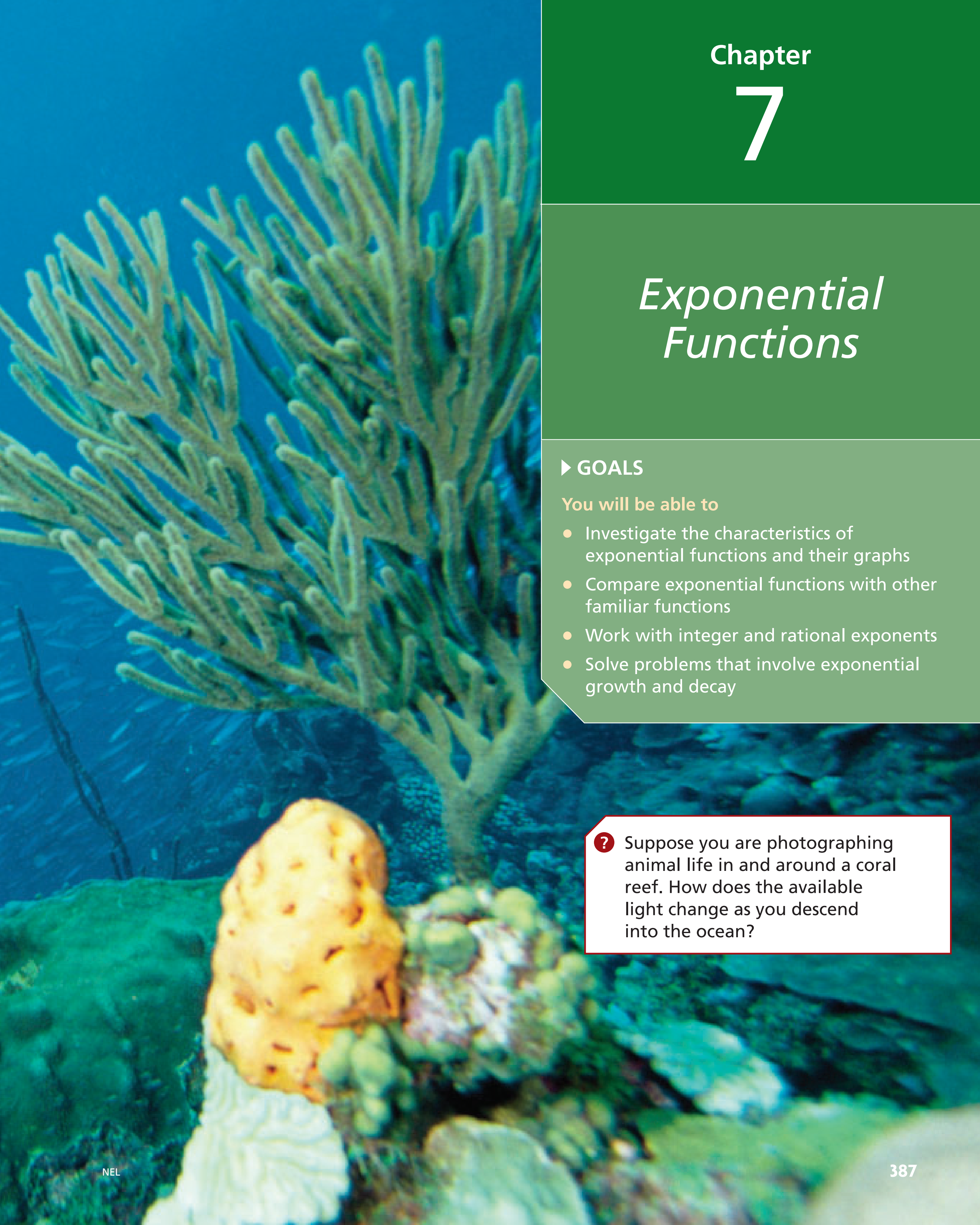


Chapter

7

Exponential Functions

GOALS

You will be able to

- Investigate the characteristics of exponential functions and their graphs
- Compare exponential functions with other familiar functions
- Work with integer and rational exponents
- Solve problems that involve exponential growth and decay

? Suppose you are photographing animal life in and around a coral reef. How does the available light change as you descend into the ocean?

7 Getting Started

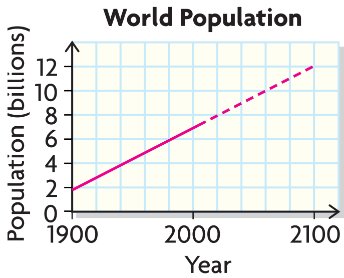
WORDS You Need to Know

1. Match each term with its picture or example.

a) circumference **b)** surface area **c)** exponent **d)** base **e)** power **f)** rational numbers **g)** increasing function **h)** decreasing function

i) 2^5

ii)

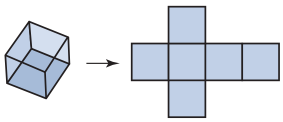


iii) 8^2

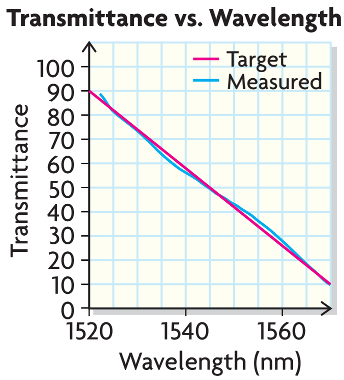
iv) $-\frac{3}{4}$

v) 8^2

vi)



vii)



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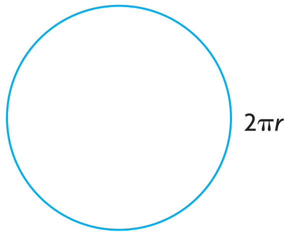


SKILLS AND CONCEPTS You Need

Square Root and Cube Root Estimates

The square root of a number is the number that multiplies by itself to give the required value. The square root symbol is $\sqrt{\ }$. The cube root of a number is the number that multiplies itself three times to give a required value. The cube root symbol is $\sqrt[3]{\ }$.

EXAMPLES

a) $\sqrt{36} = 6$ since $6 \times 6 = 36$

b) $\sqrt{0.25} = 0.5$ since $0.5 \times 0.5 = 0.25$

c) $\sqrt[3]{125} = 5$ since $5 \times 5 \times 5 = 125$

d) $\sqrt[3]{64} = 4$ since $4 \times 4 \times 4 = 64$

Estimating Square Roots and Cube Roots

EXAMPLE

Is the square root of 17 closer to 5 or to 4?

Solution: $5 \times 5 = 25$ and
$4 \times 4 = 16$

17 is closer to 16 than to 25, so $\sqrt{17}$ is closer to 4.

EXAMPLE

Is the cube root of 110 closer to 4 or 5?

Solution: $4 \times 4 \times 4 = 64$ and
$5 \times 5 \times 5 = 125$

110 is closer to 125 than to 64, so $\sqrt[3]{110}$ is closer to 5.

2. Select the best answer from those given.

a) Which is the better estimate of $\sqrt{34}$? 5.2 5.9

b) Which is the better estimate of $\sqrt{50}$? 7.1 7.8

c) Which is the better estimate of $\sqrt[3]{109}$? 4.4 4.8

d) Which is the better estimate of $\sqrt[3]{300}$? 6.4 6.7

Evaluating Powers

5^2 is called a power. 5 is the base and 2 is the exponent.

$$5^2 = 5 \times 5 \quad \text{(expanded form)}$$
$$= 25$$

Study Aid

- For help, see Essential Skills Appendix, A-3.

Exponent Laws

Rule	Written Description	Algebraic Description	Worked Example in Standard Form
Multiplication	To multiply powers with the same base, keep the base the same and add the exponents.	$b^m \times b^n = b^{m+n}$	$3^2 \times 3^3 = 3^{2+3}$ $= 3^5$ $= 243$
Division	To divide powers with the same base, keep the base the same and subtract the exponents.	$b^m \div b^n = b^{m-n}$	$4^5 \times 4^2 = 4^{5-2}$ $= 4^3$ $= 64$
Power of a Power	To simplify a power of a power, keep the base the same and multiply the exponents.	$(b^m)^n = b^{m \times n}$	$(2^3)^2 = 2^{3\times2}$ $= 2^6$ $= 64$
Power of a Product	To simplify a power of a product, apply the exponent to both numbers in the product.	$(a \times b)^m = a^m \times b^m$	$(5 \times 3)^2 = 5^2 \times 3^2$ $= 25 \times 9$ $= 225$
Power of a Quotient	To simplify a power of a quotient, apply the exponent to both numbers in the quotient.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{7}{3}\right)^2 = \frac{7^2}{3^2}$ $= \frac{49}{9}$

3. Write each of the following in expanded form, and then evaluate.

a) 5^4 **b)** $(-4)^3$ **c)** -2^2 **d)** $(5 \times 2)^3$ **e)** $\left(\frac{3}{4}\right)^2$

PRACTICE

Evaluate questions 4 to 8 without using a calculator.

4. Evaluate.

a) 8^3 **c)** 5^6 **e)** 4^5

b) 11^4 **d)** 19^2 **f)** 2^{10}

Study Aid

- For help, see Essential Skills Appendix.

Question	Appendix
5, 6, 7, and 9	A-3
10	A-2
11	A-12

5. Evaluate.

a) $(-5)^2$ **c)** $(-2)^3$ **e)** $(-10)^4$

b) -5^2 **d)** -2^3 **f)** -10^4

6. Evaluate.

a) $(-3^3)^3$ **c)** $[(-3)^4]^2$ **e)** $(-3^3)^2$

b) $[(-3)^3]^3$ **d)** $(-3^4)^2$ **f)** $(-3^2)^3$

7. Evaluate.

a) $3^2 - 4^2$ **c)** $(1 + 7^2)^2$ **e)** $5^2 \times (-2)^3$

b) $10^2 - 15^1 + 5^2$ **d)** $(6^2 - 4^2)^2$ **f)** $8^2 \div (-4)^3$

8. Evaluate.

a) $\sqrt{25} + \sqrt{16}$ **b)** $\frac{\sqrt{100}}{\sqrt{25}}$ **c)** $\sqrt{\sqrt{81}}$

9. Determine the exponent that makes each of the following true.

a) $2^x = 16$ **c)** $3^y = 27$ **e)** $(-2)^n = -8$

b) $17^m = 17$ **d)** $4^x = 64$ **f)** $5^c = 125$

10. Evaluate.

a) $\frac{4}{7} - \frac{3}{4}$ **c)** $\frac{2}{3}\left(\frac{5}{4}\right)$ **e)** $\frac{4}{9}\left(\frac{9}{5} - \frac{3}{10}\right)$

b) $\frac{7}{9} \div \frac{4}{5}$ **d)** $\frac{2}{3} + \left(\frac{5}{4}\right)$ **f)** $\left(\frac{9}{10}\right)\frac{3}{7} \div \frac{3}{14}$

11. Determine the first and second finite differences for each set of data. State whether each set represents a linear or a quadratic relationship.

a)

x	y
−3	14
−2	10
−1	6
0	2
1	−2
2	−6
3	−10

b)

x	y
−3	11.5
−2	6.5
−1	3.5
0	2.5
1	3.5
2	6.5
3	11.5

c)

x	y
−6	15
−4	−3
−2	−13
0	−15
2	−9
4	5
6	27

APPLYING What You Know

Comparing Soccer Ball Sizes

Soccer is an old game played worldwide. Early soccer balls were made from pig bladders that varied in size.

Today, the Fédération Internationale de Football Association, or FIFA, determines the "qualities and measurements" of the ball.

According to FIFA, the ball must
- be spherical
- be made of leather or other suitable material
- have a circumference that is 68 cm – 70 cm
- have a mass of 410 g – 450 g
- have a pressure of 600 g/cm^2 – 1100 g/cm^2

? How do changes in the circumference of the ball affect its volume and surface area?

A. Use the minimum circumference of 68 cm. Determine the radius of the ball to the nearest tenth of a centimetre where $\pi \doteq 3.14$.

B. Repeat part A for the maximum circumference of 70 cm.

C. Use each radius from parts A and B to calculate the volume of each sphere to the nearest tenth of a cubic centimetre.

D. Calculate the difference in volume of the two spheres.

E. Use each radius from parts A and B to calculate the surface area of each sphere to the nearest tenth of a square centimetre.

F. Calculate the difference in surface area of the two spheres.

G. Is the difference in surface area greater than the difference in volume? Explain how you made your comparison.

Communication *Tip*

$C(r) = 2\pi r$

$V(r) = \frac{4}{3}\pi r^3$

$A(r) = 4\pi r^2$

7.1 Collecting Exponential Data

YOU WILL NEED

- some water and a way to heat it (a microwave oven or a kettle)
- thermometer (a temperature probe if linked to a graphing calculator)
- stopwatch or clock with a second hand
- graph paper (if you are not using a graphing calculator)
- Styrofoam or ceramic cup

GOAL

Collect data that lead to an exponential relationship and study its characteristics.

INVESTIGATE the Math

Hot objects left to cool have temperatures that drop quickly over time. A hot cup of tea is left on a teacher's desk as he is called away unexpectedly. He returns to find that his tea is lukewarm.

? How long does it take for a hot liquid to cool to room temperature?

A. Predict how long it will take for the liquid to cool. Sketch a graph of what you think the cooling curve of temperature versus time might look like.

B. Fill a Styrofoam or ceramic cup with a hot liquid.

C. Record the room temperature with a thermometer (or temperature probe).

D. Place the thermometer (or temperature probe) into the cup of hot liquid and wait until it gives a steady reading. Record the temperature in Trial 0 in a table like the one shown. Then take about 20 readings, one every minute, and record the temperatures in your table.

Trial	Time after Initial Reading (min)	Temperature (°C)
0	0	
1	1	
2	2	
3	3	

E. Using time as the independent variable, draw a scatter plot of the data. Draw a smooth curve that you think fits the data.

F. Describe the shape of the curve from left to right.

G. On your graph, draw a horizontal line that represents the room temperature you recorded. Use your curve to estimate the time it will take for the liquid to cool to room temperature.

Reflecting

H. Did the temperature fall at a steady rate? Describe how it fell.

I. Is this relation a function? Explain.

J. State the domain and range of the relation. Explain how they relate to the changes in time and temperature.

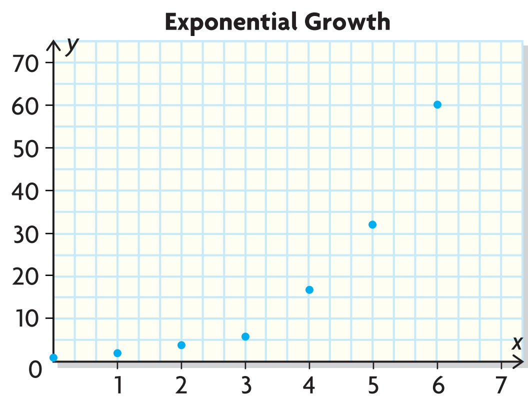
In Summary

Key Ideas

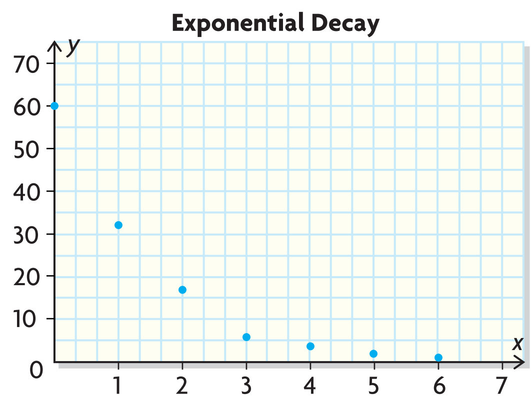
- A scatter plot of data that appears to have a rapidly increasing or decreasing nonlinear pattern can be modelled by an exponential function.
- The range of an exponential function with a positive base always has a lower limit.
- The domain of the exponential model may need to be restricted for the situation you are dealing with.

Need to Know

- Situations that show increases in the value of a function, where the increases grow larger with time in a predictable way, can be modelled by exponential functions. This is called *exponential growth*.



- Situations that show decreases in the value of a function, where the decreases grow smaller with time in a predictable way, can be modelled by exponential functions. This is called *exponential decay*.



FURTHER Your Understanding

1. What do you think would happen to the curve if you took many more measurements beyond the 20 min? Explain your reasoning.

2. Describe how you think the shape of the curve would change if
   - the initial water temperature were higher
   - the room temperature were much lower

   Sketch both situations on the same set of axes, along with the curve of your estimated data, to illustrate your thinking.

3. Marnie's grandmother won the "Really Big Lottery." Her prize is an initial payment of $5000 and a payment on her birthday for the next 25 years. Each year the payment doubles in size.

Year	Annual Payment ($1000)
1	5
2	10
3	20
4	
5	
⋮	
10	

a) Complete the table of values for the first 10 years.
b) Create a scatter plot of your data and draw a curve of best fit.
c) Compare your payment curve with the cooling curve. Discuss the similarities and the differences.

7.2 The Laws of Exponents

GOAL

Investigate the rules for simplifying numerical expressions involving products, quotients, and powers of powers.

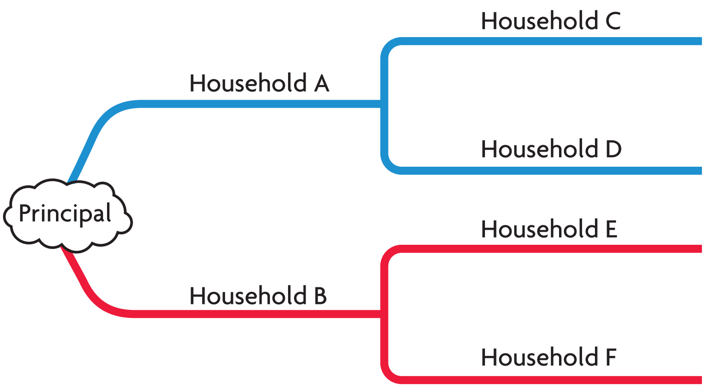
YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

The Senior Girls hockey team is due back from a tournament, but a snowstorm delays their flight. The principal calls the parents of two girls to inform them of the delay. The parents call two other households to tell them the news. Each of these households call two other households, and the pattern continues.

The number of people who are contacted in each round of calls is summarized in the table.



Round of Calls	Number of People Contacted	Power of Two
0	1	
1	2	
2	4	2^2
3	8	2^3
4	16	
5	32	
6	64	

? What are some of the relationships between powers of two in the table?

A. Copy and complete the table.

B. Select any two numbers (other than the last two) from the second column of the table. Find their product. Write the two numbers and their product as powers of two.

C. Repeat part B for two other numbers.

D. What is the relationship between the exponents of the powers that you multiplied and the exponent of the resulting power?

E. Select any two numbers (other than the first two) from the second column. Divide the greater value by the lesser one. Write the two numbers and their quotient as powers of two.

F. Repeat part E using two other numbers.

G. What is the relationship between the exponents of the powers that you divided and the exponent of the quotient?

Reflecting

H. Suggest a rule for multiplying powers with the *same* base. Create an example that shows this rule.

I. Suggest a rule for dividing powers with the *same* base. Create an example that shows this rule.

APPLY the Math

EXAMPLE 1 Connecting a power of a power to multiplication

Evaluate $(5^2)^4$.

Dylan's Solution

$(5^2)^4 = (5^2)(5^2)(5^2)(5^2)$ ← $(5^2)^4$ means four "5^2"s multiplied together.

$= 5^{2+2+2+2}$ ← Since the four powers of 5 are multiplied together and all have the same base, I added the exponents.

$= 5^8$

$(5^2)^4 = 5^{2\times4} = 5^8$ ← This leads to the answer. I noticed that I could get the same result if I had multiplied the exponents in the original question.

If a numerical expression contains several operations, evaluate by following the order of operations.

EXAMPLE 2 Selecting a strategy to evaluate an expression involving powers

Evaluate $3^4(3^8) \div 3^7$.

Lesley's Solution: Applying Exponent Rules

$$\begin{aligned} 3^4(3^8) \div 3^7 &= 3^{4+8} \div 3^7 \\ &= 3^{12} \div 3^7 \\ &= 3^{12-7} \\ &= 3^5 \\ &= 243 \end{aligned}$$

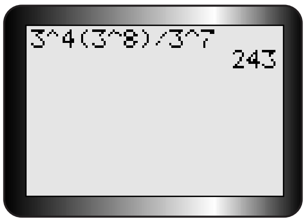
Since all of the powers in the expression have the same base, I followed the order of operations. I used exponent rules to multiply the first two powers and then divide the result.

I added the exponents for the first two powers and then subtracted the third power's exponent from the result.

Graphing calculators perform calculations following the order of operations. You can enter the expression as it appears and the calculator will determine the result.

Meredith's Solution: Using a Calculator

Evaluate $3^4(3^8) \div 3^7$.



I used my calculator to evaluate the result.

Tech *Support*

Most scientific calculators have an exponent key that can be used to evaluate powers, such as x^y. Check your calculator's manual to help you identify this key. For help using a graphing calculator to evaluate powers, see Technical Appendix, B-14.

EXAMPLE 3 Selecting a strategy for evaluating a rational expression involving powers

Evaluate $\frac{6(6^7)}{(6^3)^2}$.

Tom's Solution: Applying the Exponent Rules

$$\frac{6(6^7)}{(6^3)^2} = \frac{6^{1+7}}{6^{3\times 2}}$$

I evaluated the numerator and denominator separately by using the exponent rule for multiplication in the numerator and the power rule in the denominator.

$$= \frac{6^8}{6^6}$$

$$= 6^{8-6}$$

Then I divided the numerator by the denominator. Since the bases are the same, I subtracted the exponents.

$$= 6^2$$

$$= 36$$

Bronwyn's Solution: Using a Calculator

Evaluate $\frac{6(6^7)}{(6^3)^2}$.

I entered the expression with a set of brackets around the numerator and a different set around the denominator. I did this to make sure that the calculator evaluated the numerator and denominator separately before dividing.

In Summary

Key Ideas

- To multiply powers with the *same* base, add the exponents.

$$b^n \times b^m = b^{n+m}$$

- To divide powers with the *same* base, subtract the exponents.

$$b^n \div b^m = b^{n-m}$$

- To simplify a power of a power, multiply the exponents.

$$(b^n)^m = b^{n \times m}$$

Need to Know

- Use the order of operations to simplify expressions involving powers. If there are powers of powers in an expression, simplify them first. Then simplify the multiplications and divisions in order from left to right. For example,

$$\begin{aligned}(2^3)^2 \times 2^5 \div 2^7 &= 2^{3\times2} \times 2^5 \div 2^7\\ &= 2^6 \times 2^5 \div 2^7\\ &= 2^{6+5-7}\\ &= 2^4\\ &= 16\end{aligned}$$

- In a rational expression, simplify the numerator and denominator first. Then divide the numerator by the denominator.
- Remember, these rules apply only to powers that have the same base (e.g., $2^3 \times 2^5 = 2^8$).
- When evaluating powers, it is common practice to write answers as rational numbers.

CHECK Your Understanding

1. Write each expression as a single power.

a) $4^3 \times 4^5$ **c)** $(7^2)(7)$

b) $13^3 \times 13^{11}$ **d)** $(12^3)(12^3)(12^3)$

2. Write each expression as a single power.

a) $7^4 \div 7^3$ **b)** $6^{16} \div 6^{11}$ **c)** $\frac{9^3}{9}$ **d)** $\frac{5^6}{5^3}$

3. Write each expression as a single power.

a) $(2^3)^4$ **b)** $(12^3)^3$ **c)** $(10^7)^4$ **d)** $((3^2)^3)^4$

PRACTISING

4. **C** Create three examples to help a classmate learn about the following relationships:

a) the result of multiplying powers with the same base
b) the result of dividing powers with the same base
c) the result of raising a power to an exponent

5. **K** Simplify. Write as a single power.

a) $3(3^5) \div 3^3$
b) $10^9 \div (10^3 \times 10^2)$
c) $(7^8 \div 7^5)(7^2)$
d) $\dfrac{(6^2)(6^{11})}{6^8}$
e) $\dfrac{9^{12}}{9(9^{10})}$
f) $\dfrac{(8^7)(8^3)}{8^6(8^2)}$

6. Simplify. Write as a single power.

a) $(2^5)^3 \times 2^3$
b) $5^9 \div (5^3)^2$
c) $(7^8)(7^5)^2$
d) $\dfrac{(8^2)^5}{8^8}$
e) $\dfrac{10(10^9)}{(10^2)^3}$
f) $\dfrac{(4^7)^3}{4^9(4^{11})}$

7. Simplify. Write as a single power.

a) $10(10^5)(10^3) \div (10^3)^2$
b) $\dfrac{(8^8)(8^3)^3}{8^3(8^{11})}$
c) $\left(\dfrac{13(13^{12})}{13^7}\right)^2$
d) $\dfrac{(5^4)^2(5^5)^2}{5^2(5^{13})}$

8. Simplify, then evaluate without using a calculator.

a) $\left(\dfrac{4}{3}\right)\left(\dfrac{4}{3}\right)^2$
b) $\left(\dfrac{1}{9}\right)^4 \div \left(\dfrac{1}{9}\right)^2$
c) $\left(\left(\dfrac{2}{5}\right)^2\right)^2$
d) $\left(\dfrac{5}{4}\right)^5\left(\dfrac{5}{4}\right)^3 \div \left(\dfrac{5}{4}\right)^6$

9. Simplify.

a) $x^4(x^2)^2$
b) $\dfrac{(m^5)^2}{m^8}$
c) $(y(y^6))^3$
d) $((a^2)^2)^2$
e) $a^2\,a^2\,a^2$
f) $\dfrac{b(b^5)b^4}{b^5}$

10. Write each power in simplified form.

a) 4^3 as a base 2 power
b) 9^5 as a base 3 power
c) 27^5 as a base 3 power
d) $(-8)^4$ as a base -2 power
e) $\left(\dfrac{1}{4}\right)^3$ as a base $\dfrac{1}{2}$ power
f) $\left(\dfrac{1}{25}\right)^4$ as a base $\dfrac{1}{5}$ power

11. Simplify, then evaluate without using a calculator.

a) $(2^2)^3 \div 4^2$

b) $\frac{9^2}{(3^2)^2}$

c) $\frac{(10^2)^5}{100^3}$

d) $5^2\left(\frac{(5^4)^3}{5^{10}}\right)$

12. Clare was asked to simplify the expression $3^2(2^2)^2$. This is her solution:

$$\begin{aligned}3^2(2^2)^2 &= 3^2(2^4)\\ &= 6^6\\ &= 46\,656\end{aligned}$$

Her solution is incorrect.

a) Identify her error.

b) Determine the correct answer. Show your steps.

13. A a) Explain the steps involved in simplifying the expression $\frac{x^4(y(y^5))}{xy^4}$.

b) Simplify the expression.

c) If $x = -2$ and $y = 3$, evaluate the expression.

14. T a) What can you conclude about the numbers a and b if $3^a \times 3^b = 3^n$ and n is an even number?

b) If 5^m is a perfect square, what can you conclude about m?

Extending

15. a) Evaluate $[(-5)^2]^3$ and $(-5^2)^3$. Do these expressions have the same value? Justify your answer.

b) Evaluate $(-5)^2$ and $(-5)^3$. Make a conjecture about the sign of a base and how the exponent may affect the value of the power.

16. Simplify, then determine the number that makes each statement true.

a) $(n^2)^5 \div n^5 = 243$

b) $\left(\frac{m^7}{m^6}\right)^2 = 196$

17. Simplify.

a) $(4x^3)^2$

b) $(5x^3y^4)^3$

c) $\left(\frac{3x^3}{2y^4}\right)^2\left(\frac{2y^2}{3x^4}\right)^3$

18. If $A = \frac{(2x^2)^3}{(x^3)(4x)}$, determine a possible value for x where

a) $0 < A < 1$

b) $A = 1$

c) $A > 1$

d) $A < 0$

7.3

Working with Integer Exponents

GOAL

Evaluate numerical expressions involving integer exponents.

INVESTIGATE the Math

A yeast culture grows by doubling its number of cells every 20 min. Bill is working with someone else's lab notes. The notes include a table of the number of yeast cells counted, in thousands, every 20 min since noon. However, the first three readings are missing.

Time	Number of Cells (thousands)	Power of Two
1:00 p.m.		
1:20		
1:40		
2:00 p.m.	8	2^3
2:10	16	2^4
2:20	32	2^5

? How can Bill determine the number of cells that were present at noon?

A. Copy the table. Fill in the missing times in the first column.

B. What is the pattern of the numbers in the second column? Work backward to fill in the second column to 1:00 p.m.

C. What is the pattern of the powers of two in the third column? Work backward to fill in the third column to 1:00 p.m.

D. What does the pattern suggest about the value of 2^0?

E. Continue to work backward to complete each column.

F. What are the signs of the exponents of the first three entries in the third column?

G. Write the values in the second column as fractions in lowest terms. Write their denominators as powers of two.

H. Compare the fractions you wrote in part G with the powers of two in the first three entries.

I. How many cells did this culture have at noon?

Reflecting

J. How does the pattern in the table suggest that the value of 2^0 must be 1?

K. How can you write a power with a negative exponent so that the exponent is positive?

L. Do you think that the rules for multiplying and dividing powers with the same base still apply if the exponents are zero or negative? Create some examples and test your conjecture with and without a calculator.

APPLY the Math

EXAMPLE 1 Representing the value of a number with a zero exponent

Explain why a number with a zero exponent has a value of 1.

Alifiyah's Solution: Starting with a Value of 1

$$1 = \frac{7^4}{7^4}$$

$$= 7^{4-4}$$

$$= 7^0$$

Therefore, $7^0 = 7^{4-4} = 1$

I know that any number divided by itself is 1. I chose the power 7^4. Using this power, I wrote 1 as a rational number. I used the rule for dividing powers of the same base and subtracted the exponents. The result is a power that has an exponent of zero.

This shows that a number with a zero exponent is equal to 1.

Matt's Solution: Starting with the Exponent 0

$$6^0 = 6^{3-3}$$
$$= \frac{6^3}{6^3}$$
$$= 1$$

I began by looking at 6^0. I can write this as 6^{3-3}. (Any number other than 3 will work as well.)

Subtracting exponents means that I am dividing the two powers. This leads to the answer 1.

In an expression with a negative exponent, the sign of the exponent changes when you take the reciprocal of the original base.

EXAMPLE 2 Representing a number raised to a negative exponent

Evaluate $(-3)^{-4}$.

Liz's Solution

$$(-3)^{-4} = \left(-\frac{1}{3}\right)^4$$

I know that a number raised to a negative exponent is equal to a power of the reciprocal of the original base, raised to the positive exponent. I changed -3 to its reciprocal and changed the sign of the exponent.

$$= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$

$$= \frac{1}{81}$$

I multiplied.

EXAMPLE 3 Representing a number with a rational base and a negative exponent

Evaluate $\left(\frac{5}{6}\right)^{-2}$.

Jill's Solution: Using Division

$$\left(\frac{5}{6}\right)^{-2} = \frac{1}{\left(\frac{5}{6}\right)^{2}}$$

I wrote a 1 over the entire expression and changed −2 to 2.

$$= \frac{1}{\frac{25}{36}}$$

I evaluated the power.

$$= 1 \times \frac{36}{25}$$

I divided 1 by the fraction by taking its reciprocal and multiplying.

$$= \frac{36}{25}$$

Martin's Solution: Using the Reciprocal

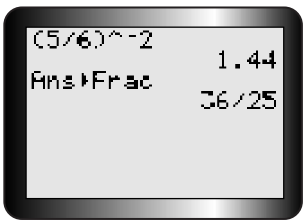
$$\left(\frac{5}{6}\right)^{-2} = \left(\frac{6}{5}\right)^{2}$$

I evaluated this expression by taking the reciprocal of the base. When I did this, I changed the sign of the exponent (from negative to positive).

$$= \frac{6^2}{5^2}$$

The rational expression is squared. So I squared the numerator and squared the denominator.

$$= \frac{36}{25}$$



I also tried evaluating this expression on my calculator. It gave me a decimal answer.

To compare the answer with the rational number, I changed the decimal into an equivalent fraction.

Tech *Support*

You can use the "Frac" command, which is found in the MATH menu on the graphing calculator, to change most decimals into equivalent fractions.

EXAMPLE 4 — Selecting a strategy for evaluating an expression with negative exponents

Evaluate $\dfrac{5^{-4}(5^{8})}{(5^{-3})^{2}}$.

Lesley's Solution: Applying the Exponent Laws

$$\frac{5^{-4}(5^{8})}{(5^{-2})^{2}} = \frac{5^{4}}{5^{-4}}$$
$$= 5^{4-(-4)}$$
$$= 5^{8}$$
$$= 390\ 625$$

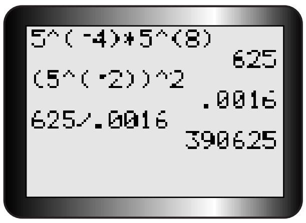
Since this expression is written as a fraction, I evaluated the numerator and denominator separately. The numerator has two powers of 5 multiplied, so I added their exponents. The denominator is a power of a power, so I multiplied these exponents.

To divide the numerator by the denominator I subtracted the exponents (since they have the same base).

Aneesh's Solution: Using a Calculator

$$\frac{5^{-4}(5^{8})}{(5^{-2})^{2}}$$

I used brackets to separate the numerator from the denominator.



I could also have done the calculation in parts. I could have evaluated the numerator and the denominator first and then divided the numerator by the denominator.

In Summary

Key Ideas

- A power with a negative exponent and an integer base is equivalent to the reciprocal of that base to the opposite exponent.

$$b^{-n} = \frac{1}{b^n}, \text{ where } b \neq 0$$

- A power with a negative exponent and a fractional base is equivalent to the reciprocal of that base to the opposite exponent.

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n, \text{ where } a \neq 0, b \neq 0$$

- Any number (or expression) to the exponent 0 is equal to 1.

$$b^0 = 1, \text{ where } b \neq 0$$

Need to Know

- When simplifying numerical expressions involving powers, present the answer as an integer, a fraction, or a decimal.
- When simplifying algebraic expressions involving exponents, it is common practice to present the answer with positive exponents.

CHECK Your Understanding

1. Rewrite each expression as an equivalent expression with a positive exponent.

a) 4^{-6} **b)** $\left(\frac{7}{3}\right)^{-5}$ **c)** $\frac{1}{8^{-2}}$ **d)** $(-3)^{-2}$

2. Evaluate each expression without using a calculator.

a) $(8)^0$ **b)** 5^{-3} **c)** $\left(\frac{3}{2}\right)^{-3}$ **d)** $(-2)^{-4}$

3. Use your calculator to evaluate each expression.

a) 8^{-2} **b)** 4^{-3} **c)** $\left(\frac{5}{2}\right)^{-3}$ **d)** $\left(-\frac{1}{2}\right)^{-3}$

PRACTISING

4. Evaluate.
K

a) 10^{-2}
b) $(-4)^{-2}$
c) $\left(\frac{1}{2}\right)^{-5}$
d) $\left(\frac{1}{7}\right)^{-3}$
e) $\frac{1}{(-9)^2}$
f) $(-5)^0$

5. Simplify. Write each expression as a single power with a positive exponent.

a) $9^7 \times 9^{-3}$
b) $6^{-3} \div 6^{-5}$
c) $8^6 \div 8^{-5}$
d) $17^{-4} \div 17^{-6}$
e) $(-3)^{-8} \times (-3)^9$
f) $(-4)^{-5} \times (-4)^5$

6. Simplify. Write each expression as a single power with a positive exponent.

a) $2^4(2^2) \div 2^{-6}$
b) $-5 \times (-5^4)^{-3}$
c) $\frac{(-12^3)^{-1}}{(-12)^7}$
d) $\left(\frac{3^4}{3^6}\right)^{-1}$
e) $\frac{9^4(9^3)}{9^{12}}$
f) $((7^2)^{-3})^{-4}$

7. Simplify. Write each expression as a single power with a positive exponent.

a) $\frac{11^{-2}(11^3)}{(11^{-2})^4}$
b) $\left(\frac{9^{-2}}{(9^2)^2}\right)^2$
c) $\left(\frac{4^{-3}}{4^{-2}}\right)^{-3}$
d) $\left(\frac{10}{10^{-3}}\right)^2\left(\frac{10^5}{10^7}\right)$
e) $\frac{(-8^{-1})(-8^{-5})}{(-8^{-2})^3}$
f) $\left(\frac{(5^3)^2}{5(5^6)}\right)^{-1}$

8. Simplify, then evaluate each expression. Leave answers as fractions or integers.

a) $13^3 \times 13^{-4}$
b) $\frac{3^{-2}}{3^{-6}}$
c) $\left(\frac{10^{-3}}{10^{-5}}\right)^2$
d) $6^{-2}(6^{-2})^{-1}$
e) $\frac{-2(-2^{-3})}{(-2)^4}$
f) $\left(\frac{5^{-2}}{5}\right)^{-1}$

9. Evaluate. Leave answers as fractions or integers.

a) $3^{-2} - 9^{-1}$
b) $4^{-2} + 3^0 - 2^{-3}$
c) $8^{-2} + (4^{-1})^2$
d) $\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{3}\right)^{-1}$
e) $12(4^0 - 3^{-2})$
f) $\frac{4^2}{2^5}$

10. Scientific notation can be used to represent very large and very small numbers. The diameter of Earth is about 1.276×10^7 m, while the diameter of a plant cell is about 1.276×10^{-5} m. Explain why negative exponents are used in scientific notation to represent very small numbers.
A

11. Simplify. Write each expression as a single power with a positive exponent.

a) $x^4(x^{-2})^2$ **c)** $\left(\dfrac{w^2}{w^4}\right)^{-3}$ **e)** $a^{-2} \times a^3 \times a^4$

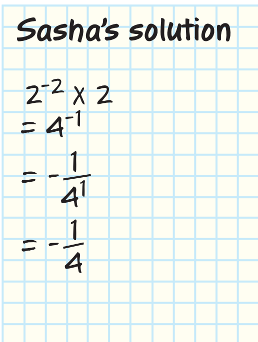
b) $\dfrac{(m^5)^2}{m^{-8}}$ **d)** $((a^{-2})^2)^{-3}$ **f)** $\dfrac{b(b^{-5})b^{-2}}{b(b^8)}$

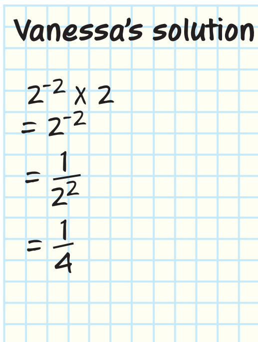
12. Determine the value of the variable that makes each of the following true.
T

a) $12^x = 1$ **c)** $10^n = 0.0001$ **e)** $2^b = \dfrac{1}{32}$

b) $10^m = 100\ 000$ **d)** $2^k = \dfrac{1}{2}$ **f)** $2^{2a} = 64$

13. Francesca is helping her friends Sasha and Vanessa study for a quiz. They are working on simplifying $2^{-2} \times 2$. Francesca notices errors in each of her friends' solutions, shown here:
C





a) Explain where each student went wrong.
b) Write the correct solution.

Extending

14. Consider the following powers: 2^{12}, 4^6, 8^4, 16^3.
a) Use your calculator to show that the powers above are equivalent.
b) Can you think of a way to explain why each power above is equivalent to the preceding power without referring to your calculator?
c) Create a similar list using 3 as the base.

15. If $x = -2$ and $y = 3$, write the following three expressions in order from least to greatest.

$$\frac{y^{-4}(x^2)^{-3}y^{-3}}{x^{-5}(y^{-4})^2}, \quad \frac{x^{-3}(y^{-1})^{-2}}{(x^{-5})(y^4)}, \quad (y - 5)(x^5) - 2(y^2)(x - 3) - 4$$

7.4 Working with Rational Exponents

YOU WILL NEED

- graph paper

GOAL

Determine the meaning of a power with a rational exponent, and evaluate expressions containing such powers.


THE SEATTLE POST-INTELLIGENCER

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KLONDYKE

COOPER & LEVY

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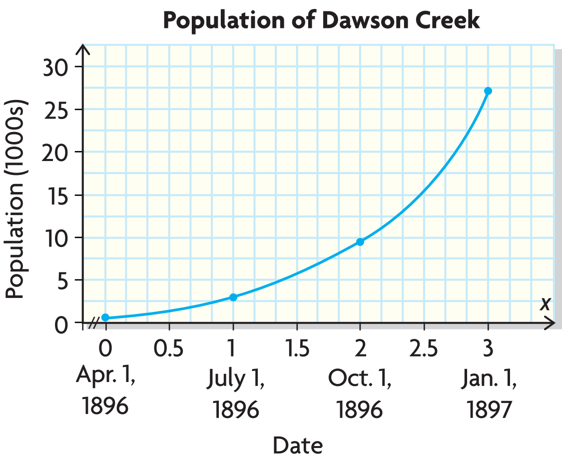
INVESTIGATE the Math

On August 16, 1896, gold was discovered near Dawson, in the Yukon region of Canada.

The population of Dawson City experienced rapid growth during this time. The population was approximately 1000 in April and 3000 in July (and grew to 30 000 at one point). The population is given in the table and is also shown on the graph.



Date	x	Population (1000s)
Apr. 1, 1896	0	1, or 3^0
July 1, 1896	1	3, or 3^1
Oct. 1, 1896	2	9, or 3^2
Jan. 1, 1897	3	27, or 3^3

? About how many people were in Dawson City in mid-May and mid-August of 1896?

A. What x-value can be used to represent the middle of May? Explain how you know.

B. Use the graph to estimate the population in the middle of May.

C. What power can be used to represent the population in the middle of May? Multiply this power by itself. What do you notice?

D. Use the exponent button on your calculator to calculate $3^{\frac{1}{2}}$. How does this value relate to your estimate in part B?

E. Use the graph to estimate the population at the beginning of May (when $x = \frac{1}{3}$). Calculate $3^{\frac{1}{3}}$ and compare it with your estimate.

F. Calculate $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$. What do you notice?

G. What is another way to express $3^{\frac{1}{3}}$ using a **radical?**

radical

the indicated root of a quantity; for example, $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 2^3 = 8$

H. Write a power of 3 that would estimate the population in the middle of August. Evaluate the power with your calculator and check your answer by using the graph shown.

Reflecting

I. Explain why $x^{\frac{1}{2}}$ is equivalent to $\sqrt{x}$.

J. Make a conjecture about powers with the exponent $\frac{1}{n}$.

K. Do the rules for multiplying powers with the same base still apply if the exponents are rational? Use the numbers in your table to investigate.

APPLY the Math

EXAMPLE 1 Representing a power with a positive rational exponent

Evaluate $65^{\frac{1}{3}}$.

Elaine's Solution

$65^{\frac{1}{3}} = \sqrt[3]{65}$ ← I know that the exponent $\frac{1}{3}$ is a way of representing the 3rd root of the base. I wrote it in radical notation.

$\sqrt[3]{65} \doteq 4$

The answer is a number whose cube is 65.

I know that $4 \times 4 \times 4 = 64$.
The answer must be close to 4.

I used my calculator to get a more accurate answer.

When an exponent is written as a decimal, it is often easier to evaluate the power if the exponent is written as its equivalent fraction.

EXAMPLE 2 Representing a power with a decimal rational exponent

Evaluate $32^{0.2}$.

Tosh's Solution

$32^{0.2} = 32^{\frac{1}{5}}$

I rewrote the power changing the exponent from 0.2 to its equivalent fraction.

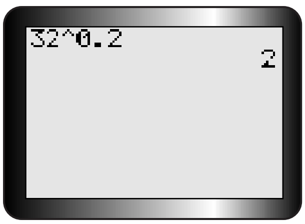
$$0.2 = \frac{2}{10} = \frac{1}{5}$$

$= \sqrt[5]{32}$

I know that an exponent $\frac{1}{5}$ is a way of representing the 5th root of the number. I wrote it in radical notation.

$= 2$

Since $2^5 = 32$, the 5th root of 32 must be 2.



I entered the power into my calculator to verify my answer.

If a power involves a negative rational exponent, then you can write the exponent as the product of a fraction and an integer.

example 3 Representing a power with a negative rational exponent

Evaluate $(-27)^{-\frac{2}{3}}$.

George's Solution: Using $\frac{1}{3}$ and -2 as Exponents

$$(-27)^{-\frac{2}{3}} = ((-27)^{\frac{1}{3}})^{-2}$$

I separated the exponent into two parts, since the exponent $-\frac{2}{3}$ can be written as a product: $\frac{1}{3} \times (-2)$.

$$= \frac{1}{((-27)^{\frac{1}{3}})^2}$$

I expressed $\left((-27)^{\frac{1}{3}}\right)^{-2}$ as a rational number, using 1 as the numerator and $\left((-27)^{\frac{1}{3}}\right)^2$ as the denominator.

$$= \frac{1}{((-27)^{\frac{1}{3}} \times (-27)^{\frac{1}{3}})}$$

$$= \frac{1}{(\sqrt[3]{-27} \times \sqrt[3]{-27})}$$

I determined the cube root of -27. I know that $(-3) \times (-3) \times (-3) = -27$.

$$= \frac{1}{(-3 \times -3)}$$

$$= \frac{1}{9}$$

Nadia's Solution: Using $-\frac{1}{3}$ and 2 as Exponents

$$(-27)^{-\frac{2}{3}} = ((-27)^{-\frac{1}{3}})^{-2}$$

I separated the exponent into two parts. The exponent $-\frac{2}{3}$ can be written as a product: $-\frac{1}{3} \times 2$

$$= \left(\frac{1}{(-27)^{\frac{1}{3}}}\right)^2$$

$$= \left(\frac{1}{\sqrt[3]{-27}}\right)^2$$

I expressed $(-27)^{\frac{1}{3}}$ as a radical.

$$= \left(\frac{1}{-3}\right)^2$$

I evaluated the root and squared the result.

$$= \frac{1}{9}$$

Anjali's Solution: Using a Calculator

$(-27)^{-\frac{2}{3}}$

I used my calculator to determine the answer and then changed it from a decimal to a fraction.

The strategies you have used to evaluate numerical expressions involving integer exponents also apply when exponents are rational.

EXAMPLE 4 Selecting a strategy to evaluate an expression involving rational exponents

Simplify, then evaluate $\dfrac{(8^{\frac{1}{6}})^7}{8^{\frac{1}{2}}8^{\frac{1}{3}}}$.

Aaron's Solution

$$\frac{(8^{\frac{1}{6}})^7}{8^{\frac{1}{2}}8^{\frac{1}{3}}} = \frac{8^{\frac{7}{6}}}{8^{\frac{5}{6}}}$$

The numerator is a power of a power, so I multiplied the exponents: $\frac{1}{6} \times 7 = \frac{7}{6}$

The denominator is a product of powers, so I added the exponents: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

To divide powers (numerator by denominator), I subtracted the exponents.

$$= 8^{\frac{7}{6}-\frac{5}{6}}$$

$$= 8^{\frac{2}{6}}$$

$$= 8^{\frac{1}{3}}$$

I wrote the fraction in lowest terms.

$$= \sqrt[3]{8}$$

$$= 2$$

I found the cube root of 8.

In Summary

Key Ideas

- A power with a rational exponent is equivalent to a radical. The rational exponent $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in \mathbf{N}$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $b \neq 0$.
- If $m \neq 1$ and if m and n are both positive integers, then $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$, where $b \neq 0$.

Need to Know

- The exponent laws that apply to powers with integer exponents also apply to powers with rational exponents.
- The power button on a scientific calculator can be used to evaluate rational exponents.
- Some roots of negative numbers cannot be determined. For example, -16 does not have a real-number square root, since $(-4)^2 = (-4) \times (-4) = +16$. Odd roots can have negative bases, but even ones cannot.
- Since radicals can be written as powers with rational exponents:
  - Their products are equivalent to the products of powers. This means that $\sqrt{a} \times \sqrt{b} \times \sqrt{c} = \sqrt{a \times b \times c}$, because $a^{\frac{1}{2}} \times b^{\frac{1}{2}} \times c^{\frac{1}{2}} = (abc)^{\frac{1}{2}}$, where a, b, and $c > 0$.
  - Their quotients are equivalent to the quotient of powers. This means that $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, because $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$, where a, b, and $c > 0$.

CHECK *Your Understanding*

1. Write in radical form. Then evaluate without using a calculator.

a) $49^{\frac{1}{2}}$ **c)** $81^{\frac{1}{4}}$ **e)** $16^{0.25}$

b) $(-125)^{\frac{1}{3}}$ **d)** $100^{\frac{1}{2}}$ **f)** $-(144)^{0.5}$

2. Write in exponent form. Then evaluate.

a) $\sqrt[10]{1024}$ **c)** $\sqrt[3]{27^4}$ **e)** $\sqrt[4]{16}$

b) $\sqrt[5]{1024}$ **d)** $\left(\sqrt[3]{-216}\right)^5$ **f)** $(\sqrt{25})^{-1}$

3. Use your calculator to evaluate each expression to the nearest hundredth.

a) $6^{\frac{2}{5}}$ **c)** $\sqrt[15]{4421}$ **e)** $10^{\frac{-2}{3}}$

b) $0.0625^{\frac{1}{4}}$ **d)** $144^{0.25}$ **f)** $200^{-0.4}$

PRACTICE

4. Evaluate each expression. Use the fact that $2^5 = 32$ and $3^4 = 81$.
K

a) $-32^{\frac{1}{5}}$ **c)** $\sqrt[10]{(-32)^2}$ **e)** -81^{-025}

b) $\sqrt[5]{-32}$ **d)** $(-2)^5$ **f)** $\sqrt[4]{81}$

5. The volume of a cube is 0.027cm^3. What does $\sqrt[3]{0.027}$ represent in the situation?
A

6. Write each expression as a single power with a positive exponent.

a) $7^{\frac{1}{2}} \times 7^2$ **c)** $(16^{\frac{2}{3}})^6$ **e)** $(10^{\frac{5}{8}})^{-2}$

b) $3^4 \div 3^{\frac{1}{2}}$ **d)** $\dfrac{12^{\frac{2}{3}}}{12^{\frac{1}{6}}}$ **f)** $2^{0.5} \times 2^2 \times 2^{2.5}$

7. Write each expression as a single power with a positive exponent.

a) $3^{-\frac{1}{2}} \div 3^{\frac{3}{2}}$ **c)** $8^{\frac{5}{2}} \div 8^{-\frac{5}{2}}$ **e)** $9^{0.5} \times 3^0$

b) $(5^{-\frac{2}{5}})^{10}$ **d)** $4^{0.3} \div 4^{0.8} \times 4^{-0.7}$ **f)** $2^3 \times 4^{-2} \div 8$

8. Given that $10^{0.5} \doteq 3.16$, determine the value of $10^{1.5}$ and $10^{-0.5}$. Explain your reasoning.

9. Write each expression using powers, then simplify. Evaluate your simplified expression.

a) $\dfrac{\sqrt{200}}{\sqrt{2}}$ **c)** $\dfrac{\sqrt{98}}{\sqrt{2}}$ **e)** $\dfrac{\sqrt{15}\sqrt{10}}{\sqrt{6}}$

b) $\sqrt{3}\sqrt{6}\sqrt{2}$ **d)** $\dfrac{\sqrt{12}}{\sqrt{8}\sqrt{6}}$ **f)** $3\sqrt{12}\sqrt{3}$

10. Write each expression as a single power with positive exponents.

a) $\dfrac{6^{\frac{3}{2}} \times 6^5}{6}$ **c)** $\dfrac{12^{-\frac{3}{4}}}{12^{\frac{-1}{2}}}$ **e)** $4^{\frac{2}{3}} \div 4^{\frac{-1}{2}} \times 4^{\frac{5}{6}}$

b) $\dfrac{(8^2)^{\frac{1}{2}}}{8(8^2)}$ **d)** $\dfrac{(11^{-\frac{3}{4}})(11^{\frac{5}{8}})}{11^{\frac{3}{2}}}$ **f)** $\left(\left(16^{-\frac{1}{2}}\right)^2\right)^{-\frac{1}{4}}$

11. Simplify. Express final answers in radical form.

a) $4^{-\frac{3}{8}}(4^2)$ **c)** $10^{\frac{9}{4}}(10^{-2})$ **e)** $5^{\frac{1}{2}} \times 5^{-1}$

b) $\dfrac{9^{\frac{4}{3}}}{9^{\frac{7}{10}}}$ **d)** $(8^{\frac{1}{5}})(8^{-\frac{2}{15}})$ **f)** $4^3 \div 4^{\frac{3}{4}}$

12. Rewrite each of the following expressions using a rational exponent. Then evaluate using your calculator. Express answers to the nearest thousandth.

a) $\sqrt[3]{120}$ **c)** $\sqrt[5]{13^{-2}}$ **e)** $(\sqrt[3]{216})^{-2}$

b) $25^{0.75}$ **d)** $10^{-0.8}$ **f)** $(\sqrt[3]{-15})^{-2}$

13. Evaluate $-(8)^{\frac{1}{3}}$ and $-(4)^{\frac{1}{2}}$ using your calculator. Compare the results.

14. Simplify. Write each answer with positive exponents.

a) $m\left(m^{\frac{2}{3}}\right)$
c) $\left(c^{3}\right)^{\frac{5}{6}}$
e) $\dfrac{s\left(s^{0.25}\right)}{\left(s^{1.5}\right)^{0.3}}$
b) $\dfrac{x}{x^{-\frac{4}{3}}}$
d) $\left(b^{\frac{8}{9}}\right)^{\frac{9}{4}}$
f) $m\left(m^{\frac{2}{3}}\right)m^{-\frac{5}{3}}$

15. Simplify. Write each answer with positive exponents.

a) $\dfrac{t\left(t^{-\frac{8}{5}}\right)}{t^{\frac{3}{5}}}$
c) $\dfrac{\left(y^{5}\right)^{-\frac{9}{5}}}{\left(y^{-\frac{3}{2}}\right)^{4}}$
e) $\left(x^{\frac{1}{3}} \div x^{\frac{2}{3}}\right)^{-3}$
b) $\dfrac{\left(x^{\frac{7}{4}}\right)^{\frac{1}{2}}}{x^{-\frac{5}{6}}}$
d) $\left(\dfrac{a^{\frac{1}{2}}a^{\frac{3}{2}}}{\left(a^{-2}\right)^{\frac{1}{2}}}\right)$
f) $\left(\left(b^{-8}\right)^{\frac{-1}{2}}\right)^{\frac{-3}{4}}$

16. Evaluate $64^{-\frac{5}{3}}$ without a calculator. Explain each of the steps in your evaluation.

17. **T** Determine the value of the variable that makes each of the following true. Express each answer to the nearest hundredth.

a) $1.05 = \sqrt[3]{M}$
c) $N^{\frac{1}{5}} - 3 = 0$
e) $x^{\frac{2}{3}} = 4$
b) $2.5 = \sqrt[4]{T}$
d) $\dfrac{x^{5}}{x^{2}} = 125$
f) $y^{-0.25} = \dfrac{1}{3}$

18. Write in exponential form. Use the exponent laws to simplify and then evaluate.

a) $\sqrt{1000} \times \sqrt[3]{1000} \div \sqrt[6]{1000}$
b) $\dfrac{\left(\sqrt{64}\right)^{2}}{\sqrt[3]{64}}$

19. **C** Use your knowledge of exponents to express $32^{\frac{4}{5}}$ in two other ways. Which one is easier to evaluate if you do not use a calculator?

Extending

20. Simplify.

a) $\dfrac{4 + 4^{-1}}{4 - 4^{-1}}$
b) $\dfrac{5^{-2} - 5^{-1}}{5^{-2} + 5^{-1}}$
c) $\dfrac{\sqrt{4^{3}}\left(\sqrt[5]{4^{4}}\right)}{\sqrt{2^{10}}}$

21. Write as a radical: $\left(21^{6}\right)^{-\frac{1}{4}}$

22. If $a = 2$ and $b = -1$, which expression has the greater value?

a) $\dfrac{a^{-2b}a^{-b+2}}{\left(a^{-2}\right)^{b}}$
b) $\dfrac{\left(a^{b}\right)^{-3}a^{-1(-2b)}}{\left(a^{-b}\right)^{3}}$

7

Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 7.3, Example 1.

Q: What is the value of an experession with a zero exponent?

A: When there is a zero exponent, the value is 1.

EXAMPLE

$$12^0 = 1 \qquad \left(\frac{7}{11}\right)^0 = 1$$

$$(-6)^0 = 1 \qquad \left(-3(8) - \frac{2}{5}\left(\frac{13}{20}\right)\right)^0 = 1$$

Study Aid

- See Lesson 7.3, Examples 2, 3, and 4.
- Try Mid-Chapter Review Questions 2 and 3.

Q: What does it mean when a power has a negative exponent and how do you evaluate this kind of power?

A: A power with a negative exponent is equivalent to a power whose base is the reciprocal of the original base, and whose exponent is the opposite of the original exponent.

To evaluate such a power, take the reciprocal of the base and change the sign of the exponent. Then, multiply the base by itself the number of times indicated by the exponent.

EXAMPLE

$$5^{-3} = \left(\frac{1}{5}\right)^3 = \frac{1^3}{5^3} = \frac{1}{125}$$

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = \frac{16}{9}$$

Study Aid

- See Lesson 7.4, Examples 1 to 4.
- Try Mid-Chapter Review Questions 4 to 8.

Q: What does it mean when a power has a rational exponent, and how do you evaluate this kind of power?

A: The denominator of a rational exponent indicates the required root of the base. The numerator has the same meaning as an integer exponent. Using the fact that powers with rational exponents can be expressed as powers of powers, this can be evaluated in two steps, in two different ways.

EXAMPLE

$$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = (3)^2 = 9$$

$$27^{\frac{2}{3}} = (27^2)^{\frac{1}{3}} = (243)^{\frac{1}{3}} = \sqrt[3]{243} = 9$$

PRACTICE Questions

Lesson 7.2

1. Write as a single power. Express your answers with positive exponents.

a) $5(5^4)$

b) $\frac{8^8}{8^6}$

c) $(16^2)^5$

d) $\frac{(-4)^6(-4)^3}{((-4)^9)^2}$

e) $\left(\frac{1}{10}\right)^6\left(\frac{1}{10}\right)^{-4}$

f) $\left(\frac{(7)^2}{(7)^4}\right)^{-5}$

Lesson 7.3

2. Write each power with only positive exponents.

a) x^{-2}

b) $(m^{-4})^2$

c) $b^{-3} \times b^{-2}$

d) $\left(\frac{1}{y}\right)^{-2}$

e) $(n^{-7})^{-2}$

f) $y^{-3} \div y$

3. Evaluate without using a calculator.

a) $5^{-2} + 10^{-1}$

b) $4^0 + 8^{-2} - 2^{-2}$

c) $9^{-1} - (3^{-1})^2$

d) $(6^{-2})^{-1} + \left(\frac{1}{3}\right)^{-2}$

e) $\left(-\frac{1}{2}\right)^3 + 4^{-3}$

f) $25^{-1} + \left(-\frac{5}{2}\right)^{-2}$

Lesson 7.4

4. Evaluate without using a calculator.

a) $\left(\frac{2}{3}\right)^{-1}$

b) $\left(-\frac{2}{5}\right)^{-3}$

c) $\left(\frac{81}{16}\right)^{\frac{1}{2}}$

d) $\left(64^{\frac{1}{3}}\right)^4$

e) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

f) $[(2^2)(4^2)]^{-1}$

5. Simplify. Write each expression with only positive exponents.

a) $a^{\frac{1}{5}} \times a^{\frac{2}{3}}$

b) $\frac{b^2}{b^{\frac{3}{2}}}$

c) $\frac{c^{-3}}{c^2}$

d) $\frac{d^{-3}}{d^{-5}}$

e) $e(e^{-5})^{-2}$

f) $\left(f^{-\frac{2}{3}}\right)^{\frac{5}{8}}$

6. Copy and complete the table.

	Exponential Form	Radical Form	Evaluation of Expression
a)	$100^{\frac{1}{2}}$		
b)	$16^{0.25}$		
c)		$\sqrt{121}$	
d)	$(-27)^{\frac{5}{3}}$		
e)	$49^{2.5}$		
f)		$\sqrt[10]{1024}$	
g)		$\sqrt[3]{\left(\frac{1}{2}\right)^9}$	

7. Evaluate. Express each answer to three decimals.

a) $\sqrt[6]{2400}$

b) $120^{0.8}$

c) $9^{\frac{-6}{5}}$

d) $0.5^{-0.5}$

e) $\sqrt[9]{-1024}$

f) 0.2^{-2}

8. Use trial and error to determine the value of the variable that makes each of the following true.

a) $\sqrt[3]{x} = 125$

b) $m^{\frac{3}{2}} = 64$

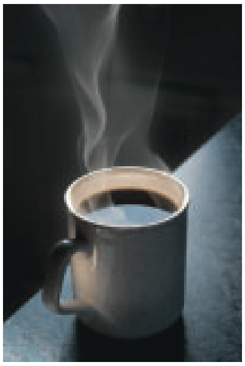
c) $p^{-3} = \frac{1}{27}$

d) $\sqrt{x^3} = 8$

7.5 Exploring the Properties of Exponential Functions

YOU WILL NEED

- graphing calculator
- graph paper



GOAL

Examine the features of the graphs of exponential functions and compare them with graphs of linear and quadratic functions.

EXPLORE the Math

The temperature of a cup of hot liquid as it cools with time is modelled by an exponential function.

? What are the characteristics of the graph of the exponential function $f(x) = b^x$, and how does this graph compare with the graphs of quadratic and linear functions?

A. Copy and complete the tables of values for the functions $g(x) = x$, $h(x) = x^2$, and $k(x) = 2^x$.

x	−3	−2	−1	0	1	2	3	4	5
$g(x) = x$									

x	−3	−2	−1	0	1	2	3	4	5
$h(x) = x^2$									

x	−3	−2	−1	0	1	2	3	4	5
$k(x) = 2^x$									

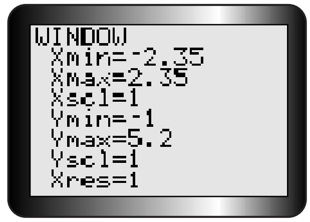
B. Add two rows to each table and calculate the first and second differences. Discuss the difference patterns for each type of function.

C. Graph each function on graph paper, and draw a smooth curve or line through each set of points. Label each with the appropriate equation.

D. State the domain and range of each function.

E. For each function, describe how the y-values change as the x-values increase and decrease.

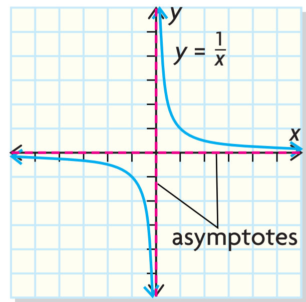
F. Use a graphing calculator to graph the functions $y = 2^x$, $y = 5^x$, and $y = 10^x$. Use the WINDOW settings shown to graph these functions on the same axes.



G. For each function, state
- the domain and range
- the intercepts
- the equations of any asymptotes

asymptote

a line that a curve approaches, but never reaches on some part of its domain



H. Use the trace key to examine the y-values as x increases and as x decreases. Which curve increases faster as you trace to the right? Which one decreases faster as you trace to the left?

I. Delete $y = 5^x$ and $y = 10^x$, and replace them with $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$. Graph these functions on the same axes.

J. For each new function, state
- the domain and range
- the intercepts
- the equations of any asymptotes

K. Describe how the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$ differ from the graph of $y = 2^x$.

L. What happens when the base of an exponential function is negative. Try $y = (-2)^x$. Discuss your findings.

Reflecting

M. Describe how the graph of an exponential function differs from the graph of a linear and a quadratic function.

N. How do the first and second differences of exponential functions differ from those of linear and quadratic functions? How can you tell that a function is exponential from its finite differences?

O. What type of function is $f(x) = b^x$ when $b = 1$?

P. Investigate the graphs of the exponential function $f(x) = b^x$ for various values of b, listing all similarities and differences in their features (such as the domain, the range, and any intercepts and asymptotes). Generalize their features for the cases $b > 1$ and $0 < b < 1$.

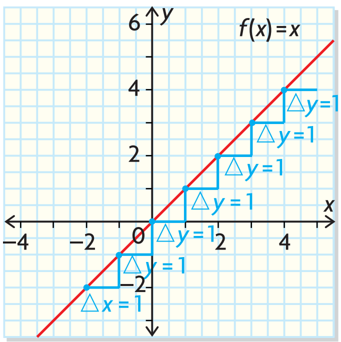
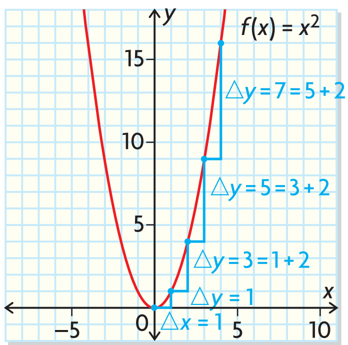
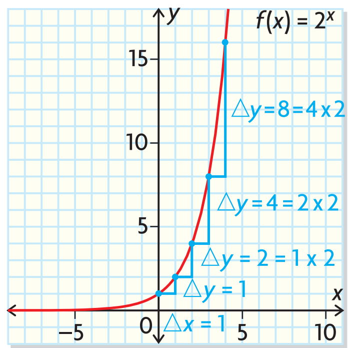
Tech *Support*

For help tracing functions on the graphing calculator, see Technical Appendix, B-2.

In Summary

Key Ideas

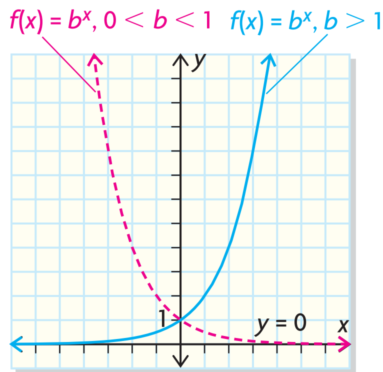
- Linear, quadratic, and exponential functions have unique difference patterns that allow them to be recognized.

Linear	Quadratic	Exponential
Linear functions have constant first differences.	Quadratic functions have first differences that are related by an addition pattern. As a result, their second differences are constant.	Exponential functions have first differences that are related by a multiplication pattern. As a result, their second differences are not constant.
		

- The exponential function $f(x) = b^x$ is
  - an increasing function representing rapid growth when $b > 1$
  - a decreasing function representing rapid decay when $0 < b < 1$

Need to Know

- The exponential function $f(x) = b^x$ has the following characteristics:
  - The function is exponential only if $b > 0$ and $b \neq 1$; its domain is the set of real numbers, and its range is the set of all positive real numbers.
  - If $b > 1$, the greater the value, the faster the growth.
  - If $0 < b < 1$, the lesser the value, the faster the decay.
  - The function has a horizontal *asymptote*, which is the *x*-axis.
  - The function has a *y*-intercept of 1.



FURTHER Your Understanding

1. Use each table of values to identify each function as linear, quadratic, or exponential. Justify your answer.

a)

x	y
−2	0.3
−1	0.6
0	1.2
1	2.4
2	4.8
3	9.6
4	19.2

b)

x	y
0.2	2
0.6	5
1.0	8
1.4	11
1.8	14
2.2	17
2.6	20

c)

x	y
−6	29
−1	20
4	13
9	8
14	5
19	4
24	5

2. For each exponential function,

i) determine the y-intercept

ii) sketch the function

iii) state the domain and range

iv) state the equation of the horizontal asymptote

a) $y = 3^x$

b) $y = 0.25^x$

c) $y = -(2^x)$

d) $y = 2(0.3)^x$

e) $y = (2^x) - 3$

f) $y = 4(0.5)^x + 5$

3. You are given the functions $f(x) = 2x + 5$, $g(x) = (x - 2)^2 + 3$, and $h(x) = 4^x$.

a) State which function is exponential. Explain how you know.

b) Use a difference table to justify your answer to part (a).

c) Which function is linear and which is quadratic? Explain how you know.

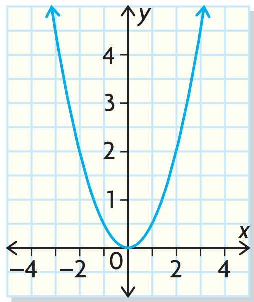
d) Use difference tables to justify your answer to part (c).

4. Select the function that matches each graph.

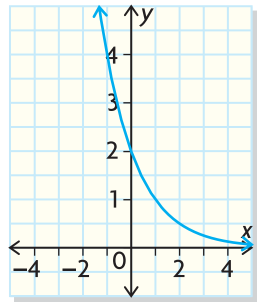
a) $y = 0.5x + 1$ **c)** $y = 0.5x^2$ **e)** $y = 2^x$

b) $y = 0.5^x$ **d)** $y = 8^x$ **f)** $y = 2(0.5)^x$

i)



iv)



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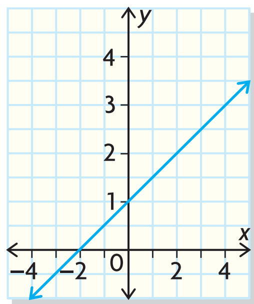

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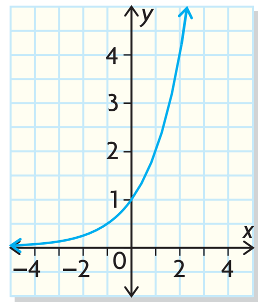

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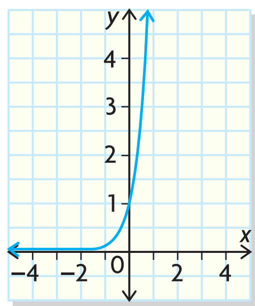

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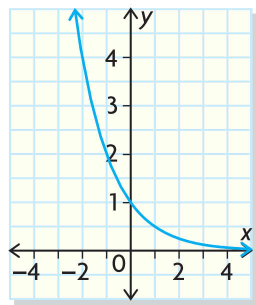


7.6 Solving Problems Involving Exponential Growth

GOAL

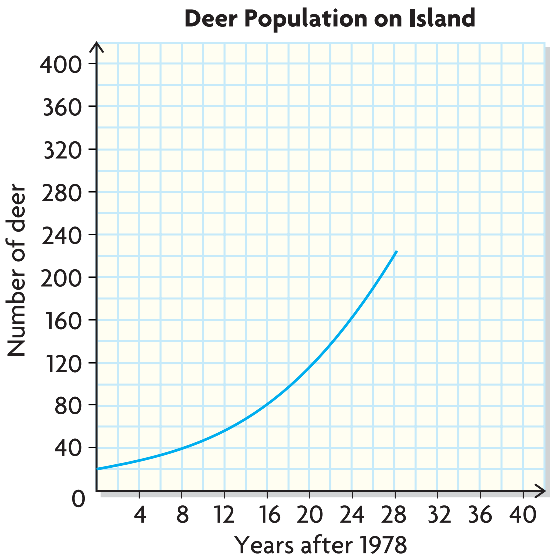
Use exponential functions to model and solve problems involving exponential growth.

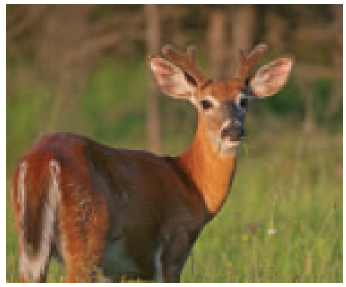
YOU WILL NEED

- graphing calculator
- graph paper

LEARN ABOUT the Math

In 1978, researchers placed a group of 20 deer on a large island in the middle of a lake. The deer had no natural predators on the island. Researchers collected population data and created a graphical model.





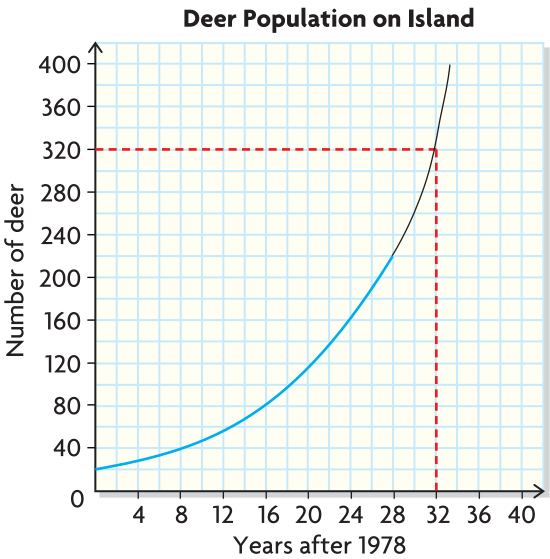
The researchers estimate that the island has enough resources to support a population of 320 deer.

? In what year will the deer population be too large for the island?

EXAMPLE 1 Making a prediction from data

Use the graph to estimate when the deer population will be greater than 320.

Liu's Solution: Using a Graph



I sketched an extension of the population curve. I extrapolated by drawing a horizontal line starting at 320 and extended this line until it touched the curve. From there, I drew a vertical line down to the years after axis. The population will hit 320 thirty-two years after 1978, or in 2010.

Petra's Solution: Using a Table of Values

Year	Number of Deer
0	20
4	30
8	40
12	60
16	*40 × 2 = 80*
20	*60 × 2 = 120*
24	*80 × 2 = 160*
28	*120 × 2 = 240*
32	*160 × 2 = 320*

I looked at some points in the table and noticed that the population doubled from 0 to 8 years and also from 4 to 12 years. It seems that every 8 years the population doubles.

I assumed that this doubling would continue and used this pattern to extend my table.

If the trend continues, then the population should reach 320 about 32 years after 1978. The year would be 2010.

Reflecting

A. What feature(s) of the graph indicate that this set of data might represent an exponential relationship?

B. Describe two ways that you can use data from a graph to determine whether or not there is an exponential relationship.

C. What are some of the advantages and disadvantages of making predictions from a graph?

APPLY the Math

EXAMPLE 2 Making a prediction from an algebraic model

There are 5000 yeast cells in a culture. The number of cells grows at a rate of 25% per day. The function that models the growth of the yeast cells is $N(d) = N_0(1 + r)^d$, where N is the number of yeast cells d days after the culture is started, N_0 is the initial population, and r is the growth rate. How many cells will there be one week later?

Joelle's Solution

Initial population, $N_0 = 5000$

Growth rate, $r = 25\% = 0.25$

Number of days, $d = 7$

I made a list of all of the given information.

$N(d) = N_0(1 + r)^d$

$N(7) = 5000(1 + 0.25)^7$

$= 5000(1.25)^7$

$\doteq 5000(4.768\ 371\ 582)$

$\doteq 23\ 841$

I substituted the given values into the function and solved for N.

I realized that the answer would be approximate since 1.25^7 is a decimal with many digits.

One week later, there are approximately 23 841 yeast cells.

EXAMPLE 3 Selecting a problem-solving strategy to make a prediction

According to the U.S. Census Bureau's World Population Clock, the number of people in the world in 1950 was 2.5 billion. The population has grown at a rate of approximately 1.7% per year since then. Create an algebraic model to predict the population in the future.

Daniel's Solution

Growth each year is 1.7% (or $1.7 \div 100 = 0.017$).

To determine the population in the next year, I calculated 1.7% of 2.5 billion, and added it to the original population.

In 1950, $P_0 = 2.5$ billion;
In 1951, growth $= 0.017 \times P_0 = 0.0425$

To make the notation easier, I used *subscripts*. For example, the population in year zero (1950) can be represented by P_0, and for year 1 (1951) it is P_1.

$P_1 = P_0 + 0.0425$

$P_1 = 2.5 + 0.0425 = 2.5425$ billion people

or

$$\begin{aligned} P_1 &= 2.5 + 0.017(2.5) \\ &= 2.5(1 + 0.017) \\ &= 2.5(1.017) \\ &= 2.5425 \end{aligned}$$

So, $P_1 = P_0(1.017)$

In 1952, $P_2 = P_1(1.017)$

$$\begin{aligned} P_2 &= [P_0(1.017)](1.017) \\ &= P_0(1.017)^2 \end{aligned}$$

Similarly,
In 1953, $P_3 = P_0(1.017)^3$

After a few calculations, I realized that I could do this in one step. I multiplied the population by 1 + 1.7% (or 1.017) to find the population in the next year.

If 1.017 is the growth rate each year, then I can find the population for any year if I know the population for the previous year. So I can calculate P_1 from P_0, P_2 from P_1, and so on. I can also calculate P_2 from P_0 just by substituting the formula for P_1.

The number of the year is the same as the exponent on 1.017.

$P_n = P_0(1.017)^n$

where P_n is the population n years after 1950, and $P_0 = 2.5$ billion

If I continue this pattern, I can calculate the population for any year after 1950 (provided the growth rate stays the same). I tried the year 2020 as an example.

In 2020, $n = 70$

$$\begin{aligned} P_{70} &= 2.5(1.017)^{70} \\ &\doteq 8.1 \text{ billion} \end{aligned}$$

Since 2020 is 70 years after 1950 and the population was 2.5 billion in 1950, I calculated that the population in 2020 should be approximately 8.1 billion people.

In Summary

Key Ideas

- A graph can be used to estimate answers to problems involving exponential growth by interpolating and extrapolating where necessary.
- If you are given a function that models exponential growth, it can be used to make accurate predictions.
- If you know the initial amount and growth rate, the exponential function $P(n) = P_0(1 + r)^n$ can be used as a model to solve problems involving exponential growth where
  - $P(n)$ is the final amount or number
  - P_0 is the initial amount or number
  - r is the rate of growth
  - n is the number of growth periods

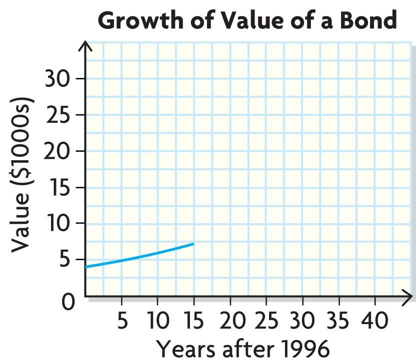
Need to Know

- The growth rate needs to be expressed as a fraction or decimal, *then* added to 1 in the exponential-growth equation.
- The units for the growth rate and for the number of growth periods must be compatible. For example, if a population growth rate is given as "per hour," then the number of growth periods in the equation is measured in hours.

CHECK Your Understanding

1. A brochure for a financial services company has a graph showing the value of a $4000 savings bond since 1996. Suppose the bond continued to increase in value at the same rate.

a) How much will the bond be worth in 2008?
b) How much would it be worth in 2025?
c) Is it possible to determine the length of time needed for the savings bond to double its value from the graph? Explain.



2. Algae in a pond grow at a rate of 10% per week. Currently the algae cover one-quarter of the pond. An algebraic model of this situation is $C(w) = 0.25(1 + 0.1)^w$, where $C(w)$ is the percent covered after w weeks.

a) Explain what 0.25 represents in the equation.
b) Explain what 0.1 represents in the equation.
c) Determine how much of the pond (to the nearest percent) will be covered 10 weeks from now.

3. According to the 1991 census, the region of Niagara, Ontario, had 364 552 residents. For the next few years, the population grew at a rate of 2.2% per year. For planning purposes, the regional government needs to determine the population of the region in 2010. The algebraic model for this case is $P(n) = P_0(1 + r)^n$.

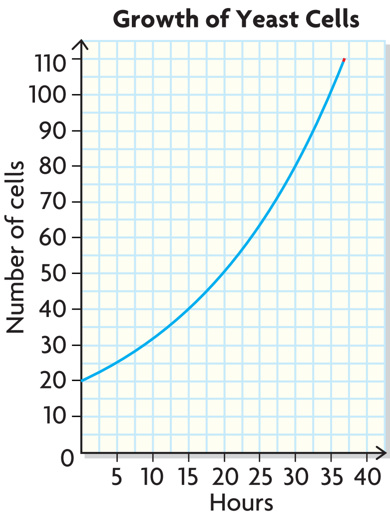
a) What is the initial population, P_0?
b) What is the growth rate, r?
c) How many growth periods, n, are there?
d) Write the algebraic model for this situation.
e) Use the model to determine the population in 2010.

PRACTISING

4. a) In each of the algebraic models that follow, identify
- the initial amount
- the growth rate
- the number of growth periods

i) $M(n) = (18)(1.05)^{22}$
ii) $P(n) = 64\,000(1.1)^{12}$
iii) $N(n) = 750(2)^4$

b) Evaluate each equation in part (a) to two decimal places.



5. The size of a yeast culture is measured each hour, and the results are displayed on the graph shown.

a) From the graph, estimate the initial number of yeast cells.
b) Estimate the number of cells after 30 h.
c) Use the initial amount to estimate the length of time required for the number of cells to double.
d) Use an amount (other than the initial amount) to determine the length of time required for the number of cells present at that time to double.
e) Describe what you noticed about the two doubling times. Check what you found with a different amount from the graph.

6. K The number of guppies in an aquarium is modelled by the function $N(t) = 12(1 + 0.04)^t$, where t is measured in weeks.

a) Describe what each part of the equation represents.
b) Determine the number of guppies in the aquarium after 10 weeks.
c) Will this equation *always* model the population in the aquarium?

7. In each case, write an equation that models the situation described.
   a) An antique is purchased for \$5000 in 1990. It appreciates in value by 3.25% each year.
   b) A town had 2500 residents in 1990. It grew at a rate of 0.5% per year for t years.
   c) A single bacterium of a particular type takes one day to double. The population is P after t days.

8. Mari invests \$2000 in a bond that pays 6% per year.
   a) Write an equation that models the growth of her investment.
   b) How much money does she have if she cashes the bond at the end of the 4th year?
   c) How much will the bond be worth at the end of the 5th year? How can you determine the amount earned during the 4th year?
   d) Determine the amounts Mari will earn at the end of the 20th and 21st years to find the amount earned during the 20th year.
   e) Compare the money earned in the 4th and 20th years. What does this tell you about exponential growth?

9. **A** Five hundred yeast cells in a bowl of warm water doubled in number every 40 min.
   a) Create a graph of the number of yeast cells versus time.
   b) Use the graph to determine how long it would take for the total number of cells to triple (to the nearest minute).
   c) Describe how you can adapt your graph to determine the number of cells for the time *before* they were monitored.

10. An ant colony triples in number every month. Currently, there are 24 000 in the nest.
   a) What is the monthly growth rate of the colony? What is the initial population?
   b) Write an equation that models the number of ants in the colony, given the number of months.
   c) Use your equation to predict the size of the colony in three months.
   d) Use your equation to predict the size of the colony five months ago.

11. **C** The number of franchises of a popular café has been growing exponentially since the first store opened in 1971. Since then, the number of stores has grown at a rate of 33% per year.
   a) Explain how you could create an algebraic model that gives the number of stores in any year after 1971. Discuss how the information in the problem relates to your algebraic model.
   b) Use your model to predict the number of stores in 2010.

12. The population of Upper Canada between 1784 and 1830 grew at a rate of about 8% each year.
T

Year	Year Since 1784	Population (1000s)
1784	0	6
1791	7	10
1811	27	77
1824	40	150
1827	43	177
1830	46	213

a) Use a graphing calculator and enter the years since 1784 in L1 and population in L2, and create a scatter plot in a suitable window.
b) Consider an exponential equation that would fit this data to the form $P(n) = P_0(1 + r)^n$. Estimate the values of P_0 and r, and write the equation. Enter your equation into the equation editor and graph it.
c) Does your equation fit the data?
d) The population in 1842 was 487 000. Does your equation "predict" this number? Explain.

Extending

13. The height, $h(t)$, of a tree with respect to its age, t, in years is given by the formula

$$h(t) = \frac{20}{1 + 200(10)^{-0.3t}}$$

a) Determine the height of a 6-year-old tree.
b) Use technology to graph the function, h. Explain why the shape of the curve is appropriate for this situation.

14. The city of Mississauga has experienced rapid growth in recent years. It had a population of 234 975 in 1975 and 610 700 in 2000. Determine the annual growth rate of the population over the 1975–2005 period.

15. An old riddle says: Water lilies in a pond double each day. It takes 30 days for the lilies to completely cover the pond. On what day was the pond half full of water lilies?

7.7 Problems Involving Exponential Decay

GOAL

Use exponential functions to model and solve problems involving exponential decay.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

Scuba divers who photograph shipwrecks or coral reefs need to adjust to decreasing amounts of light as they descend into the ocean.

Frank knows that light intensity reduces by 2% for each metre below the water surface. His underwater camera requires a minimum of 60% of the light at the surface to operate without an additional light source. He has seen coral that he would like to photograph 23 m below the surface. His user manual gives the function $I(n) = I_0(0.98)^n$ for determining light intensity, where I_0 is the initial amount of light at the surface and n is the depth below the surface, in metres.

? Is Frank able to take pictures at a depth of 23 m without an additional light source?

EXAMPLE 1 Reasoning to solve an exponential decay problem

Determine whether Frank's dive to 23 m will require him to use an additional light source to take pictures.

Kira's Solution: Using an Algebraic Model

$I(n) = I_0\,(0.98)^n$

Since light intensity *decreases* with increasing depth, the decay factor must be less than 1. Since the light intensity diminishes by 2% per metre, I can represent the amount of light 1 m deep as 98% of the light intensity at the surface (100% − 2%). For every metre below the surface, I multiply the surface light intensity by 0.98.

$I(23) = 100(0.98)^{23}$

I substituted 23 m for n and 100 for I_0 and evaluated.

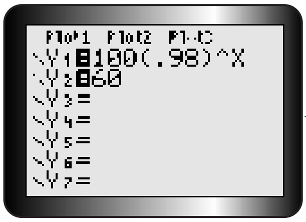
$I(23) \doteq 100(0.628\,347\,28)$

$I(23) \doteq 62.8\%$

$I(23)$ is above 60%.

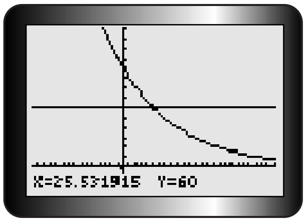
Frank's camera will not require an additional light source.

David's Solution: Using a Calculator to Create a Graphical Model



I entered the formula from the manual into the equation editor. I used 100% to represent the initial light intensity, I_0.

Since I was looking for the intensity to be above 60%, I entered a second function to represent that level.



I graphed both functions and found the point of intersection.

The light intensity is 60% at a depth of approximately 25.3 m. Since light intensity *decreases* as you dive deeper, there will be enough light at depths shallower than 25.3 m.

Frank will not need additional light to photograph at 23 m.

Tech *Support*

For help determining the point(s) of intersection between two functions, see Technical Appendix, B-11.

Reflecting

A. What part of the exponential function $I(n) = 100(0.98)^n$ indicates that it models exponential decay?

B. Compare the exponential-decay graph with the exponential-growth graph. Explain the differences.

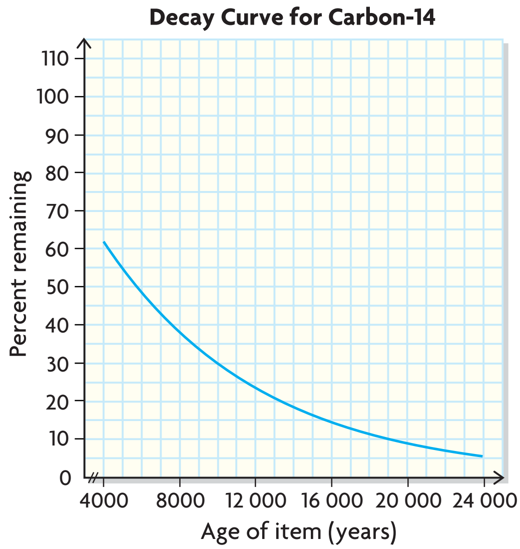
C. Explain the differences in the approaches that Kira and David took to answer the question.

APPLY the Math

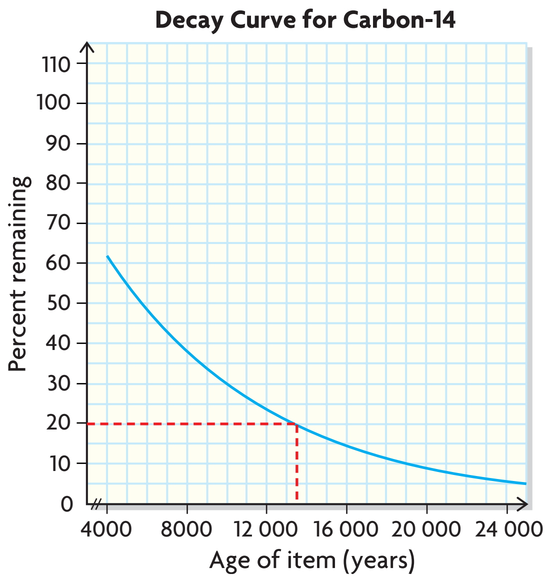
EXAMPLE 2 Connecting graphs to half-life models

Archaeologists use carbon-14 to estimate the age of the artifacts they discover. Carbon-14 has a half-life of 5730 years. This means that the amount of carbon-14 in an artifact will be reduced by half every 5730 years.

An ancient animal bone was found near a construction site. Tests were conducted, and the bone was found to contain 20% of the amount of carbon-14 in a present day bone. Use the decay curve shown to estimate the age of the bone.



Eric's Solution



I looked at 20% on the vertical axis and read across to the curve. I then read directly down to the time axis. This gave me an answer of approximately 13 500 years.

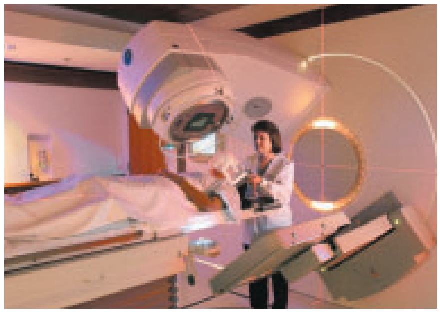
I estimate that the bone is 13 500 years old.

EXAMPLE 3 Connecting functions to half-life models

A hospital uses cobalt-60 in its radiotherapy treatment for cancer patients. Cobalt-60 has a half-life of 5.2 years. This means that every 5.2 years, 50% of the original sample of cobalt-60 has decayed. The hospital has 80 g of cobalt-60. It requires at least 64 mg for the therapy machine to provide an effective amount of radiation.

a) How often must the staff replace the cobalt in their machine?

b) How much of the orignal sample will there be after 1 year?



Liz's Solution

a) $P(n) = P_0(1 - r)^n$

$P(n) = 80(0.5)^n$

I created a function to model this situation. Since the rate of decay is $\frac{1}{2}$, I used the decimal 0.5. I used 80 for the initial amount, since that is the starting mass. $P(n)$ represents the amount of cobalt-60 remaining after n periods of 5.2 years.

n	M
1	$80(0.5)^1 = 40$
0.5	56.5685
0.4	60.6287
0.3	64.98
0.32	64.1

I need to determine the exponent to get an answer of about 64 mg. I know that one half-life $(n = 1)$ gives an answer of 40 mg. So I need a number less than 1 to start. I made a table to record my guesses. $n = 0.3$ is quite close, but too high, and $n = 0.4$ is too low. So I tried $n = 0.32$, which gave me a good approximation.

$n \doteq 0.32$

$\text{time} = 0.32 \times 5.2$

To determine the number of years the cobalt-60 will last, I converted the number of half-lives, 0.32, to years by multiplying by 5.2, the length of one half-life.

Therefore, $t \doteq 1.664$ years, or 20 months.

The hospital should replenish its supply of cobalt-60 every 20 months to maintain an effective amount of radiation.

b) $P(n) = 80(0.5)^n$

I used the function I created above.

$n = 1 \div 5.2 = 0.19$

I need to determine the number of half-lives, n, that 1 year is equivalent to. I divided 1 by 5.2, since each half-life is 5.2 years.

$P(0.19) = 80(0.5)^{0.19}$

$= 70.13$

I substituted 0.19 for n and evaluated.

About 70.13 g of cobalt-60 will remain after 1 year.

In Summary

Key Ideas

- You can use a graph to estimate answers to problems involving exponential decay by interpolating and extrapolating where necessary.
- If you are given a function that models exponential decay, it can be used to make accurate predictions.
- If you know the initial amount and the decay rate, the exponential function $P(n) = P_0(1 - r)^n$ can be used as a model to solve problems involving exponential decay where
  - $P(n)$ is the final amount or number
  - P_0 is the initial amount or number
  - r is the rate of decay (a number between 0 and 1)
  - n is the number of decay periods

Need to Know

- One way to tell the difference between growth and decay is to consider whether or not the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
- If the decay rate is given as a percent, it needs to be converted into a fraction or decimal and then subtracted from 1 in the exponential-decay equation.
- The periods for the decay rate and the number of decay periods must be compatible. For example, if light intensity decreases per metre, then the number of decay periods in the equation is measured in metres.

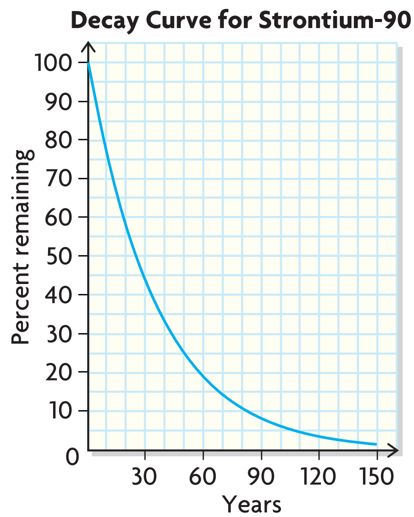
CHECK Your Understanding

1. a) In each of the algebraic models that follow, identify
- the initial amount
- the decay rate
- the number of decay periods

i) $M(n) = 100(1 - 0.25)^{28}$
ii) $P(n) = 32\,000(1 - 0.44)^{12}$
iii) $N(n) = 500(1 - 0.025)^{20}$

b) Evaluate each expression to the nearest hundredth of a unit.

2. The radioactive decay curve for strontium-90 is shown.
a) Estimate the percent of strontium-90 remaining after 20 years.
b) How much time is required for a sample to decay to 25% of its original mass?



3. A new car costs \$24 000. It loses 18% of its value each year after it is purchased. This type of loss is called *depreciation*. The value of the car is given by $V(n) = V_0(1 - 0.18)^n$, where V_0 is the original value of the car, and n is the number of years after the car was purchased.

a) Use the formula to determine how much of the car's initial value is lost after 5 years.

b) Use the formula to determine the value of the car after 30 months.

PRACTISING

4. Solve each equation. Express each answer to the nearest hundredth.

a) $y = 52(0.87)^7$

b) $N(t) = 100(0.99)^6$

c) $P(t) = 512\left(\frac{1}{2}\right)^9$

5. After being filled, a basketball loses 3.2% of its air every day. An equation that models this situation is $V(d) = V_0(1 - 0.032)^d$.

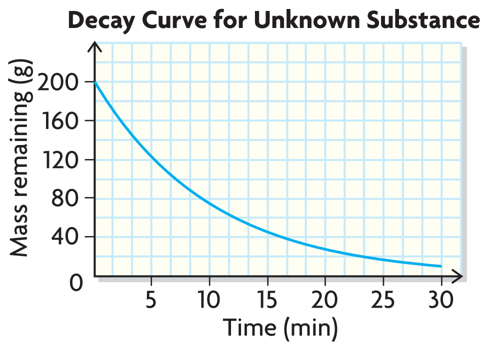
K

a) Describe what each part of the equation represents.

b) The initial volume of air in the ball was 840 cm^3. Determine the volume after 4 days.

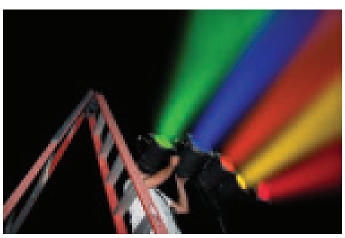
c) Will this model be valid after several weeks? Explain why or why not.

6. A lab has 200 grams of an unknown radioactive substance. The scientists in the lab measure the mass of the substance each minute and plot the curve shown.



a) How many grams of the substance remain after 18 min?

b) Use the graph to determine the half-life of the substance.



7. Gels used to change the colour of spotlights each reduce the intensity of the light by 4%. The algebraic model for this situation is $I = 100(0.96)^n$.

a) Describe what each part of the equation represents.

b) Determine the intensity of the spotlight if three gels are used.

c) How many gels would reduce the intensity by more than 75%?

8. The value of a car after it is purchased depreciates according to the formula $V(n) = 25\,000(0.85)^n$, where $V(n)$ is the car's value in the nth year after it was purchased.

A

a) What is the purchase price of the car?
b) What is the annual rate of depreciation?
c) What is the car's value at the end of 3 years?
d) How much value does the car lose in its first year?
e) How much value does it lose in its fifth year?
f) After how many years will the value of the car be half of the original purchase price?

9. A hot cup of coffee cools according to the equation

$$T(t) = 68\left(\frac{1}{2}\right)^{\frac{t}{22}} + 18$$

where $T(t)$ is the temperature, in degrees Celsius, and t is the time in minutes.

a) Which part of the equation indicates that it models exponential decay?
b) What value of t makes the exponent in the equation equal to 1?
c) What is the significance of this value?
d) What was the initial temperature of the coffee?
e) Determine the temperature of the coffee after 40 min, to the nearest degree.

10. Old Aboriginal wooden tools were found at an archaeological dig in the Brantford area. Carbon-14 dating was used to determine the age of the tools. Carbon-14 has a half-life of 5730 years. The general equation that models radioactive decay is

$$A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

where the initial radioactivity of the tools is 100%, t is time in years, $A(t)$ is the radioactivity of the tools today, and H is the half-life of the radioactive substance.

a) Archaeologists guessed that the tools were about 6000 years old. What percent of the present-day radioactivity would the tools emit if that were the case? Express your answer to the nearest tenth of a percent.
b) The tools' actual radioactivity was 56% of the radioactivity of the same type of present-day material. Determine the age of the wood to the nearest hundred years.

11. Write an equation to model each situation. In each case, describe how **C** you can tell that it is an example of exponential decay.

a) The value of my car was \$25 000 when I purchased it. The car depreciates at a rate of 25% each year.

b) The radioactive isotope U_{238} has a half-life of 4.5×10^9 years.

c) Light intensity in a pond is reduced by 12% per metre of depth, relative to the light intensity at the surface.

12. The population of a small mining town was 13 700 in 2000. Each **T** year, the population decreases by an average of 5%.

a) Explain why the situation described is an example of exponential decay.

b) Write an exponential function that models this situation. Explain each part of the equation.

c) Use your equation to estimate when the population will drop below 5000.

Bounce	Maximum Height after Bounce (m)
0	5
1	$5 \times 0.80 = 4$
2	$4 \times 0.80 = 3.2$
3	
4	

13. A rubber ball is dropped from a height of 5 m. It bounces to a height that is 80% of its previous maximum height after each bounce.

a) Complete the table to determine the height of the ball after the 4th bounce.

b) The equation that models the maximum height of the ball after each bounce is $H(n) = 5(0.80)^n$. Use the equation to verify the height of the ball after 4 bounces

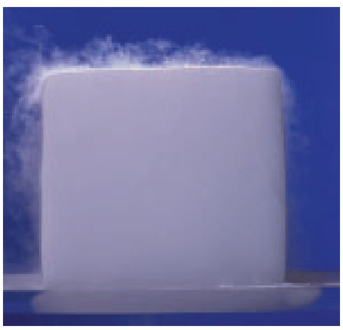
c) Determine the height of the ball after 7 bounces.

Extending

14. A blue shirt loses 0.5% of its colour every time it is washed. Once the shirt has lost 15% of its colour, it is no longer desirable to wear.

a) Describe two methods for determining how many times the shirt can be washed before it must be discarded.

b) Determine how many times the shirt can be washed before it must be discarded.



15. Frozen carbon dioxide changes directly from a solid to a gas in a process known as *sublimation*. For this reason, it is sometimes called "dry ice." Suppose a 50 kg block of dry ice lost 10 kg in 24 h due to sublimation.

a) Determine the percent lost in 24 h.

b) Write an equation that models the mass, M, of dry ice left after d days.

c) Use your equation to determine the amount of dry ice left after 54 h.

Curious Math

Disappearing Coins

You drop 100 pennies onto a table and remove all of those that turn up "tails." You repeat this over and over for the coins that remain. What function models this situation?

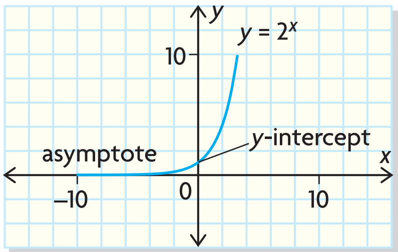
1. Drop your pennies onto a table. Remove all the coins that turned up "tails."
2. Record your data in a table like the one shown.

Trial #	Number of Coins	Number of Tails Removed	Number of Coins that Remain
0	100		
1			
2			

3. Repeat parts 1 and 2 with the remaining coins until you have no coins left.
4. Create a scatter plot of number of coins versus trial number. What type of function does this appear to be?
5. Compare your results with other classmates. How many trials are needed until all the coins are gone?
6. Determine the equation of a function that shows the relationship between the number of coins remaining and the trial number.
7. If you were to use thumbtacks instead of coins and took away the tacks that landed tip up, do you think the scatter plot and function would be the same? Explain.

7 Chapter Review

FREQUENTLY ASKED Questions



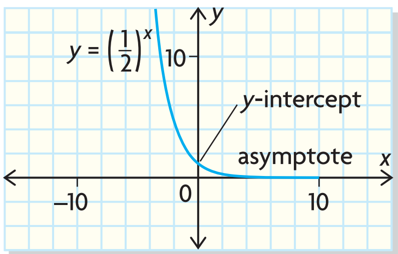
Q: How can you identify an exponential function from

- **its equation?**
- **its graph?**
- **a table of values?**

A: The basic exponential function has the form $f(x) = b^x$, where $b > 0$.

The shape of its graph depends upon the parameter, b.

If $b > 1$, the curve increases as x increases.
If $0 < b < 1$, the curve decreases as x increases.



Each of the functions has the x-axis as its horizontal *asymptote*.

If you create a difference table for an exponential function, you will notice that the differences are never constant, as they are for linear and quadratic functions. Instead, the differences are related by a multiplication pattern.

x	$f(x) = 2^x$	First Differences	Second Differences
0	1	1	
1	2	2	1
2	4	4	2
3	8	8	4
4	16	16	8
5	32	32	16
6	64		

Study *Aid*

- See Lesson 7.5, Key Ideas.
- Try Chapter Review Question 7.

Q: How can exponential functions model growth and decay? How can you use them to solve problems?

A: Exponential functions can be used to model situations that show repeated multiplication by the same factor. Here are some examples:

Growth	Decay
Population $P(n) = P_0(1 + r)^n$	Depreciation of assets $V(n) = V_0 (1 - r)^n$
Cell division (bacteria, yeast cells, etc.) $P(n) = P_0 (1 + r)^n$	Radioactivity or half-life $Q(n) = 100\left(\frac{1}{2}\right)^n$
Compound interest $A(n) = P(1 + i)^n$	Light intensity in water $V(n) = 100(1 - r)^n$

If you are given a graph, you can estimate growth and decay by interpolating and extrapolating as needed.

If you are given a function, you can make accurate predictions by substituting the given information into the function and solving for the unknown quantity.

You can develop a function based on the general exponential function and use it to make accurate predictions if you know the initial amount, P_0, and the rate of growth or decay, r.

Exponential Growth	Exponential Decay
$P(n) = P_0(1 + r)^n$	$P(n) = P_0(1 - r)^n$

n is the number of growth/decay periods and $P(n)$ is the amount in the future.

Study Aid

- See Lessons 7.6 and 7.7, Examples 1, 2, and 3 in each.
- Try Chapter Review Questions 8 to 13.

PRACTICE Questions

Lesson 7.2

1. Write as a single power. Express answers with positive exponents.

a) $3^4 \times 3^8 \times 3$

b) $\frac{(-5)^6}{(-5)^4}$

c) $\frac{11^5}{11^9}$

d) $\left((-9)^2\right)^5$

e) $\frac{4^7 4^5}{4^{12}}$

f) $\frac{6^{10}}{(6^6)^2}$

Lesson 7.3

2. Write as a single power. Express answers with positive exponents.

a) $12^{-6} \times 12^8 \times 12^0$

b) $\frac{\left((-8)^6\right)^{-2}}{\left((-8)^{-4}\right)^3}$

c) $\frac{(20^{-1})^8}{20^2 20^6}$

d) $\left(10(10^3)^{-1}\right)^{-2}$

3. Write as a single power. Express answers with positive exponents.

a) $\frac{a^5}{a^3}$

b) $(b)(b^4)(b^2)$

c) $\frac{c^3}{c^9}$

d) $(d^6)^3$

e) $\frac{e(e^5)}{e^7}$

f) $(f^{-3})^{-2}$

Lesson 7.4

4. Copy and complete the table.

	Exponential Form	Radical Form	Evaluation of Expression
a)	$36^{\frac{1}{2}}$		
b)	$16^{\frac{5}{4}}$		
c)		$\sqrt[5]{1024}$	
d)	$16\,807^{0.2}$		
e)		$\sqrt[3]{-216^4}$	

5. Use your calculator to evaluate each expression. Express answers to two decimals.

a) $125^{0.33}$

b) $\sqrt[3]{-1953.125}$

c) $\sqrt[7]{-180}$

d) $16^{\frac{2}{3}}$

e) $10^{-\frac{3}{2}}$

f) $\sqrt[12]{1.9}$

6. Evaluate each expression without using a calculator.

a) $-125^{\frac{1}{3}}$

b) $81^{0.25}$

c) $\sqrt[3]{27}$

d) $16^{\frac{3}{2}}$

e) $256^{-\frac{5}{4}}$

f) $\sqrt[5]{-32}$

Lesson 7.5

7. Calculate the finite differences for each table of values. Then use the finite differences to classify each function as linear, quadratic, or exponential.

a)

x	y
1	−1
2	3
3	9
4	17
5	27
6	39

b)

x	y
−2	3
−1	8
0	13
1	18
2	23
3	28

c)

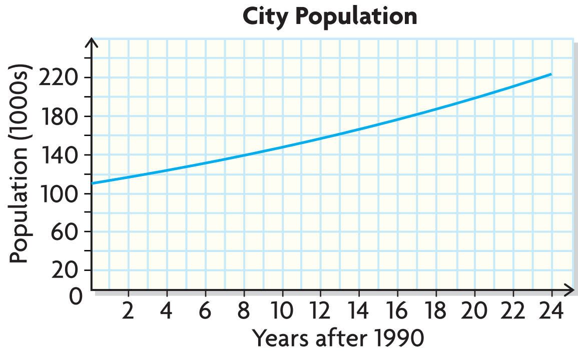
x	y
−1	0.25
1	0.5
3	1
5	2
7	4
9	8

d)

x	y
−2	10
−1	30
0	90
1	270
2	810
3	2430

Lesson 7.6

8. A city has a population increase of 3% per year from 1990 to 2007. In 1990 the population was 110 000.



a) Use the graph to estimate the population in 2006.

b) How long did it take for the population to increase by 30 000 from its 1990 value?

c) Estimate the *doubling period* of the population if it continues to grow at this rate. Explain your steps.

9. A biologist measures 1000 yeast cells in a culture at 12:00 noon. She predicts that, under current conditions, the culture should double every 50 min.

a) Sketch a graph of the population of this culture for 200 min after noon.

b) Use your graph to determine the population of the culture in 2.5 h.

c) How can you adapt the graph to determine what the population of cells was 3 h *before* noon?

10. In each case, write a function that models the situation described.

a) A rare coin is purchased for \$2500 in 2000. It appreciates in value by 5% each year for t years.

b) A school has 750 students in 2003. It grew at a rate of 2% per year for t years.

c) Evaluate the functions you found in parts (a) and (b) when $t = 10$. Explain what these numbers represent.

11. Simon has a Wayne Gretzky rookie hockey card. He bought it in 2000 for \$500 on e-Bay. He estimates that it will appreciate in value by 7% each year. The dealer told him the function $V(t) = 500(1 + 0.07)^t$ can be used to estimate the card's future value.

a) Explain how the numbers in the equation are related to the situation.

b) How much will the card be worth in 2020?

c) How long will it take for the card to double its value?

Lesson 7.7

12. A police diver is searching a harbour for stolen goods. The equation that models the intensity of light per metre of depth is $I(n) = 100(0.94)^n$.

a) Describe each part of the equation.

b) Determine the amount of sunlight the diver will have at a depth of 16 m, relative to the intensity at the surface.

13. Copy and complete the table.

Function	Exponential Growth or Decay	Initial Value (y-intercept)	Growth/ Decay Rate
$V(t) = 125(0.78)^t$			
$P(t) = 0.12(1.05)^t$			
$A(x) = (2)^x$			
$Q(x) = 0.85\left(\frac{1}{3}\right)^x$			

14. Jerry invests \$500 in a bond that pays 5% per year. He will need the money for college in 3 years.

a) Write an equation that models the growth of the money.

b) Use the equation to determine how much Jerry will have at the end of 3 years.

c) How much money did his \$500 earn in the 3 years?

d) Jerry thinks that if he keeps his money invested for twice as long (6 years), he will earn twice as much. Is this true? Explain your reasoning.

7 Chapter Self-Test

1. Evaluate without using a calculator.

a) 5^{-3} c) $8^{\frac{1}{3}}$ e) -7^0

b) $\left(\frac{3}{4}\right)^{-2}$ d) $16^{-0.75}$ f) $100^{\frac{-3}{2}}$

2. Write as a single power. Express answers with a positive exponent.

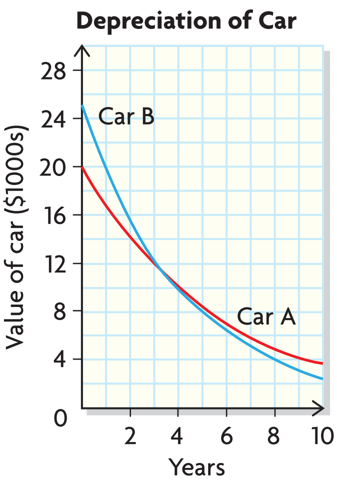
a) $(6)^{-\frac{1}{3}} \times (6)^{\frac{5}{6}}$ c) $\frac{10}{10^{-4}}$ e) $a^7(a^6)^{-2}$

b) $4\left(\frac{1}{4}\right)^{-4}$ d) $\frac{7^8}{(7^2)^3}$ f) $\frac{b^3(b^{-2})}{b^4}$

3. Write $\sqrt[6]{4^3}$ in exponent form, then evaluate.

4. Sketch the graph of each function. If applicable, label the x- and y-intercepts and asymptotes.

a) $y = 2^x$ b) $y = 0.5^x$



5. The values of two different automobiles over time are shown in the graph.

a) Compare the initial value of each car with its value through the first 6 years of ownership.

b) Which car has the higher depreciation rate? Explain your reasoning.

6. An archaeologist discovers an ancient settlement. To determine the age of the settlement, she measures the radioactivity of a fragment of bone recovered at the site. Carbon-14 has a half-life of 5730 years. The algebraic model for the radioactivity of carbon-14 is

$$A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Determine the radioactivity of the bone, to the nearest percent, if it is 12 000 years old.

7. The population of a small town has increased at a rate of 1.5% per year since 1980. The town had a population of 1600 that year.

a) Write the equation that models the growth in population of the town. Describe each part of your equation.

b) Use your equation to determine the population of the town in 2008.

7 Chapter Task

The Half-Life of Caffeine

Caffeine is a stimulant found in many products, such as coffee, tea, pop, and chocolate. When you drink a cup of tea or eat a chocolate bar, you ingest caffeine. Your body breaks down caffeine slowly. As with other drugs, caffeine has a half-life in the body. The half-life of caffeine in a typical nonsmoking adult is 5.5 h.

The caffeine content of many popular foods and drinks is listed in the table at the right. You can use the information from the table to write an equation to model the amount of caffeine in your system after drinking or eating foods that contain caffeine.

A. A cup of brewed coffee has approximately 130 mg of caffeine in it. Suppose you drink a cup of drip coffee at 9 a.m. How much caffeine is left in your body at
i) 12:00 noon? **ii)** 8 p.m.? **iii)** midnight?

B. Suppose that in addition to your 9 a.m. cup of coffee at noon, you ate a chocolate bar and drank a cup of green tea. How can you calculate the additonal amount of caffeine in your system? Determine the total amount of caffeine in your body at noon.

C. If you had nothing else to eat or drink that contained caffeine from noon onward, predict the amount of caffeine in your body at 8 p.m.

D. Make a list of the food or drinks with caffeine you have had today. Write the time you had each food or drink. Calculate the amount of caffeine that will be left in your body at 10 p.m. tonight. Use a table like the one below to record your data.

Time Ingested	Food or Drink	Caffeine Content (mg)	Number of Hours until 10 p.m.

Caffeine Content	(mg)
250 mL of Coffee:	
Drip	165
Brewed	130
Instant	95
Decaffeinated	4
250 mL of Tea:	
Brewed	45
Instant	35
Green tea	30
Soft drinks (mean)	43
Chocolate bar	20
Energy drinks	80
Espresso, double (2 oz.)	70

Task *Checklist*

- ✔ Did you show the calculations you used to determine the caffeine levels for part A?
- ✔ Did you remember to include the caffeine from the 9 a.m. cup of coffee in your calculations for part B?
- ✔ Did you remember to write an equation that models this situation?
- ✔ Did you include the table you created for part D?