Section 1.1—Radical Expressions: Rationalizing Denominators

Now that we have reviewed some concepts that will be needed before beginning the introduction to calculus, we have to consider simplifying expressions with radicals in the denominator of radical expressions. Recall that a rational number is a number that can be expressed as a fraction (quotient) containing integers. So the process of changing a denominator from a radical (square root) to a rational number (integer) is called **rationalizing the denominator**. The reason that we rationalize denominators is that dividing by an integer is preferable to dividing by a radical number.

In certain situations, it is useful to rationalize the numerator. Practice with rationalizing the denominator prepares you for rationalizing the numerator.

There are two situations that we need to consider: radical expressions with one-term denominators and those with two-term denominators. For both, the numerator and denominator will be multiplied by the same expression, which is the same as multiplying by one.

EXAMPLE 1 **Selecting a strategy to rationalize the denominator**

Simplify $\frac{3}{4\sqrt{5}}$ by rationalizing the denominator.

Solution

$$\begin{aligned}\frac{3}{4\sqrt{5}} &= \frac{3}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} && \text{(Multiply both the numerator and denominator by } \sqrt{5}\text{)}\\ &= \frac{3\sqrt{5}}{4 \times 5} && \text{(Simplify)}\\ &= \frac{3\sqrt{5}}{20}\end{aligned}$$

When the denominator of a radical fraction is a two-term expression, you can rationalize the denominator by multiplying by the **conjugate**.

An expression such as $\sqrt{a} + \sqrt{b}$ has the conjugate $\sqrt{a} - \sqrt{b}$.

Why are conjugates important? Recall that the linear terms are eliminated when expanding a difference of squares. For example,

$$\begin{aligned}(a - b)(a + b) &= a^2 + ab - ab - b^2\\ &= a^2 - b^2\end{aligned}$$

If a and b were radicals, squaring them would rationalize them.

Consider this product: $\left(\sqrt{m} + \sqrt{n}\right)\left(\sqrt{m} - \sqrt{n}\right)$, m, n rational

$$= \left(\sqrt{m}\right)^2 - \sqrt{mn} + \sqrt{mn} - \left(\sqrt{n}\right)^2$$

$$= m - n$$

Notice that the result is rational!

EXAMPLE 2 Creating an equivalent expression by rationalizing the denominator

Simplify $\frac{2}{\sqrt{6} + \sqrt{3}}$ by rationalizing the denominator.

Solution

$$\frac{2}{\sqrt{6} + \sqrt{3}} = \frac{2}{\sqrt{6} + \sqrt{3}} \times \frac{\sqrt{6} - \sqrt{3}}{\sqrt{6} - \sqrt{3}}$$

(Multiply both the numerator and denominator by $\sqrt{6} - \sqrt{3}$)

$$= \frac{2\left(\sqrt{6} - \sqrt{3}\right)}{6 - 3}$$

(Simplify)

$$= \frac{2\left(\sqrt{6} - \sqrt{3}\right)}{3}$$

EXAMPLE 3 Selecting a strategy to rationalize the denominator

Simplify the radical expression $\frac{5}{2\sqrt{6} + 3}$ by rationalizing the denominator.

Solution

$$\frac{5}{2\sqrt{6} + 3} = \frac{5}{2\sqrt{6} + 3} \times \frac{2\sqrt{6} - 3}{2\sqrt{6} - 3}$$

(The conjugate $2\sqrt{6} + \sqrt{3}$ is $2\sqrt{6} - \sqrt{3}$)

$$= \frac{5\left(2\sqrt{6} - 3\right)}{4\sqrt{36} - 9}$$

(Simplify)

$$= \frac{5\left(2\sqrt{6} - 3\right)}{24 - 9}$$

$$= \frac{5\left(2\sqrt{6} - 3\right)}{15}$$

(Divide by the common factor of 5)

$$= \frac{2\sqrt{6} - 3}{3}$$

The numerator can also be rationalized in the same way as the denominator was in the previous expressions.

EXAMPLE 4 Selecting a strategy to rationalize the numerator

Rationalize the numerator of the expression $\frac{\sqrt{7} - \sqrt{3}}{2}$.

Solution

$$
\begin{aligned}
\frac{\sqrt{7} - \sqrt{3}}{2} &= \frac{\sqrt{7} - \sqrt{3}}{2} \times \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} + \sqrt{3}} && \text{(Multiply the numerator and denominator by } \sqrt{7} + \sqrt{3}\text{)} \\
&= \frac{7 - 3}{2\left(\sqrt{7} + \sqrt{3}\right)} && \text{(Simplify)} \\
&= \frac{4}{2\left(\sqrt{7} + \sqrt{3}\right)} && \text{(Divide by the common factor of 2)} \\
&= \frac{2}{\sqrt{7} + \sqrt{3}}
\end{aligned}
$$

IN SUMMARY

Key Ideas

- To rewrite a radical expression with a one-term radical in the denominator, multiply the numerator and denominator by the one-term denominator.

$$
\begin{aligned}
\frac{\sqrt{a}}{\sqrt{b}} &= \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} \\
&= \frac{\sqrt{ab}}{b}
\end{aligned}
$$

- When the denominator of a radical expression is a two-term expression, rationalize the denominator by multiplying the numerator and denominator by the conjugate, and then simplify.

$$
\begin{aligned}
\frac{1}{\sqrt{a} - \sqrt{b}} &= \frac{1}{\sqrt{a} - \sqrt{b}} \times \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} \\
&= \frac{\sqrt{a} + \sqrt{b}}{a - b}
\end{aligned}
$$

Need to Know

- When you simplify a radical expression such as $\frac{\sqrt{3}}{5\sqrt{2}}$, multiply the numerator and denominator by the radical only.

$$
\begin{aligned}
\frac{\sqrt{3}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} &= \frac{\sqrt{6}}{5(2)} \\
&= \frac{\sqrt{6}}{10}
\end{aligned}
$$

- $\sqrt{a} + \sqrt{b}$ is the conjugate $\sqrt{a} - \sqrt{b}$, and vice versa.

Exercise 1.1

PART A

1. Write the conjugate of each radical expression.

a. $2\sqrt{3} - 4$
b. $\sqrt{3} + \sqrt{2}$
c. $-2\sqrt{3} - \sqrt{2}$
d. $3\sqrt{3} + \sqrt{2}$
e. $\sqrt{2} - \sqrt{5}$
f. $-\sqrt{5} + 2\sqrt{2}$

2. Rationalize the denominator of each expression. Write your answer in simplest form.

a. $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{2}}$
b. $\dfrac{2\sqrt{3} - 3\sqrt{2}}{\sqrt{2}}$
c. $\dfrac{4\sqrt{3} + 3\sqrt{2}}{2\sqrt{3}}$
d. $\dfrac{3\sqrt{5} - \sqrt{2}}{2\sqrt{2}}$

PART B

K 3. Rationalize each denominator.

a. $\dfrac{3}{\sqrt{5} - \sqrt{2}}$
b. $\dfrac{2\sqrt{5}}{2\sqrt{5} + 3\sqrt{2}}$
c. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$
d. $\dfrac{2\sqrt{5} - 8}{2\sqrt{5} + 3}$
e. $\dfrac{2\sqrt{3} - \sqrt{2}}{5\sqrt{2} + \sqrt{3}}$
f. $\dfrac{3\sqrt{3} - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2}}$

4. Rationalize each numerator.

a. $\dfrac{\sqrt{5} - 1}{4}$
b. $\dfrac{2 - 3\sqrt{2}}{2}$
c. $\dfrac{\sqrt{5} + 2}{2\sqrt{5} - 1}$

C 5. a. Rationalize the denominator of $\dfrac{8\sqrt{2}}{\sqrt{20} - \sqrt{18}}$.

b. Rationalize the denominator of $\dfrac{8\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$.

c. Why are your answers in parts a and b the same? Explain.

6. Rationalize each denominator.

a. $\dfrac{2\sqrt{2}}{2\sqrt{3} - \sqrt{8}}$
b. $\dfrac{2\sqrt{6}}{2\sqrt{27} - \sqrt{8}}$
c. $\dfrac{2\sqrt{2}}{\sqrt{16} - \sqrt{12}}$
d. $\dfrac{3\sqrt{2} + 2\sqrt{3}}{\sqrt{12} - \sqrt{8}}$
e. $\dfrac{3\sqrt{5}}{4\sqrt{3} - 5\sqrt{2}}$
f. $\dfrac{\sqrt{18} + \sqrt{12}}{\sqrt{18} - \sqrt{12}}$

A 7. Rationalize the numerator of each of the following expressions:

a. $\dfrac{\sqrt{a} - 2}{a - 4}$
b. $\dfrac{\sqrt{x + 4} - 2}{x}$
c. $\dfrac{\sqrt{x + h} - x}{x}$