\vec{x} = (4, 23)$ can be written as a linear combination of either set of vectors, $\{(-1, 4), (2, 5)\}$ or $\{(1, 0), (-2, 1)\}$.

Solution

In each case, the procedure is the same, and so we will show the details for just one set of calculations. We are looking for solutions to the following separate equations: $a(-1, 4) + b(2, 5) = (4, 23)$ and $c(1, 0) + d(-2, 1) = (4, 23)$.

Multiplying, $(-a, 4a) + (2b, 5b) = (4, 23)$ (Properties of scalar multiplication)

$(-a + 2b, 4a + 5b) = (4, 23)$ (Properties of vector addition)

Since the vector on the left side is equal to that on the right side, we can write

① $-a + 2b = 4$

② $4a + 5b = 23$

This forms a linear system that can be solved using the method of elimination.

③ $-4a + 8b = 16$, after multiplying equation ① by 4.

Adding equation ② and equation ③ gives, $13b = 39$, so $b = 3$ and, by substitution, $a = 2$.

Therefore, $2(-1, 4) + 3(2, 5) = (4, 23)$.
The calculations for the second linear combination are done in the same way as the first, and so $c - 2d = 4$ and $d = 23$. Substituting gives $c = 50$ and $d = 23$. Therefore, $50(1, 0) + 23(-2, 1) = (4, 23)$. You should verify the calculations on your own.

In R^2, it is possible to take any pair of noncollinear (non-parallel) vectors as a spanning set, provided that $(0, 0)$ is not one of the two vectors.

EXAMPLE 2

Reasoning about spanning sets in R^2

Show that the set of vectors $\{(2, 3), (4, 6)\}$ does not span R^2.

Solution

Since these vectors are scalar multiples of each other, i.e., $(4, 6) = 2(2, 3)$, they cannot span R^2. All linear combinations of these two vectors produce only vectors that are scalar multiples of (2, 3). This is shown by the following calculation:

$$\begin{aligned} a(2, 3) + b(4, 6) = (2a + 4b, 3a + 6b) &= (2(a + 2b), 3(a + 2b)) \\ &= (a + 2b)(2, 3) \end{aligned}$$

This result means that we cannot use linear combinations of the set of vectors $\{(2, 3), (4, 6)\}$ to obtain anything but a multiple of $(2, 3)$. As a result, the only vectors that can be created are ones in either the same or opposite direction of (2, 3). There is no linear combination of these vectors that would allow us to obtain, for example, the vector $(3, 4)$.

When we say that a set of vectors spans R^2, we are saying that every vector in the plane can be written as a linear combination of the two given vectors. In Example 1, we did not prove that either set of vectors was a spanning set. All that we showed was that the given vector could be written as a linear combination of a set of vectors. It is true in this case, however, that both sets do span R^2. In the following example, it will be shown how a set of vectors in R^2 can be proven to be a spanning set.

EXAMPLE 3

Proving that a given set of vectors spans R^2

Show that the set of vectors $\{(2, 1), (-3, -1)\}$ is a spanning set for R^2.

Solution

In order to show that the set spans R^2, we write the linear combination $a(2, 1) + b(-3, -1) = (x, y)$, where (x, y) represents *any* vector in R^2. Carrying out the same procedure as in the previous example, we obtain

① $2a - 3b = x$

② $a - b = y$

Again the process of elimination will be used to solve this system of equations.

① $2a - 3b = x$

③ $2a - 2b = 2y$, after multiplying equation ② by 2

Subtracting eliminates a, $-3b - (-2b) = x - 2y$

Therefore, $-b = x - 2y$ or $b = -x + 2y$. By substituting this value of b into equation 2, we find $a = -x + 3y$. Therefore, the solution to this system of equations is $a = -x + 3y$ and $b = -x + 2y$.

This means that, whenever we are given the components of any vector, we can find the corresponding values of a and b by substituting into the formula. Since the values of x and y are unique, the corresponding values of a and b are also unique. Using this formula to write $(-3, 7)$ as a linear combination of the two given vectors, we would say $x = -3$, $y = 7$ and solve for a and b to obtain

$$a = -(-3) + 3(7) = 24$$

and

$$b = -(-3) + 2(7) = 17$$

$$24(2, 1) + 17(-3, -1) = (-3, 7)$$

So the vector $(-3, 7)$ can be written as a linear combination of $(2, 1)$ and $(-3, -1)$. Therefore, the set of vectors $\{(2, 1), (-3, -1)\}$ spans R^2.

Examining Linear Combinations of Vectors in R^3

In the previous section, the set of unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ was introduced as unit vectors lying along the positive x-, y-, and z-axes, respectively. This set of vectors is referred to as the standard basis for R^3, meaning that every vector in R^3 can be written uniquely as a linear combination of these three vectors. (It should be pointed out that there is an infinite number of sets containing three vectors that could also be used as a basis for R^3.)

EXAMPLE 4 **Representing linear combinations in R^3**

Show that the vector $(2, 3, -5)$ can be written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$ and illustrate this geometrically.

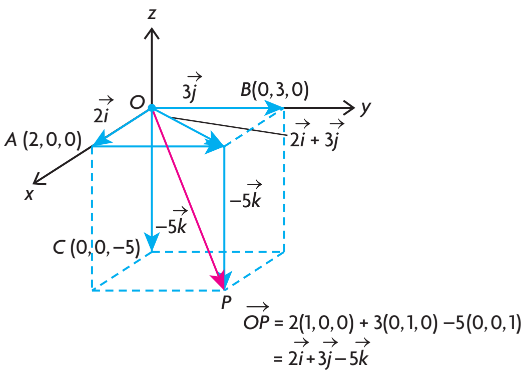
Solution

Writing the given vector as a linear combination

$$\begin{aligned}(2, 3, -5) &= 2(1, 0, 0) + 3(0, 1, 0) - 5(0, 0, 1)\\ &= 2\vec{i} + 3\vec{j} - 5\vec{k}\end{aligned}$$

This is exactly what we would expect based on the work in the previous section.

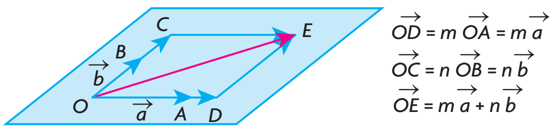
Geometrically, the linear combination of the vectors can be visualized in the following way.



The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are basis vectors in R^3. This has the same meaning for R^3 that it has for R^2. As before, every vector in R^3 can be uniquely written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$. Stated simply, $\vec{OP} = (a, b, c)$ can be written as $\vec{OP} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$.

The methods for working in R^3 are similar to methods we have already seen at the beginning of this section. A natural question to ask is, "Suppose you are given two noncollinear (non-parallel) vectors in R^3; what do these vectors span?"

In the diagram, the large parallelogram is meant to represent an infinite plane extending in all directions. On this parallelogram are drawn the nonzero vectors $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ with their tails at the origin, O. When we write the linear combination $m\vec{a} + n\vec{b}$, $\vec{OE}$ is the resulting vector and is the diagonal of the smaller parallelogram, $ODEC$. Since each of the scalars, m and n, can be any real number, an infinite number of vectors, each unique, will be generated from this linear combination. All of these vectors lie on the plane determined by $\vec{a}$ and $\vec{b}$. It should be noted that if we say $\vec{OE}$ lies on the plane, the point E also lies on the plane so that when we say $\vec{OA}$, $\vec{OB}$, and $\vec{OE}$ lie on the plane, this is effectively saying that the points A, B, and E also lie on the plane. When two or more points or vectors lie on the same plane they are said to be **coplanar**.



Spanning Sets

1. Any pair of nonzero, noncollinear vectors will span R^2.
2. Any pair of nonzero, noncollinear vectors will span a plane in R^3.

EXAMPLE 5 Selecting a linear combination strategy to determine if vectors lie on the same plane

a. Given the two vectors $\vec{a} = (-1, -2, 1)$ and $\vec{b} = (3, -1, 1)$, does the vector $\vec{c} = (-9, -4, 1)$ lie on the plane determined by $\vec{a}$ and $\vec{b}$? Explain.

b. Does the vector $(-9, -5, 1)$ lie in the plane determined by the first two vectors?

Solution

a. This question is asking whether $\vec{c}$ lies in the span of $\vec{a}$ and $\vec{b}$. Stated algebraically, are there values of m and n for which $m(-1, -2, 1) + n(3, -1, 1) = (-9, -4, 1)$?

Multiplying, $(-m, -2m, m) + (3n, -n, n) = (-9, -4, 1)$
or $(-m + 3n, -2m - n, m + n) = (-9, -4, 1)$

Equating components leads to

① $-m + 3n = -9$
② $2m + n = 4$
③ $m + n = 1$

The easiest way of dealing with these equations is to work with equations ① and ③. If we add these equations, m is eliminated and $4n = -8$, so $n = -2$. Substituting into equation ③ gives $m = 3$. We must verify that these values give a consistent answer in the remaining equation. Checking in equation ②: $2(3) + (-2) = 4$.

Since $(-9, -4, 1)$ can be written as a linear combination of $(-1, -2, 1)$ and $(3, -1, 1)$, i.e., $(-9, -4, 1) = 3(-1, -2, 1) - 2(3, -1, 1)$, it lies in the plane determined by the two given vectors.

b. If we carry out calculations identical to those in the solution for part a., the only difference would be that the second equation would now be $2m + n = 5$, and substituting $m = 3$ and $n = -2$ would give $2(3) + (-2) = 4 \neq 5$. Since we have an inconsistent result, this implies that the vector $(-9, -5, 1)$ does not lie on the same plane as $\vec{a}$ and $\vec{b}$.

In general, when we are trying to determine whether a vector lies in the plane determined by two other nonzero, noncollinear vectors, it is sufficient to solve any pair of equations and look for consistency in the third equation. If the result is consistent, the vector lies in the plane, and if not, the vector does not lie in the plane.

IN SUMMARY

Key Ideas

- In R^2, $\overrightarrow{OP} = (a, b) = a(1, 0) + b(0, 1) = a\vec{i} + b\vec{j}$. $\vec{i}$ and $\vec{j}$ span R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.
- In R^3, $\overrightarrow{OP} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$. $\vec{i}$, $\vec{j}$, and $\vec{k}$ span R^3 since every vector in R^3 can be written uniquely as a linear combination of these three vectors.

Need to Know

- Any pair of nonzero, noncollinear vectors will span R^2.
- Any pair of nonzero, noncollinear vectors will span a plane in R^3. This means that every vector in the plane can be expressed as a linear combination involving this pair of vectors.

Exercise 6.8

PART A

1. A student writes $2(1, 0) + 4(-1, 0) = (-2, 0)$ and then concludes that $(1, 0)$ and $(-1, 0)$ span R^2. What is wrong with this conclusion?
2. It is claimed that $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$ is a set of vectors spanning R^3. Explain why it is not possible for these vectors to span R^3.
3. Describe the set of vectors spanned by $(0, 1)$. Say why this is the same set as that spanned by $(0, -1)$.
4. In R^3, the vector $\vec{i} = (1, 0, 0)$ spans a set. Describe the set spanned by this vector. Name two other vectors that would also span the same set.
5. It is proposed that the set $\{(0, 0), (1, 0)\}$ could be used to span R^2. Explain why this is not possible.
6. The following is a spanning set for R^2:
   $\{(-1, 2), (2, -4), (-1, 1), (-3, 6), (1, 0)\}$.
   Remove three of the vectors and write down a spanning set that can be used to span R^2.

PART B

7. Simplify each of the following linear combinations and write your answer in component form: $\vec{a} = \vec{i} - 2\vec{j}$, $\vec{b} = \vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$
   a. $2(2\vec{a} - 3\vec{b} + \vec{c}) - 4(-\vec{a} + \vec{b} - \vec{c}) + (\vec{a} - \vec{c})$
   b. $\frac{1}{2}(2\vec{a} - 4\vec{b} - 8\vec{c}) - \frac{1}{3}(3\vec{a} - 6\vec{b} + 9\vec{c})$

K 8. Name two sets of vectors that could be used to span the xy-plane in R^3. Show how the vectors $(-1, 2, 0)$ and $(3, 4, 0)$ could each be written as a linear combination of the vectors you have chosen.

C 9. a. The set of vectors $\{(1, 0, 0), (0, 1, 0)\}$ spans a set in R^3. Describe this set.
   b. Write the vector $(-2, 4, 0)$ as a linear combination of these vectors.
   c. Explain why it is not possible to write $(3, 5, 8)$ as a linear combination of these vectors.
   d. If the vector $(1, 1, 0)$ were added to this set, what would these three vectors span in R^3?

10. Solve for a, b, and c in the following equation:
    $2(a, 3, c) + 3(c, 7, c) = (5, b + c, 15)$

11. Write the vector $(-10, -34)$ as a linear combination of the vectors $(-1, 3)$ and $(1, 5)$.

12. In Example 3, it was shown how to find a formula for the coefficients a and b whenever we are given a general vector (x, y).
    a. Repeat this procedure for $\{(2, -1), (-1, 1)\}$.
    b. Write each of the following vectors as a linear combination of the set given in part a.: $(2, -3)$, $(124, -5)$, and $(4, -11)$.

A 13. a. Show that the vectors $(-1, 2, 3)$, $(4, 1, -2)$, and $(-14, -1, 16)$ do not lie on the same plane.
    b. Show that the vectors $(-1, 3, 4)$, $(0, -1, 1)$, and $(-3, 14, 7)$ lie on the same plane, and show how one of the vectors can be written as a linear combination of the other two.

14. Determine the value for x such that the points $A(-1, 3, 4)$, $B(-2, 3, -1)$, and $C(-5, 6, x)$ all lie on a plane that contains the origin.

T 15. The vectors $\vec{a}$ and $\vec{b}$ span R^2. What values of m and n will make the following statement true: $(m - 2)\vec{a} = (n + 3)\vec{b}$? Explain your reasoning.

PART C

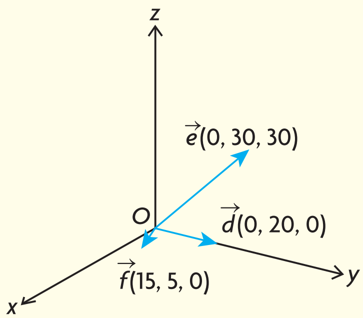
16. The vectors $(4, 1, 7)$, $(-1, 1, 6)$, and $(p, q, 5)$ are coplanar. Determine three sets of values for p and q for which this is true.

17. The vectors $\vec{a}$ and $\vec{b}$ span R^2. For what values of m is it true that $(m^2 + 2m - 3)\vec{a} + (m^2 + m - 6)\vec{b} = \vec{0}$? Explain your reasoning.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 6: FIGURE SKATING

A figure skater is attempting to perform a quadruple spin jump. He sets up his jump with an initial skate along vector $\vec{d}$. He then plants his foot and applies vertical force at an angle according to vector $\vec{e}$. This causes him to leap into the air and spin. After landing, his momentum will carry him into the wall if he does not apply force to stop himself. So he applies force along vector $\vec{f}$ to slow himself down and change direction.



a. Add vectors $\vec{d}$ and $\vec{e}$ to find the resulting vector ($\vec{a}$) for the skater's jump. The angle between $\vec{d}$ and $\vec{e}$ is 25°. If the *xy*-plane represents the ice surface, calculate the angle the skater will take with respect to the ice surface on this jump.

b. Discuss why the skater will return to the ground even though the vector that represents his leap carries him in an upward direction.

c. Rewrite the resulting vector $\vec{a}$ without the vertical coordinate. For example, if the vector has components (20, 30, 15), rewrite as (20, 30). Explain the significance of this vector.

d. Add vectors $\vec{a}$ and $\vec{f}$ to find the resulting vector ($\vec{b}$) as the skater applies force to slow himself and change direction. Explain the significance of this vector.

Key Concepts Review

In Chapter 6, you were introduced to vectors: quantities that are described in terms of both magnitude and direction. You should be familiar with the difference between a geometric vector and an algebraic vector. Consider the following summary of key concepts:

- Scalar quantities have only magnitude, while vector quantities have both magnitude and direction.
- Two vectors are equal if they have the same magnitude and direction.
- Two vectors are opposite if they have the same magnitude and opposite directions.
- When vectors are drawn tail-to-tail, their sum or resultant is the diagonal of the parallelogram formed by the vectors.
- When vectors are drawn head-to-tail, their sum or resultant is the vector drawn from the tail of the first to the head of the second.
- Multiplying a vector by a nonzero scalar results in a new vector in the same or opposite direction of the original vector with a greater or lesser magnitude compared to the original. The set of vectors formed are described as collinear (parallel vectors).
- The vector $\overrightarrow{OP}$ is called a position vector and is drawn on a coordinate axis with its tail at the origin and its head located at point P.
- In R^2, $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$ where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$.
- In R^3, $\overrightarrow{OP} = (a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$ where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$.
- In R^2, the vector between two points with its tail at $A(x_1, y_1)$ and head at $B(x_2, y_2)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- In R^3, the vector between two points with its tail at $A(x_1, y_1, z_1)$ and head at $B(x_2, y_2, z_2)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Any pair of nonzero, noncollinear vectors will span R^2.
- Any pair of nonzero, noncollinear vectors will span a plane in R^3.

Review Exercise

1. Determine whether each of the following statements is true or false. Provide a brief explanation for each answer.
   a. $|\vec{a} + \vec{b}| \geq |\vec{a}|$
   b. $|\vec{a} + \vec{b}| = |\vec{a} + \vec{c}|$ implies $|\vec{b}| = |\vec{c}|$
   c. $\vec{a} + \vec{b} = \vec{a} + \vec{c}$ implies $\vec{b} = \vec{c}$
   d. $\overrightarrow{RF} = \overrightarrow{SW}$ implies $\overrightarrow{RS} = \overrightarrow{FW}$
   e. $m\vec{a} + n\vec{a} = (m + n)\vec{a}$
   f. If $|\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$, then $|\vec{a} + \vec{b}| = |\vec{c} + \vec{d}|$.

2. If $\vec{x} = 2\vec{a} - 3\vec{b} - 4\vec{c}$, $\vec{y} = -2\vec{a} + 3\vec{b} + 3\vec{c}$, and $\vec{z} = 2\vec{a} - 3\vec{b} + 5\vec{c}$, determine simplified expressions for each of the following:
   a. $2\vec{x} - 3\vec{y} + 5\vec{z}$
   b. $3(-2\vec{x} - 4\vec{y} + \vec{z}) - (2\vec{x} - \vec{y} + \vec{z}) - 2(-4\vec{x} - 5\vec{y} + \vec{z})$

3. If $X(-2, 1, 2)$ and $Y(-4, 4, 8)$ are two points in R^3, determine the following:
   a. $\overrightarrow{XY}$ and $|\overrightarrow{XY}|$
   b. The coordinates of a unit vector in the same direction as $\overrightarrow{XY}$.

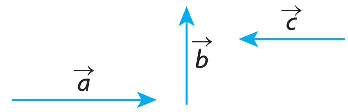
4. $X(-1, 2, 6)$ and $Y(5, 5, 12)$ are two points in R^3.
   a. Determine the components of a position vector equivalent to $\overrightarrow{YX}$.
   b. Determine the components of a *unit* vector that is in the same direction as $\overrightarrow{YX}$.

5. Find the components of the unit vector with the opposite direction to that of the vector from $M(2, 3, 5)$ to $N(8, 1, 2)$.

6. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
   a. Determine the components of the vectors representing the diagonals.
   b. Determine the angles between the sides of the parallelogram.

7. The points $A(-1, 1, 1)$, $B(2, 0, 3)$, and $C(3, 3, -4)$ are vertices of a triangle.
   a. Show that this triangle is a right triangle.
   b. Calculate the area of triangle ABC.
   c. Calculate the perimeter of triangle ABC.
   d. Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices.

8. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



a. Construct the vector $\vec{a} - \vec{b} + \vec{c}$.

b. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and if $|\vec{a}| = 4$ and $|\vec{b}| = 3$, determine $|\vec{a} + \vec{b}|$.

9. Given $\vec{p} = (-11, 7)$, $\vec{q} = (-3, 1)$, and $\vec{r} = (-1, 2)$, express each vector as a linear combination of the other two.

10. a. Find an equation to describe the set of points equidistant from $A(2, -1, 3)$ and $B(1, 2, -3)$.

b. Find the coordinates of two points that are equidistant from A and B.

11. Calculate the values of a, b, and c in each of the following:

a. $2(a, b, 4) + \frac{1}{2}(6, 8, c) - 3(7, c, -4) = (-24, 3, 25)$

b. $2\left(a, a, \frac{1}{2}a\right) + (3b, 0, -5c) + 2\left(c, \frac{3}{2}c, 0\right) = (3, -22, 54)$

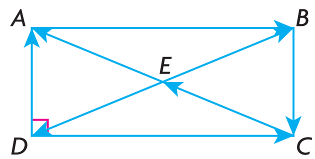
12. a. Determine whether the points $A(1, -1, 1)$, $B(2, 2, 2)$, and $C(4, -2, 1)$ represent the vertices of a right triangle.

b. Determine whether the points $P(1, 2, 3)$, $Q(2, 4, 6)$, and $R(-1, -2, -3)$ are collinear.

13. a. Show that the points $A(3, 0, 4)$, $B(1, 2, 5)$, and $C(2, 1, 3)$ represent the vertices of a right triangle.

b. Determine $\cos \angle ABC$.

14. In the following rectangle, vectors are indicated by the direction of the arrows.

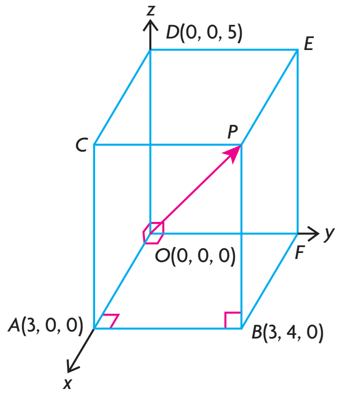


a. Name two pairs of vectors that are opposites.

b. Name two pairs of identical vectors.

c. Explain why $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{DB}|^2$.

15. A rectangular prism measuring 3 by 4 by 5 is drawn on a coordinate axis as shown in the diagram.



a. Determine the coordinates of points C, P, E, and F.

b. Determine position vectors for $\overrightarrow{DB}$ and $\overrightarrow{CF}$.

c. By drawing the rectangle containing $\overrightarrow{DB}$ and $\overrightarrow{OP}$, determine the acute angle between these vectors.

d. Determine the angle between $\overrightarrow{OP}$ and $\overrightarrow{AE}$.

16. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, and the angle between them is 30°. Determine each of the following:

a. $|\vec{d} + \vec{e}|$　　b. $|\vec{d} - \vec{e}|$　　c. $|\vec{e} - \vec{d}|$

17. An airplane is headed south at speed 400 km/h. The airplane encounters a wind from the east blowing at 100 km/h.

a. How far will the airplane travel in 3 h?

b. What is the direction of the airplane?

18. a. Explain why the set of vectors: $\{(2, 3), (3, 5)\}$ spans R^2.

b. Find m and n in the following: $m(2, 3) + n(3, 5) = (323, 795)$.

19. a. Show that the vector $\vec{a} = (5, 9, 14)$ can be written as a linear combination of the vectors $\vec{b}$ and $\vec{c}$, where $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$. Explain why $\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$.

b. Is the vector $\vec{a} = (-13, 36, 23)$ in the span of $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$? Explain your answer.

20. A cube is placed so that it has three of its edges located along the positive x-, y-, and z-axes (one edge along each axis) and one of its vertices at the origin.

    a. If the cube has a side length of 4, draw a sketch of this cube and write the coordinates of its vertices on your sketch.

    b. Write the coordinates of the vector with its head at the origin and its tail at the opposite vertex.

    c. Write the coordinates of a vector that starts at $(4, 4, 4)$ and is a diagonal in the plane parallel to the xz-plane.

    d. What vector starts at the origin and is a diagonal in the xy-plane?

21. If $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$, and $\vec{c} = 2\vec{i} + 13\vec{k}$, determine $\left|2\left(\vec{a} + \vec{b} - \vec{c}\right) - \left(\vec{a} + 2\vec{b}\right) + 3\left(\vec{a} - \vec{b} + \vec{c}\right)\right|$.

22. The three points $A(-3, 4)$, $B(3, -4)$, and $C(5, 0)$ are on a circle with radius 5 and centre at the origin. Points A and B are the endpoints of a diameter, and point C is on the circle.

    a. Calculate $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{BC}|$.

    b. Show that A, B, and C are the vertices of a right triangle.

23. In terms of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{0}$, find a vector expression for each of the following:

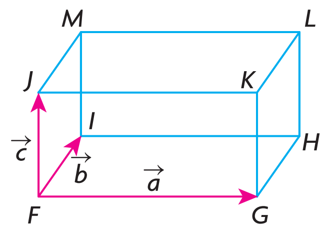
    a. $\overrightarrow{FL}$

    b. $\overrightarrow{MK}$

    c. $\overrightarrow{HJ}$

    d. $\overrightarrow{IH} + \overrightarrow{KJ}$

    e. $\overrightarrow{IK} - \overrightarrow{IH}$



24. Draw a diagram showing the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 2|\vec{b}|$ and $|\vec{b}| = |\vec{a} + \vec{b}|$ are both true. (Make sure to indicate the direction of the vectors.)

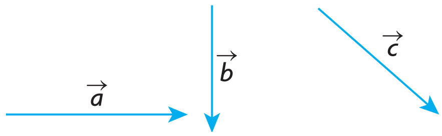
25. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, express each of the following in terms of $|\vec{a}|$ and $|\vec{b}|$:

    a. $|\vec{a} + \vec{b}|$
    b. $|\vec{a} - \vec{b}|$
    c. $|2\vec{a} + 3\vec{b}|$

26. Show that if $\vec{a}$ is perpendicular to each of the vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test

1. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are shown.



Using these three vectors, demonstrate that $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$. Name this property and explain how your answer shows this to be true.

2. $A(-2, 3, -5)$ and $B(6, 7, 3)$ are two points in R^3. Determine each of the following:

   a. $\overrightarrow{AB}$ b. $\left|\overrightarrow{AB}\right|$ c. a unit vector in the direction of $\overrightarrow{BA}$

3. The vectors $\vec{x}$ and $\vec{y}$ are each of length 3 units, i.e., $|\vec{x}| = |\vec{y}| = 3$. If $|\vec{x} + \vec{y}| = \sqrt{17}$, determine $|\vec{x} - \vec{y}|$.

4. a. If $3\vec{x} - 2\vec{y} = \vec{a}$ and $5\vec{x} - 3\vec{y} = \vec{b}$, express the vectors $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

   b. Solve for a, b, and c: $(2, -1, c) + (a, b, 1) - 3(2, a, 4) = (-3, 1, 2c)$.

5. a. Explain why the vectors $\vec{a} = (-2, 3)$ and $\vec{b} = (3, -1)$ span R^2.

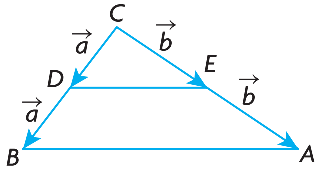
   b. Determine the values of p and q in $p(-2, 3) + q(3, -1) = (13, -9)$.

6. a. Show that the vector $\vec{a} = (1, 12, -29)$ can be written as a linear combination of $\vec{b} = (3, 1, 4)$ and $\vec{c} = (1, 2, -3)$.

   b. Determine whether $\vec{r} = (16, 11, -24)$ can be written as a linear combination of $\vec{p} = (-2, 3, 4)$ and $\vec{q} = (4, 1, -6)$. Explain the significance of your result geometrically.

7. $\vec{x}$ and $\vec{y}$ are vectors of magnitude 1 and 2, respectively, with an angle of 120° between them. Determine $|3\vec{x} + 2\vec{y}|$ and the direction of $3\vec{x} + 2\vec{y}$.

8. In triangle ABC, point D is the midpoint of $\overrightarrow{BC}$ and point E is the midpoint of $\overrightarrow{AC}$. Vectors are marked as shown. Use vectors to prove that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$.



5. **a.** 19 000 fish/year
b. 23 000 fish/year

6. **a.** **i.** 3
ii. 1
iii. 3
iv. 2
b. No, $\lim_{x \to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ must exist and they must be the same. In this case, $\lim_{x \to 4^-} f(x) = \infty$, but $\lim_{x \to 4^+} f(x) = -\infty$, so $\lim_{x \to 4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x \to 2^-} f(x) = 5$, but $\lim_{x \to 2^+} f(x) = 3$.

8. **a.** $-\frac{1}{5}$ **d.** $\frac{4}{3}$
b. 6 **e.** $\frac{1}{12}$
c. $-\frac{1}{9}$ **f.** $\frac{1}{2}$

9. **a.** $6x + 1$
b. $-\frac{1}{x^2}$

10. **a.** $3x^2 - 8x + 5$
b. $\frac{3x^2}{\sqrt{2x^3 + 1}}$
c. $\frac{6}{(x + 3)^2}$
d. $4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$
e. $\frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$
f. $5[x^2 + (2x + 1)^3]^4 \times [2x + 6(2x + 1)^2]$

11. $4x + 3y - 10 = 0$

12. 3

13. **a.** $p'(t) = 4t + 6$
b. 46 people per year
c. 2006

14. **a.** $f'(x) = 5x^4 - 15x^2 + 1$; $f''(x) = 20x^3 - 30x$
b. $f'(x) = \frac{4}{x^3}$; $f''(x) = -\frac{12}{x^4}$
c. $f'(x) = -\frac{2}{\sqrt{x^3}}$; $f''(x) = \frac{3}{\sqrt{x^5}}$
d. $f'(x) = 4x^3 + \frac{4}{x^5}$; $f''(x) = 12x^2 - \frac{20}{x^6}$

15. **a.** maximum: 82, minimum: 6
b. maximum: $9\frac{1}{3}$, minimum: 2
c. maximum: $\frac{e^4}{1 + e^4}$, minimum: $\frac{1}{2}$
d. maximum: 5, minimum: 1

16. **a.** $v(t) = 9t^2 - 81t + 162$, $a(t) = 18t - 81$
b. stationary when $t = 6$ or $t = 3$, advancing when $v(t) > 0$, and retreating when $v(t) < 0$
c. $t = 4.5$
d. $0 \leq t < 4.5$
e. $4.5 < t \leq 8$

17. 14 062.5 m^2

18. $r \doteq 4.3$ cm, $h \doteq 8.6$ cm

19. $r = 6.8$ cm, $h = 27.5$ cm

20. **a.** $140 - 2x$
b. 101 629.5 cm^3; 46.7 cm by 46.7 cm by 46.6 cm

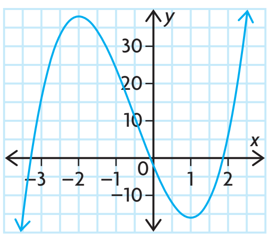
21. $x = 4$

22. \$70 or \$80

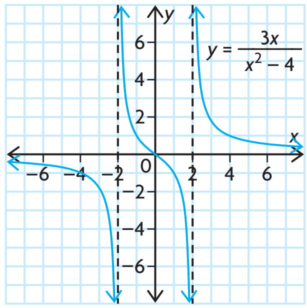
23. \$1140

24. **a.** $\frac{dy}{dx} = -10x + 20$,
$x = 2$ is critical number,
Increase: $x < 2$,
Decrease: $x > 2$
b. $\frac{dy}{dx} = 12x + 16$,
$x = -\frac{4}{3}$ is critical number,
Increase: $x > -\frac{4}{3}$,
Decrease: $x < -\frac{4}{3}$
c. $\frac{dy}{dx} = 6x^2 - 24$,
$x = \pm 2$ are critical numbers,
Increase: $x < -2$, $x > 2$,
Decrease: $-2 < x < 2$
d. $\frac{dy}{dx} = -\frac{2}{(x - 2)^2}$. The function has no critical numbers. The function is decreasing everywhere it is defined, that is, $x \neq 2$.

25. **a.** $y = 0$ is a horizontal asymptote. $x = \pm 3$ are the vertical asymptotes. There is no oblique asymptote. $\left(0, -\frac{8}{9}\right)$ is a local maximum.
b. There are no horizontal asymptotes. $x = \pm 1$ are the vertical asymptotes. $y = 4x$ is an oblique asymptote. $(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum.

26. **a.**


$y = 4x^3 + 6x^2 - 24x - 2$

b.


27. **a.** $(-20)e^{5x+1}$
b. $e^{3x}(3x + 1)$
c. $(3 \ln 6)6^{3x-8}$
d. $(\cos x)e^{\sin x}$

28. $y = 2e(x - 1) + e$

29. **a.** 5 days
b. 27

30. **a.** $2 \cos x + 15 \sin 5x$
b. $8 \cos 2x(\sin 2x + 1)^3$
c. $\frac{2x + 3 \cos 3x}{2\sqrt{x^2 + \sin 3x}}$
d. $\frac{1 + 2 \cos x}{(\cos x + 2)^2}$
e. $2x \sec^2 x^2 - 2 \tan x \sec^2 x$
f. $-2x \sin x^2 \cos(\cos x^2)$

31. about 4.8 m

32. about 8.5 m

Chapter 6

Review of Prerequisite Skills, p. 273

1. **a.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{\sqrt{3}}{2}$
b. $-\sqrt{3}$ **e.** $\frac{\sqrt{2}}{2}$
c. $\frac{1}{2}$ **f.** 1

2. $\frac{4}{3}$

3. **a.** $AB \doteq 29.7$, $\angle B \doteq 36.5°$, $\angle C \doteq 53.5°$
b. $\angle A \doteq 97.9°$, $\angle B \doteq 29.7°$, $\angle C \doteq 52.4°$

4. $\angle Z \doteq 50°$, $XZ \doteq 7.36$, $YZ \doteq 6.78$

5. $\angle R \doteq 44°$, $\angle S \doteq 102°$, $\angle T \doteq 34°$

6. 5.82 km

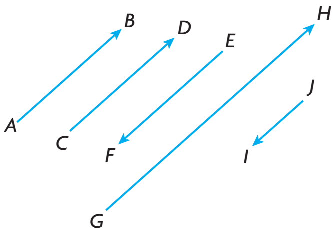
7. 8.66 km

8. 21.1 km

9. 59.4 cm^2

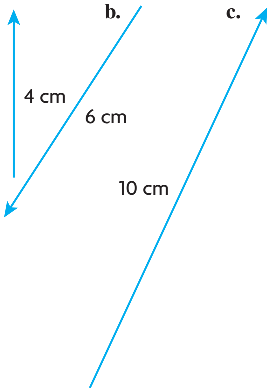
Section 6.1, pp. 279–281

1. a. False; two vectors with the same magnitude can have different directions, so they are not equal.
   b. True; equal vectors have the same direction and the same magnitude.
   c. False; equal or opposite vectors must be parallel and have the same magnitude. If two parallel vectors have different magnitude, they cannot be equal or opposite.
   d. False; equal or opposite vectors must be parallel and have the same magnitude. Two vectors with the same magnitude can have directions that are not parallel, so they are not equal or opposite.
2. The following are scalars: height, temperature, mass, area, volume, distance, and speed. There is not a direction associated with any of these qualities.
   The following are vectors: weight, displacement, force, and velocity. There is a direction associated with each of these qualities.
3. Answers may vary. For example:
   A rolling ball stops due to friction, which resists the direction of motion.
   A swinging pendulum stops due to friction resisting the swinging pendulum.
4. Answers may vary. For example:
   a. $\vec{AD} = \vec{BC}$; $\vec{AB} = \vec{DC}$; $\vec{AE} = \vec{EC}$; $\vec{DE} = \vec{EB}$
   b. $\vec{AD} = -\vec{CB}$; $\vec{AB} = -\vec{CD}$; $\vec{AE} = -\vec{CE}$; $\vec{ED} = -\vec{EB}$; $\vec{DA} = -\vec{BC}$
   c. $\vec{AC}$ & $\vec{DB}$; $\vec{AE}$ & $\vec{EB}$; $\vec{EC}$ & $\vec{DE}$; $\vec{AB}$ & $\vec{CB}$
5.

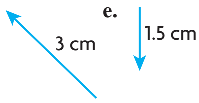


   a. $\vec{AB} = \vec{CD}$
   b. $\vec{AB} = -\vec{EF}$
   c. $|\vec{AB}| = |\vec{EF}|$ but $\vec{AB} \neq \vec{EF}$
   d. $\vec{GH} = 2\vec{AB}$
   e. $\vec{AB} = -2\vec{JI}$

6. a. b. c.



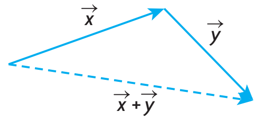
   d. e.



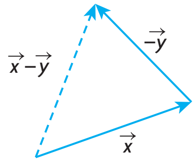
7. a. 100 km/h, south
   b. 50 km/h, west
   c. 100 km/h, northeast
   d. 25 km/h, northwest
   e. 60 km/h, east
8. a. 400 km/h, due south
   b. 70 km/h, southwesterly
   c. 30 km/h southeasterly
   d. 25 km/h, due east
9. a. i. False; they have equal magnitude, but opposite direction.
   ii. True; they have equal magnitude.
   iii. True; the base has sides of equal length, so the vectors have equal magnitude.
   iv. True; they have equal magnitude and direction.
   b. $|\vec{BD}| = \sqrt{18}$, $|\vec{BE}| = \sqrt{73}$, $|\vec{BH}| = \sqrt{82}$
10. a. The tangent vector describes James's velocity at that moment. At point A, his speed is 15 km/h and he is heading north. The tangent vector shows his velocity is 15 km/h, north.
   b. James's speed
   c. The magnitude of James's velocity (his speed) is constant, but the direction of his velocity changes at every point.
   d. C
   e. 3.5 min
   f. southwest
11. a. $\sqrt{10}$ or 3.16
   b. $(-3, 1)$
   c. $(0, -3)$
   d. $(0, 0)$

Section 6.2, pp. 290–292

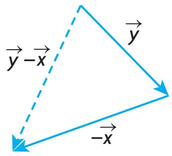
1. a.



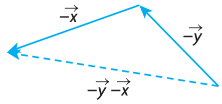
   b.



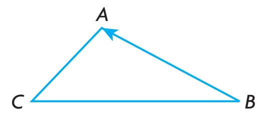
   c.



   d.

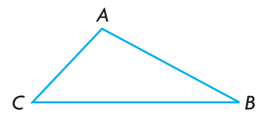


2. a.



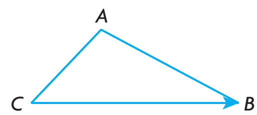
   $\vec{BA}$

   b.



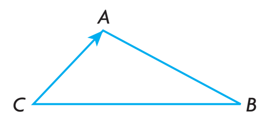
   $\vec{0}$

   c.



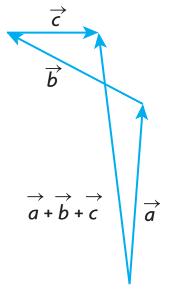
   $\vec{CB}$

   d.

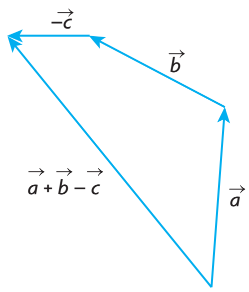


   $\vec{CA}$

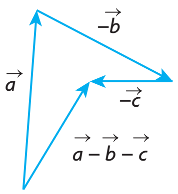
3. a.



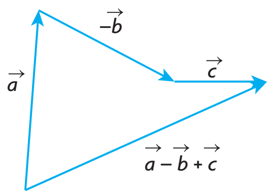
b.



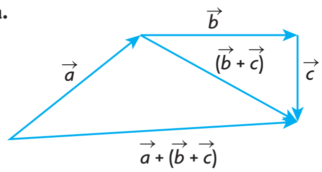
c.



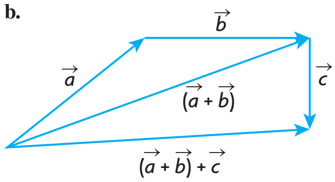
d.



4. a.



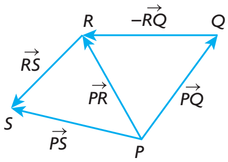
b.



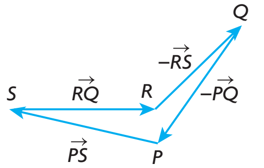
c. The resultant vectors are the same. The order in which you add vectors does not matter.

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

5. a. $=\overrightarrow{PS}$



b. $\vec{0}$



6. $\vec{x}+\vec{y}=\overrightarrow{MR}+\overrightarrow{RS}$

$=\overrightarrow{MS}$

$\vec{z}+\vec{t}=\overrightarrow{ST}+\overrightarrow{TQ}$

$=\overrightarrow{SQ}$

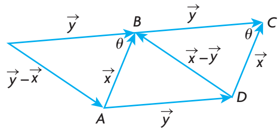
so

$(\vec{x}+\vec{y})+(\vec{z}+\vec{t})=\overrightarrow{MS}+\overrightarrow{SQ}$

$=\overrightarrow{MQ}$

7. a. $-\vec{x}$
b. $\vec{y}$
c. $\vec{x}+\vec{y}$
d. $-\vec{x}+\vec{y}$
e. $\vec{x}+\vec{y}+\vec{z}$
f. $-\vec{x}-\vec{y}$
g. $-\vec{x}+\vec{y}+\vec{z}$
h. $-\vec{x}-\vec{z}$

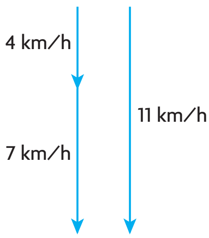
8. a.



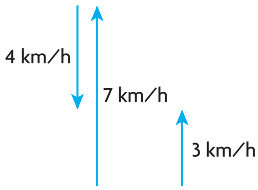
b. See the figure in part a. for the drawn vectors.
$|\vec{y}-\vec{x}|^2=|\vec{y}|^2+|\vec{x}|^2-2|\vec{y}||-\vec{x}|\cos(\theta)$ and $|-\vec{x}|=|\vec{x}|$, so $|\vec{y}-\vec{x}|^2=|\vec{x}-\vec{y}|^2$

9. a. 11 km/h

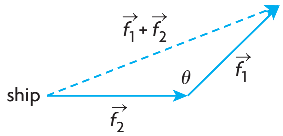
b.



c. 3 km/h



10. a.



b. The vectors form a triangle with side lengths $|\vec{f_1}|$, $|\vec{f_2}|$, and $|\vec{f_1}+\vec{f_2}|$. Find $|\vec{f_1}+\vec{f_2}|$ using the cosine law.

$|\vec{f_1}+\vec{f_2}|^2=|\vec{f_1}|^2+|\vec{f_2}|^2-2|\vec{f_1}||\vec{f_2}|\cos(\theta)$

$|\vec{f_1}+\vec{f_2}|=\sqrt{|\vec{f_1}|^2+|\vec{f_2}|^2-2|\vec{f_1}||\vec{f_2}|\cos(\theta)}$

11. 170 km/h, N28.1°W

12. $|\vec{x}+\vec{y}|=25$, $\theta \doteq 73.7°$

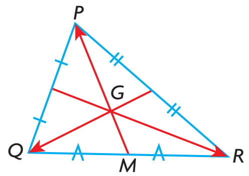
13. 0.52

14. The diagonals of a parallelogram bisect each other. So, $\overrightarrow{EA}=-\overrightarrow{EC}$ and $\overrightarrow{ED}=-\overrightarrow{EB}$.
Therefore,
$\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}=\vec{0}$.

15. Multiple applications of the Triangle Law for adding vectors show that $\overrightarrow{RM}+\vec{b}=\vec{a}+\overrightarrow{TP}$ (since both are equal to the undrawn vector $\overrightarrow{TM}$), and that $\overrightarrow{RM}+\vec{a}=\vec{b}+\overrightarrow{SQ}$ (since both are equal to the undrawn vector $\overrightarrow{RQ}$).
Adding these two equations gives
$2\overrightarrow{RM}+\vec{a}+\vec{b}=\vec{a}+\vec{b}+\overrightarrow{TP}+\overrightarrow{SQ}$
$2\overrightarrow{RM}=\overrightarrow{TP}+\overrightarrow{SQ}$

16. $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ represent the diagonals of a parallelogram with sides $\vec{a}$ and $\vec{b}$.
Since $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, and the only parallelogram with equal diagonals is a rectangle, the parallelogram must also be a rectangle.

17.

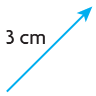


Let point M be defined as shown. Two applications of the triangle law for adding vectors show that
$\overrightarrow{GQ} + \overrightarrow{QM} + \overrightarrow{MG} = \vec{0}$
$\overrightarrow{GR} + \overrightarrow{RM} + \overrightarrow{MG} = \vec{0}$
Adding these two equations gives
$\overrightarrow{GQ} + \overrightarrow{QM} + 2\,\overrightarrow{MG} + \overrightarrow{GR} + \overrightarrow{RM} = \vec{0}$
From the given information,
$2\overrightarrow{MG} = \overrightarrow{GP}$ and
$\overrightarrow{QM} + \overrightarrow{RM} = \vec{0}$ (since they are opposing vectors of equal length), so
$\overrightarrow{GQ} + \overrightarrow{GP} + \overrightarrow{GR} = \vec{0}$, as desired.

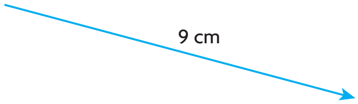
Section 6.3, pp. 298–301

1. A vector cannot equal a scalar.

2. **a.**



b.



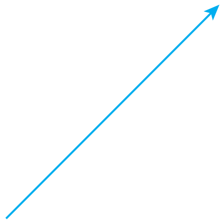
c.

d.

3. E25°N describes a direction that is 25° toward the north of due east. N65°E and “a bearing of 65°” both describe a direction that is 65° toward the east of due north.

4. Answers may vary. For example:

a.



$2\vec{v}$

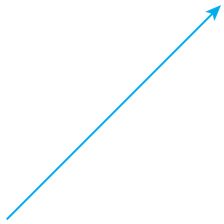
b.

$\frac{1}{2}\vec{v}$

c.

$-\frac{2}{3}\vec{v}$

d.

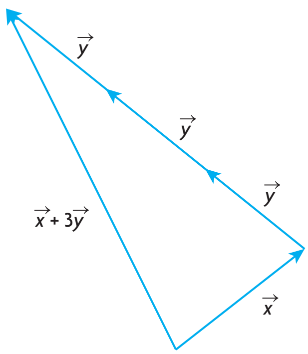


$-2\vec{v}$

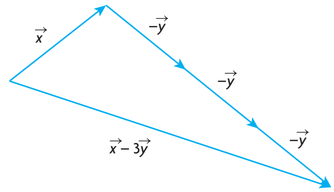
e.

$\frac{1}{|\vec{v}|}\vec{v}$

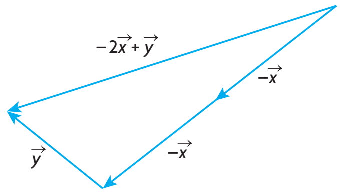
5. **a.**



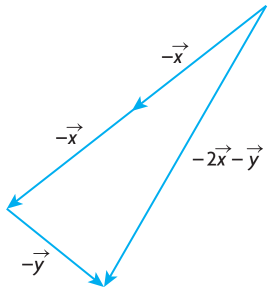
b.



c.

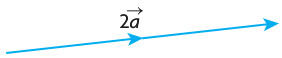


d.

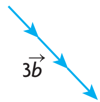


6. Answers may vary. For example:

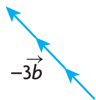
a.



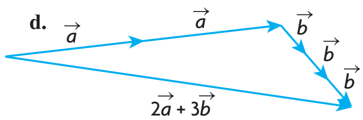
b.



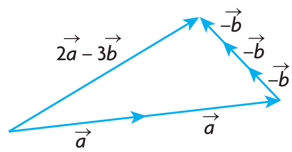
c.



d.



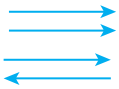
e.



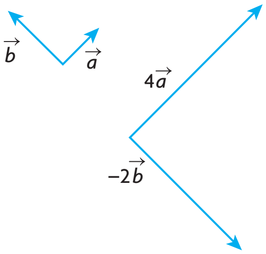
7. **a.** Answers may vary. For example: $m = 4,\ n = -3$, infinitely many

b. Answers may vary. For example: $d = 2, e = 0, f = -1$; not unique

8.



9. yes

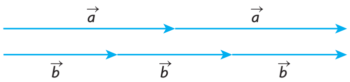


10. **a.** collinear
b. not collinear
c. not collinear
d. collinear

11. **a.** $\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the same direction as $\vec{x}$.

b. $-\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the opposite direction of $\vec{x}$.

12.



$m = \frac{2}{3}$

13. **a.** $-\frac{2}{3}\vec{a}$ **d.** $\frac{2}{3}|\vec{a}|$

b. $\frac{1}{3}\vec{a}$ **e.** $\frac{4}{3}\vec{a}$

c. $\frac{1}{3}|\vec{a}|$

14. $\sqrt{5}$ or 2.24, $\theta = 26.6°$ from $\vec{x}$ toward $2\vec{x} + \vec{y}$

15. 2.91, 9.9° from $\vec{x}$ toward $\vec{y}$

16. $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$

$|\vec{b}| = \left|\frac{1}{|\vec{a}|}\vec{a}\right|$

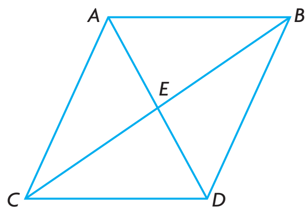
$|\vec{b}| = \frac{1}{|\vec{a}|}|\vec{a}|$

$|\vec{b}| = 1$

17. $\overrightarrow{AD} = \vec{c} + \overrightarrow{CD}$
$\overrightarrow{AD} = \vec{b} + \overrightarrow{BD}$
$2\overrightarrow{AD} = \vec{c} + \vec{b} + \overrightarrow{CD} + \overrightarrow{BD}$
But $\overrightarrow{CD} + \overrightarrow{BD} = 0$.
So, $2\overrightarrow{AD} = \vec{c} + \vec{b}$, or
$\overrightarrow{AD} = \frac{1}{2}\vec{c} + \frac{1}{2}\vec{b}$.

18. $\overrightarrow{MN} = \vec{b} - \vec{a}$
$\overrightarrow{QR} = 2\vec{b} - 2\vec{a}$
Notice that
$2\overrightarrow{MN} = 2\vec{b} - 2\vec{a}$
$= \overrightarrow{QR}$
We can conclude that $\overrightarrow{QR}$ is parallel to $\overrightarrow{MN}$ and $|\overrightarrow{QR}| = 2|\overrightarrow{MN}|$.

19.



Answers may vary. For example:
a. $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$
b. $\vec{u} = \overrightarrow{AD}$ and $\vec{v} = \overrightarrow{AE}$
c. $\vec{u} = \overrightarrow{AC}$ and $\vec{v} = \overrightarrow{DB}$
d. $\vec{u} = \overrightarrow{ED}$ and $\vec{v} = \overrightarrow{AD}$

20. **a.** $|n| = 3|m|$
b. $m = 0, n = 0$

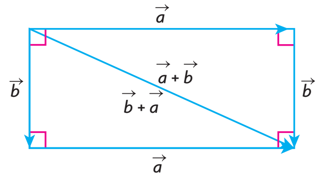
21. **a.** $\overrightarrow{CD} = \vec{b} - \vec{a}$
b. $\overrightarrow{BE} = 2\vec{b} - 2\vec{a}$
$= 2(\vec{b} - \vec{a})$
$= 2\overrightarrow{CD}$
The two are, therefore, parallel (collinear) and $\overrightarrow{BE} = 2|\overrightarrow{CD}|$.

22. Applying the triangle law for adding vectors shows that
$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$
The given information states that
$\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$
$\frac{3}{2}\overrightarrow{AB} = \overrightarrow{DC}$
By the properties of trapezoids, this gives
$\frac{3}{2}\overrightarrow{AE} = \overrightarrow{EC}$, and since
$\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC}$, the original equation gives
$\overrightarrow{AE} + \frac{3}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$
$\frac{5}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$
$\overrightarrow{AE} = \frac{2}{5}\overrightarrow{AD} + \frac{3}{5}\overrightarrow{AB}$

Section 6.4, pp. 306–307

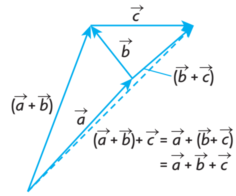
1. **a.** 0
b. 1
c. $\vec{0}$
d. 1

2.

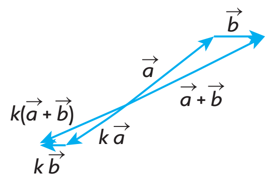


$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

3.



4. Answers may vary. For example:



5. $\overrightarrow{PQ} = \overrightarrow{RQ} + \overrightarrow{SR} + \overrightarrow{TS} + \overrightarrow{PT}$
$= (\overrightarrow{RQ} + \overrightarrow{SR}) + (\overrightarrow{TS} + \overrightarrow{PT})$
$= (\overrightarrow{SR} + \overrightarrow{RQ}) + (\overrightarrow{PT} + \overrightarrow{TS})$
$= \overrightarrow{SQ} + \overrightarrow{PS}$
$= \overrightarrow{PS} + \overrightarrow{SQ}$
$= \overrightarrow{PQ}$

6. **a.** $\overrightarrow{EC}$
b. $\vec{0}$
c. Yes, the diagonals of a rectangular prism are of equal length.

7. $-4\vec{a} + 9\vec{b} - 24\vec{c}$

8. **a.** $12\vec{i} - 17\vec{j} + 5\vec{k}$
b. $-7\vec{i} + 11\vec{j} - 4\vec{k}$
c. $= -6\vec{i} + 13\vec{j} - 7\vec{k}$

9. $\vec{x} = \frac{5}{13}\vec{a} - \frac{18}{13}\vec{b}$,
$\vec{y} = \frac{1}{13}\vec{a} + \frac{12}{13}\vec{b}$

10. $\vec{a} = \vec{x} - \vec{y}$
$= \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z} - (\vec{b} + \vec{z})$
$= \frac{2}{3}\vec{y} - \frac{2}{3}\vec{z} - \vec{b}$
$= \frac{2}{3}(\vec{y} - \vec{z}) - \vec{b}$
$= \frac{2}{3}\vec{b} - \vec{b}$
$= -\frac{1}{3}\vec{b}$

11. a. $\overrightarrow{AG} = \vec{a} + \vec{b} + \vec{c}$,
$\overrightarrow{BH} = -\vec{a} + \vec{b} + \vec{c}$,
$\overrightarrow{CE} = -\vec{a} - \vec{b} + \vec{c}$,
$\overrightarrow{DF} = \vec{a} - \vec{b} + \vec{c}$

b. $|\overrightarrow{AG}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$
$= |-\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$
$= |\overrightarrow{BH}|^2$
$\therefore |\overrightarrow{AG}| = |\overrightarrow{BH}|$

12. Applying the triangle law for adding vectors shows that
$\overrightarrow{TY} = \overrightarrow{TZ} + \overrightarrow{ZY}$
The given information states that
$\overrightarrow{TX} = 2\overrightarrow{ZY}$
$\frac{1}{2}\overrightarrow{TX} = \overrightarrow{ZY}$
By the properties of trapezoids, this gives
$\frac{1}{2}\overrightarrow{TO} = \overrightarrow{OY}$, and since
$\overrightarrow{TY} = \overrightarrow{TO} + \overrightarrow{OY}$, the original equation gives
$\overrightarrow{TO} + \frac{1}{2}\overrightarrow{TO} = \overrightarrow{TZ} + \frac{1}{2}\overrightarrow{TX}$
$\frac{3}{2}\overrightarrow{TO} = \overrightarrow{TZ} + \frac{1}{2}\overrightarrow{TX}$
$\overrightarrow{TO} = \frac{2}{3}\overrightarrow{TZ} + \frac{1}{3}\overrightarrow{TX}$

Mid-Chapter Review, pp. 308–309

1. a. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{BA} = \overrightarrow{CD}$, $\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{CB} = \overrightarrow{DA}$
There is not enough information to determine if there is a vector equal to $\overrightarrow{AP}$.
b. $|\overrightarrow{PD}| = |\overrightarrow{DA}|$
$= |\overrightarrow{BC}|$ (parallelogram)

2. a. $\overrightarrow{RV}$ c. $\overrightarrow{PS}$ e. $\overrightarrow{PS}$
b. $\overrightarrow{RV}$ d. $\overrightarrow{RU}$ f. $\overrightarrow{PQ}$

3. a. $\sqrt{3}$
b. 53°

4. $t = 4$ or $t = -4$

5. In quadrilateral $PQRS$, look at $\triangle PQR$. Joining the midpoints B and C creates a vector $\overrightarrow{BC}$ that is parallel to $\overrightarrow{PR}$ and half the length of $\overrightarrow{PR}$. Look at $\triangle SPR$. Joining the midpoints A and D creates a vector $\overrightarrow{AD}$ that is parallel to $\overrightarrow{PR}$ and half the length of $\overrightarrow{PR}$. $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ and equal in length to $\overrightarrow{AD}$. Therefore, $ABCD$ is a parallelogram.

6. a. $2\sqrt{21}$
b. 71°
c. $\frac{1}{|\vec{u} + \vec{v}|}(\vec{u} + \vec{v}) = \frac{1}{2\sqrt{21}}(\vec{u} + \vec{v})$
d. $20\sqrt{7}$

7. 3

8. $|\vec{m} + \vec{n}| = \big||\vec{m}| - |\vec{n}|\big|$

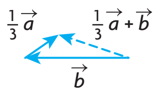
9. $\overrightarrow{BC} = -\vec{y}$, $\overrightarrow{DC} = \vec{x}$,
$\overrightarrow{BD} = -\vec{x} - \vec{y}$, $\overrightarrow{AC} = \vec{x} - \vec{y}$

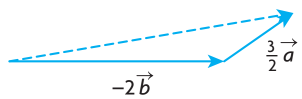
10. Construct a parallelogram with sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Since the diagonals bisect each other, $2\overrightarrow{OB}$ is the diagonal equal to $\overrightarrow{OA} + \overrightarrow{OC}$. Or
$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC}$.
So, $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$.
And $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$.
Now $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OC} - \overrightarrow{OA})$,
Multiplying by 2 gives
$2\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$.

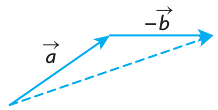
11. $\overrightarrow{BD} = 2\vec{x} + \vec{y}$
$\overrightarrow{BC} = 2\vec{x} - \vec{y}$

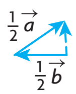
12. 460 km/h, south

13. a. $\overrightarrow{PT}$
b. $\overrightarrow{PT}$
c. $\overrightarrow{SR}$

14. a.


b.


c.


d.


15. $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
$= 3\vec{b} - \vec{a}$
$\overrightarrow{RS} = \overrightarrow{QS} - \overrightarrow{QR}$
$= -3\vec{a}$

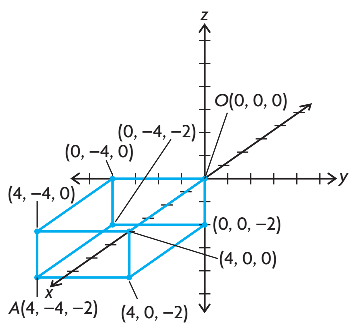
Section 6.5, pp. 316–318

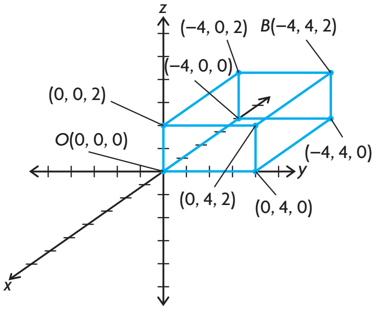
1. No, as the y-coordinate is not a real number.

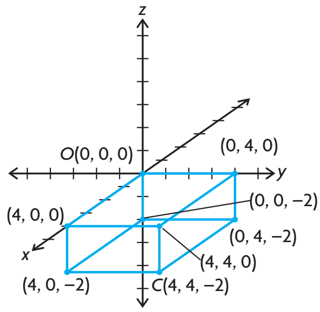
2. a. We first arrange the x-, y-, and z-axes (each a copy of the real line) in a way so that each pair of axes are perpendicular to each other (i.e., the x- and y-axes are arranged in their usual way to form the xy-plane, and the z-axis passes through the origin of the xy-plane and is perpendicular to this plane). This is easiest viewed as a "right-handed system," where, from the viewer's perspective, the positive z-axis points upward, the positive x-axis points out of the page, and the positive y-axis points rightward in the plane of the page. Then, given point $P(a, b, c)$, we locate this point's unique position by moving a units along the x-axis, then from there b units parallel to the y-axis, and finally c units parallel to the z-axis. It's associated unique position vector is determined by drawing a vector with tail at the origin $O(0, 0, 0)$ and head at P.
b. Since this position vector is unique, its coordinates are unique.
Therefore, $a = -4$, $b = -3$, and $c = -8$.

3. a. $a = 5$, $b = -3$, and $c = 8$.
b. $(5, -3, 8)$

4. This is not an acceptable vector in I^3 as the z-coordinate is not an integer. However, since all of the coordinates are real numbers, this is acceptable as a vector in R^3.

5.






6. **a.** $A(0, -1, 0)$ is located on the y-axis. $B(0, -2, 0)$ $C(0, 2, 0)$, and $D(0, 10, 0)$ are three other points on this axis.

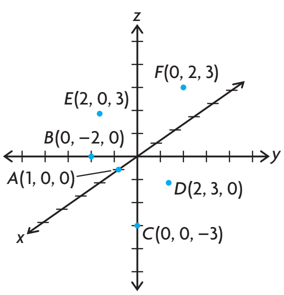
b. $\overrightarrow{OA} = (0, -1, 0)$, the vector with tail at the origin $O(0, 0, 0)$ and head at A.

7. **a.** Answers may vary. For example: $\overrightarrow{OA} = (0, 0, 1)$, $\overrightarrow{OB} = (0, 0, -1)$, $\overrightarrow{OC} = (0, 0, -5)$

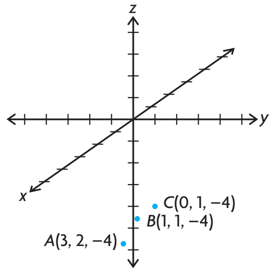
b. Yes, these vectors are collinear (parallel), as they all lie on the same line, in this case the z-axis.

c. A general vector lying on the z-axis would be of the form $\overrightarrow{OA} = (0, 0, a)$ for any real number a. Therefore, this vector would be represented by placing the tail at O and the head at the point (0, 0, a) on the z-axis.

8.

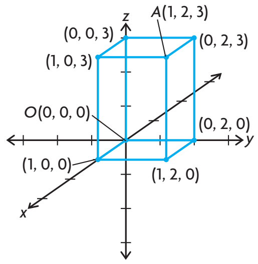


9. **a.**

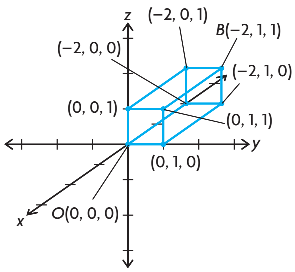


b. $z = -4$

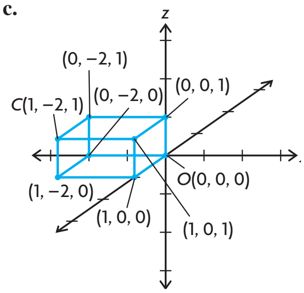
10. **a.**



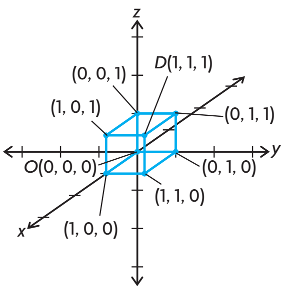
b.



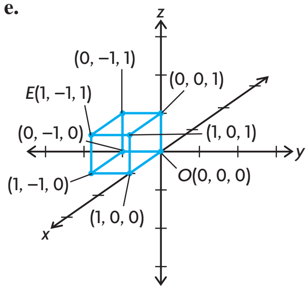
c.



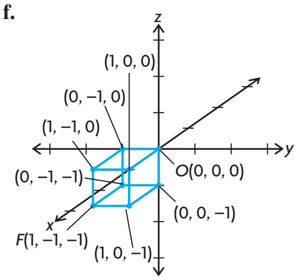
d.



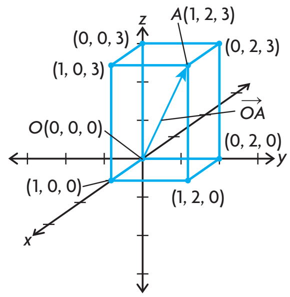
e.



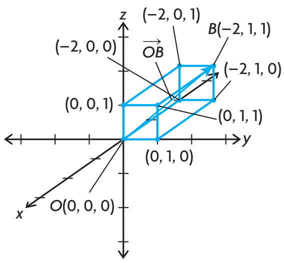
f.



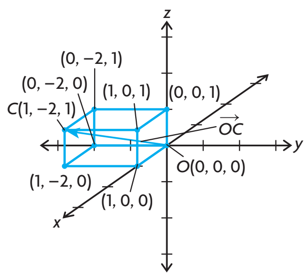
11. **a.** $\overrightarrow{OA} = (1, 2, 3)$



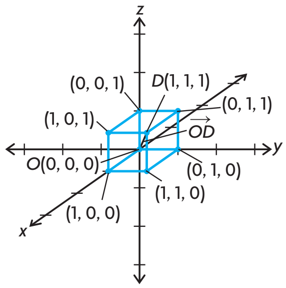
b. $\overrightarrow{OB} = (-2, 1, 1)$



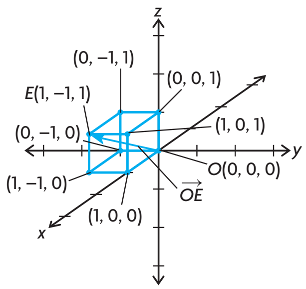
c. $\overrightarrow{OC} = (1, -2, 1)$



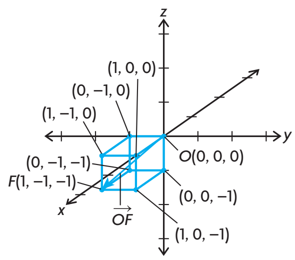
d. $\overrightarrow{OD} = (1, 1, 1)$



e. $\overrightarrow{OE} = (1, -1, 1)$



f. $\overrightarrow{OF} = (1, -1, -1)$



12. **a.** $a = 11$ $c = 5$

b. Since P and Q represent the same point in R^3, they will have the same associated position vector, i.e., $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$. So, since these vectors are equal, they will certainly have equal magnitudes, i.e., $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$.

13. xy-plane, $z = 0$; xz-plane, $y = 0$; yz-plane, $x = 0$.

14. **a.** Every point on the plane containing points M, N, and P has y-coordinate equal to 0. Therefore, the equation of the plane containing these points is $y = 0$ (this is just the xz-plane).

b. The plane $y = 0$ contains the origin $O(0, 0, 0)$, and so since it also contains the points M, N, and P as well, it will contain the position vectors associated with these points joining O (tail) to the given point (head). That is, the plane $y = 0$ contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

15. **a.** $A(-2, 0, 0)$, $B(-2, 4, 0)$, $C(0, 4, 0)$, $D(0, 0, -7)$, $E(0, 4, -7)$, $F(-2, 0, -7)$

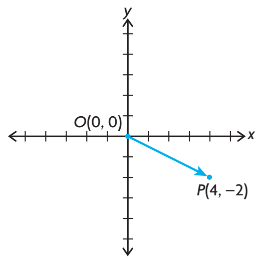
b. $\overrightarrow{OA} = (-2, 0, 0)$,
$\overrightarrow{OB} = (-2, 4, 0)$,
$\overrightarrow{OC} = (0, 4, 0)$, $\overrightarrow{OD} = (0, 0, -7)$,
$\overrightarrow{OE} = (0, 4, -7)$,
$\overrightarrow{OF} = (-2, 0, -7)$

c. 7 units

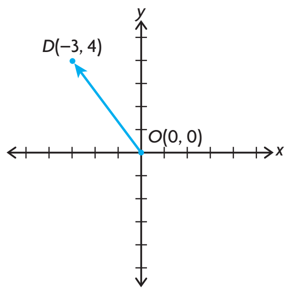
d. $y = 4$

e. Every point contained in rectangle $BCEP$ has y-coordinate equal to 4, and so is of the form $(x, 4, z)$, where x and z are real numbers such that $-2 \le x \le 0$ and $-7 \le z \le 0$.

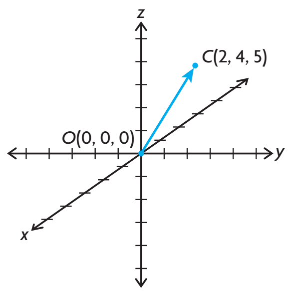
16. **a.**



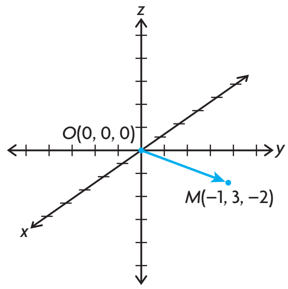
b.



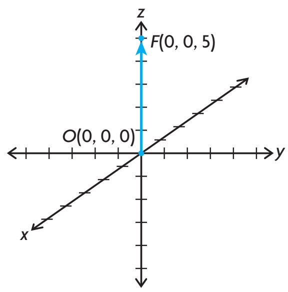
c.



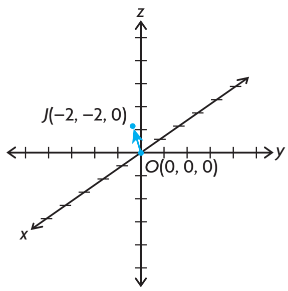
d.



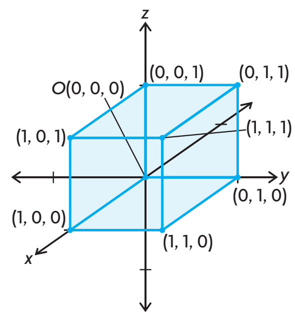
e.



f.



17.



18. First, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$ by the triangle law of vector addition, where $\overrightarrow{OA} = (5, -10, 0)$, $\overrightarrow{OB} = (0, 0, -10)$, $\overrightarrow{OP}$ and $\overrightarrow{OA}$ are drawn in standard position (starting from the origin O $(0, 0, 0)$), and $\overrightarrow{OB}$ is drawn starting from the head of $\overrightarrow{OA}$. Notice that $\overrightarrow{OA}$

lies in the xy-plane, and $\overrightarrow{OB}$ is perpendicular to the xy-plane (so is perpendicular to $\overrightarrow{OA}$). So, $\overrightarrow{OP}$, $\overrightarrow{OA}$, and $\overrightarrow{OB}$ form a right triangle and, by the Pythagorean theorem,

$|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$

Similarly, $\overrightarrow{OA} = \vec{a} + \vec{b}$ by the triangle law of vector addition, where $\vec{a} = (5, 0, 0)$ and $\vec{b} = (0, -10, 0)$, and these three vectors form a right triangle as well. So,

$$\begin{aligned}|\overrightarrow{OA}|^2 &= |\vec{a}|^2 + |\vec{b}|^2\\ &= 25 + 100\\ &= 125\end{aligned}$$

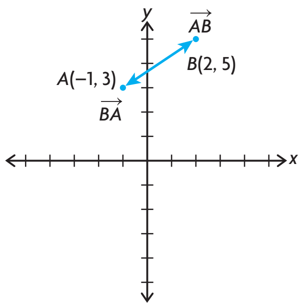
Obviously $|\overrightarrow{OB}|^2 = 100$, and so substituting gives

$$\begin{aligned}|\overrightarrow{OP}|^2 &= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\\ &= 125 + 100\\ &= 225\\ |\overrightarrow{OP}| &= \sqrt{225}\\ &= 15\end{aligned}$$

19. $(6, -5, 2)$

Section 6.6, pp. 324–326

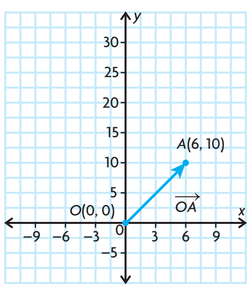
1. **a.** $\overrightarrow{AB} = (3, 2)$, $\overrightarrow{BA} = (-3, -2)$



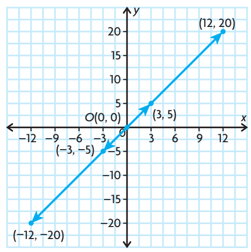
b. $|\overrightarrow{OA}| \doteq 3.16$, $|\overrightarrow{OB}| \doteq 5.39$

c. $|\overrightarrow{AB}| \doteq 3.61$, $|\overrightarrow{BA}| \doteq 3.61$

2.



a.



b. $\frac{1}{2}\overrightarrow{OA}$ and $-\frac{1}{2}\overrightarrow{OA}$, $2\overrightarrow{OA}$ and $-2\overrightarrow{OA}$

3. 5

4. **a.** $a = -3, b = 5$

b. 5.83

5. **a.** $|\vec{a}| = 61$, $|\vec{b}| = 41$

b. $|\vec{a} + \vec{b}| \doteq 100.02$, $|\vec{a} - \vec{b}| \doteq 28.28$

6. **a.** $(-2, 7)$

b. $(-30, 0)$

c. $(1, 11)$

7. **a.** $7\vec{i} - 8\vec{j}$

b. $3\vec{i} - 51\vec{j}$

c. $-29\vec{i} + 28\vec{j}$

8. **a.** about 4.12

b. about 6.71

c. about 18.38

d. about 18.38

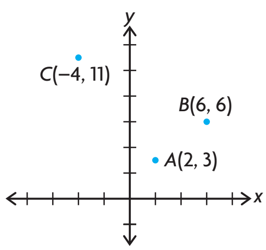
9. **a.** $\overrightarrow{AB} = (4, 2)$, $\overrightarrow{CD} = (2, 4)$, $\overrightarrow{EF} = (-6, 4)$, $\overrightarrow{GH} = (5, 0)$

b. $|\overrightarrow{AB}| \doteq 4.47$, $|\overrightarrow{CD}| \doteq 4.47$, $|\overrightarrow{EF}| \doteq 7.21$, $|\overrightarrow{GH}| = 5$

10. **a.** $\overrightarrow{OC} = (17, -3)$, $\overrightarrow{BA} = (-5, 9)$, $\overrightarrow{BC} = (6, 3)$

b. $\overrightarrow{OA} = (6, 3) = \overrightarrow{BC}$, so obviously we will have $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.

11. **a.**

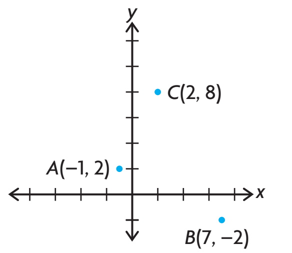


b. $\overrightarrow{AB}(4, 3)$, $|\overrightarrow{AB}| = 5$, $|\overrightarrow{AC}| = 10$, $\overrightarrow{CB} \doteq 11.18$

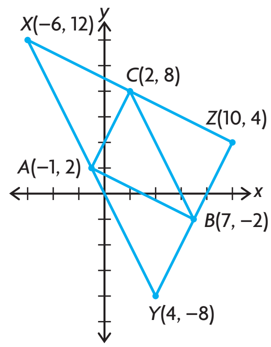
c. $|\overrightarrow{CB}|^2 = 125$, $|\overrightarrow{AC}|^2 = 100$, $|\overrightarrow{AB}|^2 = 25$

Since $|\overrightarrow{CB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{AB}|^2$, the triangle is a right triangle.

12. **a.**



b.

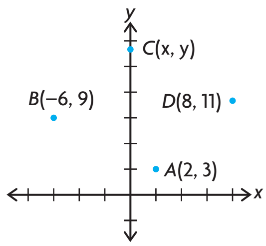


c. $X(-6, 12)$, $Y(4, -8)$, $Z(10, 4)$

13. **a.** $x = 7, y = -2$

b. $x = -12, y = 4$.

14. **a.**



b. Because $ABCD$ is a rectangle, we will have

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{AD}\\ (x, y) - (-6, 9) &= (8, 11) - (2, 3)\\ (x + 6, y - 9) &= (6, 8)\\ x + 6 &= 6\\ y - 9 &= 8\end{aligned}$$

So, $x = 0$ and $y = 17$, i.e., $C(0, 17)$.

15. **a.** $P\left(\frac{21}{10}, 0\right)$

b. $Q\left(0, -\frac{21}{4}\right)$

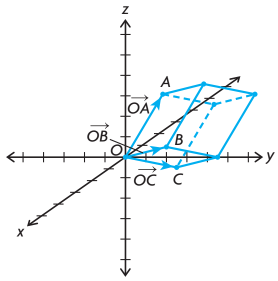
16. $\left(\frac{9}{41}, \frac{40}{41}\right)$

17. **a.** about 80.9°

b. about 35.4°

Section 6.7, pp. 332–333

1. a. $-1\vec{i} + 2\vec{j} + 4\vec{k}$
   b. about 4.58
2. $\vec{OB} = (3, 4, -4)$, $|\vec{OB}| = 6.40$
3. 3
4. a. $(-1, 6, 11)$
   b. $|\vec{OA}| = 13$, $|\vec{OB}| = 3$, $|\vec{OP}| \doteq 12.57$
   c. $\vec{AB} = (5, -2, -13)$, $|\vec{AB}| \doteq 14.07$, $\vec{AB}$ represents the vector from the tip of $\vec{OA}$ to the tip of $\vec{OB}$. It is the difference between the two vectors.
5. a. $(1, -3, 3)$
   b. $(-7, -16, 8)$
   c. $\left(-\frac{13}{2}, 2, \frac{3}{2}\right)$
   d. $(2, 30, -13)$
6. a. $\vec{i} - 2\vec{j} + 2\vec{k}$
   b. $3\vec{i} + 0\vec{j} + 0\vec{k}$
   c. $9\vec{i} + 3\vec{j} - 3\vec{k}$
   d. $-9\vec{i} - 3\vec{j} + 3\vec{k}$
7. a. about 5.10
   b. about 1.41
   c. about 5.39
   d. about 11.18
8. $\vec{x} = \vec{i} + 4\vec{j} - \vec{k}$,
   $\vec{y} = -2\vec{i} - 2\vec{j} + 6\vec{k}$
9. a. The vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$ represent the xy-plane, xz-plane, and yz-plane, respectively. They are also the vector from the origin to points $(a, b, 0)$, $(a, 0, c)$, and $(0, b, c)$, respectively.
   b. $\vec{OA} = a\vec{i} + b\vec{j} + 0\vec{k}$,
   $\vec{OB} = a\vec{i} + 0\vec{j} + c\vec{k}$,
   $\vec{OC} = 0\vec{i} + b\vec{j} + c\vec{k}$
   c. $|\vec{OA}| = \sqrt{a^2 + b^2}$,
   $|\vec{OB}| = \sqrt{a^2 + c^2}$,
   $|\vec{OB}| = \sqrt{b^2 + c^2}$
   d. $(0, -b, c)$, $\vec{AB}$ is a direction vector from A to B.
10. a. 7
    b. 13
    c. $(5, 2, 9)$
    d. 10.49
    e. $(-5, -2, -9)$
    f. 10.49
11. In order to show that $ABCD$ is a parallelogram, we must show that $\vec{AB} = \vec{DC}$ or $\vec{BC} = \vec{AD}$. This will show they have the same direction, thus the opposite sides are parallel.
    $\vec{AB} = (3, -4, 12)$
    $\vec{DC} = (3, -4, 12)$
    We have shown $\vec{AB} = \vec{DC}$ and $\vec{BC} = \vec{AD}$, so $ABCD$ is a parallelogram.
12. $a = \frac{2}{3}$, $b = 7$, $c = 0$
13. a.



    b. $V_1 = (0, 0, 0)$,
    $V_2 = (-2, 2, 5)$,
    $V_3 = (0, 4, 1)$,
    $V_4 = (0, 5, -1)$,
    $V_5 = (-2, 6, 6)$,
    $V_6 = (-2, 7, 4)$,
    $V_7 = (0, 9, 0)$,
    $V_8 = (-2, 11, 5)$
14. $(1, 0, 0)$
15. 4.36

Section 6.8, pp. 340–341

1. They are collinear, thus a linear combination is not applicable.
2. It is not possible to use $\vec{0}$ in a spanning set. Therefore, the remaining vectors only span R^2.
3. The set of vectors spanned by $(0, 1)$ is $m(0, 1)$. If we let $m = -1$, then $m(0, 1) = (0, -1)$.
4. $\vec{i}$ spans the set $m(1, 0, 0)$. This is any vector along the x-axis. Examples: $(2, 0, 0)$, $(-21, 0, 0)$.
5. As in question 2, it isn't possible to use $\vec{0}$ in a spanning set.
6. $\{(-1, 2), (-1, 1)\}$, $\{(2, -4), (-1, 1)\}$, $\{(-1, 1), (-3, 6)\}$ are all the possible spanning sets for R^2 with 2 vectors.
7. a. $14\vec{i} - 43\vec{j} + 40\vec{k}$
   b. $-7\vec{i} + 23\vec{j} - 14\vec{k}$
8. $\{(1, 0, 0), (0, 1, 0)\}$:
   $(-1, 2, 0) = -1(1, 0, 0) + 2(0, 1, 0)$
   $(3, 4, 0) = 3(1, 0, 0) + 4(0, 1, 0)$
   $\{(1, 1, 0), (0, 1, 0)\}$
   $(-1, 2, 0) = -1(1, 1, 0) + 3(0, 1, 0)$
   $(3, 4, 0) = 3(1, 1, 0) + (0, 1, 0)$
9. a. It is the set of vectors in the xy-plane.
   b. $-2(1, 0, 0) + 4(0, 1, 0)$
   c. By part a., the vector is not in the xy-plane. There is no combination that would produce a number other than 0 for the z-component.
   d. It would still only span the xy-plane. There would be no need for that vector.
10. $a = -2$, $b = 24$, $c = 3$
11. $(-10, -34) = 2(-1, 3) - 8(1, 5)$
12. a. $a = x + y$,
    $b = x + 2y$
    b. $(2, -3) = -1(2, -1) - 4(-1, 1)$
    $(124, -5) = 119(2, 1) + 114(-1, 1)$
    $(4, -11) = -7(2, 1) - 18(-1, 1)$
13. a. The statement $a(-1, 2, 3) + b(4, 1, -2) = (-14, -1, 16)$ does not have a consistent solution.
    b. $3(-1, 3, 4) - 5(0, -1, 1) = (-3, 14, 7)$
14. -7
15. $m = 2$, $n = 3$; Non-parallel vectors cannot be equal, unless their magnitudes equal 0.
16. Answers may vary. For example:
    $p = -6$ and $q = 1$,
    $p = 25$ and $q = 0$,
    $p = \frac{13}{3}$ and $q = \frac{2}{3}$
17. As in question 15, non-parallel vectors. Their magnitudes must be 0 again to make the equality true.
    $m^2 + 2m - 3 = (m - 1)(m + 3)$
    $m = 1, -3$
    $m^2 + m - 6 = (m - 2)(m + 3)$
    $m = 2, -3$
    So, when $m = -3$, their sum will be 0.

Review Exercise, pp. 344–347

1. a. false; Let $\vec{b} = -\vec{a} \neq 0$, then:
   $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})|$
   $= |0|$
   $= 0 < |\vec{a}|$
   b. true; $|\vec{a} + \vec{b}|$ and $|\vec{a} + \vec{c}|$ both represent the lengths of the diagonal of a parallelogram, the first with sides $\vec{a}$ and $\vec{b}$ and the second with sides $\vec{a}$ and $\vec{c}$; since both parallelograms have $\vec{a}$ as a side and diagonals of equal length $|\vec{b}| = |\vec{c}|$.
   c. true; Subtracting $\vec{a}$ from both sides shows that $\vec{b} = \vec{c}$.

d. true; Draw the parallelogram formed by $\vec{RF}$ and $\vec{SW}$. $\vec{FW}$ and $\vec{RS}$ are the opposite sides of a parallelogram and must be equal.

e. true; the distributive law for scalars

f. false; Let $\vec{b} = -\vec{a}$ and let $\vec{c} = \vec{d} \neq 0$. Then,
$|\vec{a}| = |-\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$
but $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})| = 0$

$|\vec{c} + \vec{b}| = |\vec{c} + \vec{c}| = |2\vec{c}|$
so $|\vec{a} + \vec{b}| \neq |\vec{c} + \vec{d}|$

2. **a.** $20\vec{a} - 30\vec{b} + 8\vec{c}$
b. $\vec{a} - 3\vec{b} - 3\vec{c}$

3. **a.** $\vec{XY} = (-2, 3, 6)$,
$|\vec{XY}| = 7$
b. $\left(-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

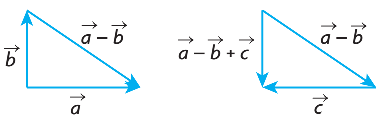
4. **a.** $(-6, -3, -6)$
b. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

5. $\left(-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

6. **a.** $\vec{OA} + \vec{OB} = (-3, 8, -8)$,
$\vec{OA} - \vec{OB} = (9, -4, -4)$
b. $\theta \doteq 84.4°$

7. **a.** $|\vec{AB}| = \sqrt{14}$,
$|\vec{BC}| = \sqrt{59}$,
$|\vec{CA}| = \sqrt{45}$
b. 12.5
c. 18.13
d. $(6, 2, -2)$

8. **a.**



b. 5

9. $\frac{1}{2}(-11, 7) + \left(-\frac{3}{2}\right)(-3, 1) = (-1, 2)$,
$\frac{1}{3}(-11, 7) + \left(-\frac{2}{3}\right)(-1, 2) = (-3, 1)$,
$3(-3, 1) + 2(-1, 2) = (-11, 7)$

10. **a.** $x - 3y + 6z = 0$ where $P(x, y, z)$ is the point.
b. $(0, 0, 0)$ and $\left(1, \frac{1}{3}, 0\right)$

11. **a.** $a = -3, b = 26.5, c = 10$
b. $a = 8, b = \frac{7}{3}, c = -10$

12. **a.** yes
b. yes

13. **a.** $|\vec{AB}|^2 = 9$, $|\vec{AC}|^2 = 3$, $|\vec{BC}|^2 = 6$
Since $|\vec{AB}|^2 = |\vec{AC}|^2 + |\vec{BC}|^2$ the triangle is right-angled
b. $\frac{\sqrt{6}}{3}$

14. **a.** $\vec{DA}, \vec{BC}$ and $\vec{EB}, \vec{ED}$
b. $\vec{DC}, \vec{AB}$ and $\vec{CE}, \vec{EA}$
c. $|\vec{AD}|^2 + |\vec{DC}|^2 = |\vec{AC}|^2$
But $|\vec{AC}|^2 = |\vec{DB}|^2$
Therefore, $|\vec{AD}|^2 + |\vec{DC}|^2 = |\vec{DB}|^2$

15. **a.** $C(3, 0, 5)$, $P(3, 4, 5)$, $E(0, 4, 5)$, $F(0, 4, 0)$
b. $\vec{DB} = (3, 4, -5)$,
$\vec{CF} = (-3, 4, -5)$
c. 90°
d. 50.2°

16. **a.** 7.74
b. 2.83
c. 2.83

17. **a.** 1236.9 km
b. S14.0°W

18. **a.** Any pair of nonzero, noncollinear vectors will span R^2. To show that (2, 3) and (3, 5) are noncollinear, show that there does not exist any number k such that $k(2, 3) = (3, 5)$.
Solve the system of equations:
$2k = 3$
$3k = 5$
Solving both equations gives two different values for k, $\frac{3}{2}$ and $\frac{5}{3}$, so (2, 3) and (3, 5) are noncollinear and thus span R^2.
b. $m = -770, n = 621$

19. **a.** Find a and b such that
$(5, 9, 14) = a(-2, 3, 1) + b(3, 1, 4)$
$(5, 9, 14) = (-2a, 3a, a) + (3b, b, 4b)$
$(5, 9, 14) = (-2a + 3b, 3a + b, a + 4b)$
i. $5 = -2a + 3b$
ii. $9 = 3a + b$
iii. $14 = a + 4b$
Use the method of elimination with i. and iii.

$$\begin{aligned} 2(14) &= 2(a + 4b) \\ 28 &= 2a + 8b \\ +\quad 5 &= -2a + 3b \\ \hline 33 &= 11b \\ 3 &= b \end{aligned}$$

By substitution, $a = 2$.
$\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$ because it can be written as a linear combination of $\vec{b}$ and $\vec{c}$.

b. If vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$, then $\vec{a}$ can be written as a linear combination of $\vec{b}$ and $\vec{c}$. Find m and n such that

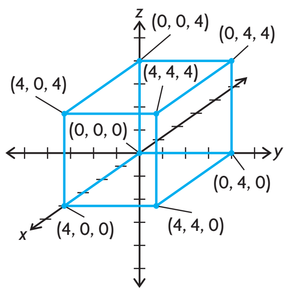
$$\begin{aligned} (-13, 36, 23) &= m(-2, 3, 1) + n(3, 1, 4) \\ &= (-2m, 3m, m) + (3n, n, 4n) \\ &= (-2m + 3n, 3m + n, m + 4n) \end{aligned}$$

Solve the system of equations:
$-13 = -2m + 3n$
$36 = 3m + n$
$23 = m + 4n$
Use the method of elimination:

$$\begin{aligned} 2(23) &= 2(m + 4n) \\ 46 &= 2m + 8n \\ +\ -13 &= -2m + 3n \\ \hline 33 &= 11n \\ 3 &= n \end{aligned}$$

By substitution, $m = 11$.
So, vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$.

20. **a.**



b. $(-4, -4, -4)$
c. $(-4, 0, -4)$
d. $(4, 4, 0)$

21. 7

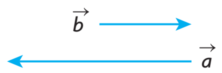
22. **a.** $|\vec{AB}| = 10$,
$|\vec{BC}| = 2\sqrt{5} = 4.47$,
$|\vec{CA}| = \sqrt{80} = 8.94$
b. If A, B, and C are vertices of a right triangle, then
$|\vec{BC}|^2 + |\vec{CA}|^2 = |\vec{AB}|^2$

$$\begin{aligned} |\vec{BC}|^2 + |\vec{CA}|^2 &= (2\sqrt{5})^2 + (\sqrt{80})^2 \\ &= 20 + 80 \\ &= 100 \\ |\vec{AB}|^2 &= 10^2 \\ &= 100 \end{aligned}$$

So, triangle ABC is a right triangle.

23. **a.** $\vec{a} + \vec{b} + \vec{c}$
b. $\vec{a} - \vec{b}$
c. $-\vec{b} - \vec{a} + \vec{c}$
d. $\vec{0}$
e. $\vec{b} + \vec{c}$

24.

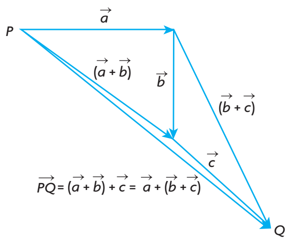


25. **a.** $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$

26. **Case 1** If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.
Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition.
We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. **a.** (8, 4, 8)
b. 12
c. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

3. $\sqrt{19}$

4. **a.** $\vec{x} = 2\vec{b} - 3\vec{a}, \vec{y} = 3\vec{b} - 5\vec{a}$
b. $a = 1, b = 5, c = -11$

5. **a.** $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.
b. $p = -2, q = 3$

6. **a.** $(1, 12, -29) = -2(3, 1, 4) + 7(1, 2, -3)$
b. $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

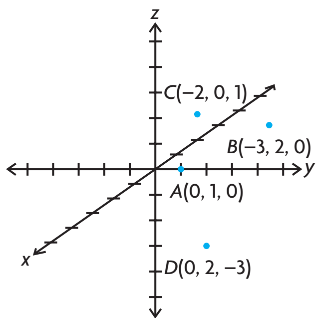
7. $\sqrt{13}$, $\theta \doteq 3.61$; 73.9° relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$
$\overrightarrow{DE} = \vec{b} - \vec{a}$
Also,
$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$
$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$
Thus,
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$

Chapter 7

Review of Prerequisite Skills, p. 350

1. $v \doteq 806$ km/h N 7.1° E
2. 15.93 units W 32.2° N
3.



4. **a.** $(3, -2, 7)$; $l \doteq 7.87$
b. $(-9, 3, 14)$; $l \doteq 16.91$
c. $(1, 1, 0)$; $l \doteq 1.41$
d. $(2, 0, -9)$; $l \doteq 9.22$

5. **a.** $(x, y, 0)$
b. $(x, 0, z)$
c. $(0, y, z)$

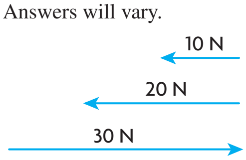
6. **a.** $\vec{i} - 7\vec{j}$
b. $6\vec{i} - 2\vec{j}$
c. $-8\vec{i} + 11\vec{j} + 3\vec{k}$
d. $4\vec{i} - 6\vec{j} + 8\vec{k}$

7. **a.** $\vec{i} + 3\vec{j} - \vec{k}$
b. $5\vec{i} + \vec{j} - \vec{k}$
c. $12\vec{i} + \vec{j} - 2\vec{k}$

Section 7.1, pp. 362–364

1. **a.** 10 N is a melon, 50 N is a chair, 100 N is a computer
b. Answers will vary.

2. **a.**



b. 180°

3. a line along the same direction

4. For three forces to be in equilibrium, they must form a triangle, which is a planar figure.

5. **a.** The resultant is 13 N at an angle of N 22.6° W. The equilibrant is 13 N at an angle of S 22.6° W.
b. The resultant is 15 N at an angle of S 36.9° W. The equilibrant is 15 N at N 36.9° E.

6. **a.** yes **b.** yes **c.** no **d.** yes

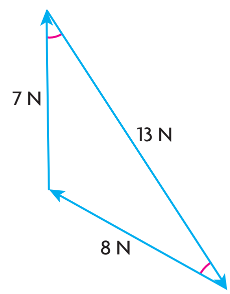
7. Arms 90 cm apart will yield a resultant with a smaller magnitude than at 30 cm apart. A resultant with a smaller magnitude means less force to counter your weight, hence a harder chin-up.

8. The resultant would be 12.17 N at 34.7° from the 6 N force toward the 8 N force. The equilibrant would be 12.17 N at 145.3° from the 6 N force away from the 8 N force.

9. 9.66 N 15° from given force, 2.95 N perpendicular to 9.66 N force

10. 49 N directed up the ramp

11. **a.**

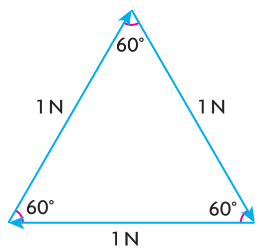


b. 60°

12. approximately 7.1 N 45° south of east

13. **a.** 7
b. The angle between f_1 and the resultant is 16.3°. The angle between $\vec{f}_1$ and the equilibrant is 163.7°.

14. **a.**



For these three equal forces to be in equilibrium, they must form an equilateral triangle. Since the resultant will lie along one of these lines, and since all angles of an equilateral triangle are 60°, the resultant will be at a 60° angle with the other two vectors.