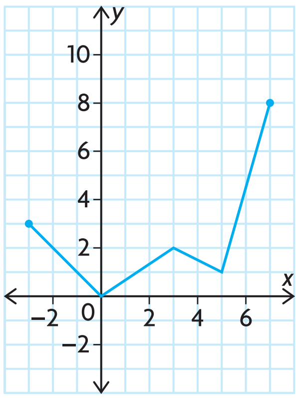
Section 1.6—Continuity

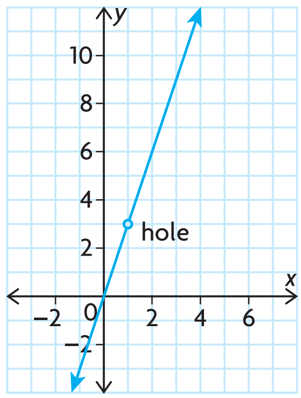
The idea of continuity may be thought of informally as the idea of being able to draw a graph without lifting one's pencil. The concept arose from the notion of a graph "without breaks or jumps or gaps."

When we talk about a function being continuous at a point, we mean that the graph passes through the point without a break. A graph that is not continuous at a point (sometimes referred to as being discontinuous at a point) has a break of some type at the point. The following graphs illustrate these ideas:

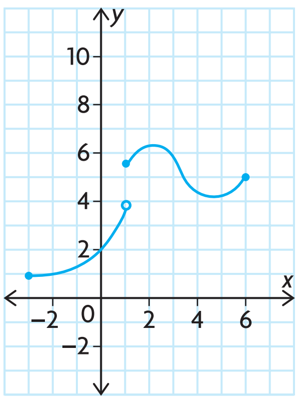
A. Continuous for all values of the domain



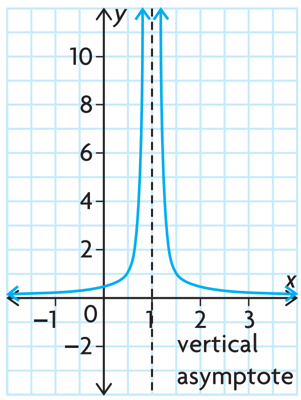
B. Discontinuous at $x = 1$ (point discontinuity)



C. Discontinuous at $x = 1$ (jump discontinuity)



D. Discontinuous at $x = 1$ (infinite discontinuity)



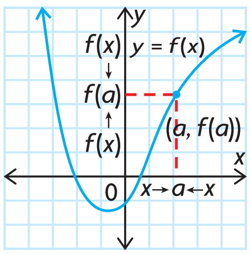
What conditions must be satisfied for a function f to be continuous at a? First, $f(a)$ must be defined. The curves in figure B and figure D above are not continuous at $x = 1$ because they are not defined at $x = 1$.

A second condition for continuity at a point $x = a$ is that the function makes no jumps there. This means that, if "x is close to a," then $f(x)$ must be close to $f(a)$. This condition is satisfied if $\lim_{x \to a} f(x) = f(a)$. Looking at the graph in figure C, on the previous page, we see that $\lim_{x \to 1} f(x)$ does not exist, and the function is therefore not continuous at $x = 1$.

We can now define the continuity of a function at a point.

Continuity at a Point

The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.



Otherwise, $f(x)$ is discontinuous at $x = a$.

The geometrical meaning of f being continuous at $x = a$ can be stated as follows: As $x \to a$, the points $(x, f(x))$ on the graph of f converge at the point $(a, f(a))$, ensuring that the graph of f is unbroken at $(a, f(a))$.

EXAMPLE 1 Reasoning about continuity at a point

a. Graph the following function:

$$f(x) = \begin{cases} x^2 - 3, \text{ if } x \le -1 \\ x - 1, \text{ if } x > -1 \end{cases}$$

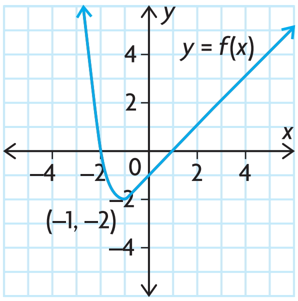
b. Determine $\lim_{x \to -1} f(x)$

c. Determine $f(-1)$.

d. Is f continuous at $x = -1$? Explain.

Solution

a.



b. From the graph, $\lim_{x \to -1} f(x) = -2$. *Note:* Both the left-hand and right-hand limits are equal.

c. $f(-1) = -2$

d. Therefore, $f(x)$ is continuous at $x = -1$, since $f(-1) = \lim_{x \to -1} f(x)$.

EXAMPLE 2 Reasoning whether a function is continuous or discontinuous at a point

Test the continuity of each of the following functions at $x = 2$. If a function is not continuous at $x = 2$, give a reason why it is not continuous.

a. $f(x) = x^3 - x$

b. $g(x) = \dfrac{x^2 - x - 2}{x - 2}$

c. $h(x) = \dfrac{x^2 - x - 2}{x - 2}$, if $x \neq 2$ and $h(2) = 3$

d. $F(x) = \dfrac{1}{(x - 2)^2}$

e. $G(x) = \begin{cases} 4 - x^2, \text{ if } x < 2 \\ 3, \text{if } x \geq 2 \end{cases}$

Solution

a. The function f is continuous at $x = 2$ since $f(2) = 6 = \lim_{x \to 2} f(x)$. (Polynomial functions are continuous at all real values of x.)

b. The function g is not continuous at $x = 2$ because g is not defined at this value.

c.
$$\begin{aligned}\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} &= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{(x - 2)} \\ &= \lim_{x \to 2} (x + 1) \\ &= 3 \\ &= h(2)\end{aligned}$$

Therefore, $h(x)$ is continuous at $x = 2$.

d. The function F is not continuous at $x = 2$ because $F(2)$ is not defined.

e. $\lim_{x \to 2^-} G(x) = \lim_{x \to 2^-} (4 - x^2) = 0$ and $\lim_{x \to 2^+} G(x) = \lim_{x \to 2^+} (3) = 3$

Therefore, since $\lim_{x \to 2} G(x)$ does not exist, the function is not continuous at $x = 2$.

INVESTIGATION

To test the definition of continuity by graphing, investigate the following:

A. Draw the graph of each function in Example 2.

B. Which of the graphs are continuous, contain a hole or a jump, or have a vertical asymptote?

C. Given only the defining rule of a function $y = f(x)$, such as $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$, explain why the graphing technique to test for continuity on an interval may be less suitable.

D. Determine where $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$ is not defined and where it is continuous.

IN SUMMARY

Key Ideas

- A function f is continuous at $x = a$ if
  - $f(a)$ is defined
  - $\lim_{x \to a} f(x)$ exists
  - $\lim_{x \to a} f(x) = f(a)$
- A function that is not continuous has some type of break in its graph. This break is the result of a hole, jump, or vertical asymptote.

Need to Know

- All polynomial functions are continuous for all real numbers.
- A rational function $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.
- A rational function in simplified form has a discontinuity at the zeros of the denominator.
- When the one-sided limits are not equal to each other, then the limit at this point does not exist and the function is not continuous at this point.

Exercise 1.6

PART A

C 1. How can looking at a graph of a function help you tell where the function is continuous?

2. What does it mean for a function to be continuous over a given domain?

3. What are the basic types of discontinuity? Give an example of each.

4. Find the value(s) of x at which each function is discontinuous.

a. $f(x) = \dfrac{9 - x^2}{x - 3}$ c. $h(x) = \dfrac{x^2 + 1}{x^3}$ e. $g(x) = \dfrac{13x}{x^2 + x - 6}$

b. $g(x) = \dfrac{7x - 4}{x}$ d. $f(x) = \dfrac{x - 4}{x^2 - 9}$ f. $h(x) = \begin{cases} -x, \text{ if } x \leq 3 \\ 1 - x, \text{ if } x > 3 \end{cases}$

PART B

K 5. Determine all the values of x for which each function is continuous.

a. $f(x) = 3x^5 + 2x^3 - x$ c. $h(x) = \dfrac{x^2 + 16}{x^2 - 5x}$ e. $g(x) = 10^x$

b. $g(x) = \pi x^2 - 4.2x + 7$ d. $f(x) = \sqrt{x + 2}$ f. $h(x) = \dfrac{16}{x^2 + 65}$

6. Examine the continuity of $g(x) = x + 3$ when $x = 2$.

7. Sketch a graph of the following function:

$$h(x) = \begin{cases} x - 1, \text{ if } x < 3 \\ 5 - x, \text{ if } x \geq 3 \end{cases}$$

Determine if the function is continuous everywhere.

8. Sketch a graph of the following function:

$$f(x) = \begin{cases} x^2, \text{ if } x < 0 \\ 3, \text{ if } x \geq 0 \end{cases}$$

Is the function continuous?

A 9. Recent postal rates for non-standard and oversized letter mail within Canada are given in the following table. Maximum dimensions for this type of letter mail are 380 mm by 270 mm by 20 mm.

100 g or Less	Between 100 g and 200 g	Between 200 g and 500 g
\$1.10	\$1.86	\$2.55

Draw a graph of the cost, in dollars, to mail a non-standard envelope as a function of its mass in grams. Where are the discontinuities of this function?

10. Determine whether $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ is continuous at $x = 3$.

11. Examine the continuity of the following function:

$$f(x) = \begin{cases} x, \text{ if } x \leq 1 \\ 1, \text{ if } 1 < x \leq 2 \\ 3, \text{ if } x > 2 \end{cases}$$

12. $g(x) = \begin{cases} x + 3, \text{ if } x \neq 3 \\ 2 + \sqrt{k}, \text{ if } x = 3 \end{cases}$

Find k, if $g(x)$ is continuous.

13. The signum function is defined as follows:

$$f(x) = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$$

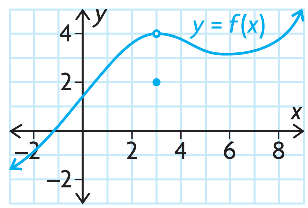
a. Sketch the graph of the signum function.

b. Find each limit, if it exists.

i. $\lim\limits_{x \to 0^-} f(x)$ ii. $\lim\limits_{x \to 0^+} f(x)$ iii. $\lim\limits_{x \to 0} f(x)$

c. Is $f(x)$ continuous? Explain.

14. Examine the graph of $f(x)$.



a. Find $f(3)$.

b. Evaluate $\lim\limits_{x \to 3^-} f(x)$.

c. Is $f(x)$ continuous on the interval $-3 < x < 8$? Explain.

15. What must be true about A and B for the function

$$f(x) = \begin{cases} \dfrac{Ax - B}{x - 2}, \text{ if } x \leq 1 \\ 3x, \text{ if } 1 < x < 2 \\ Bx^2 - A, \text{ if } x \geq 2 \end{cases}$$

if the function is continuous at $x = 1$ but discontinuous at $x = 2$?

PART C

16. Find constants a and b, such that the function

$$f(x) = \begin{cases} -x, \text{ if } -3 \leq x \leq -2 \\ ax^2 + b, \text{ if } -2 < x < 0 \\ 6, \text{ if } x = 0 \end{cases}$$

is continuous for $-3 \leq x \leq 0$.

17. Consider the following function:

$$g(x) = \begin{cases} \dfrac{x|x - 1|}{x - 1}, \text{ if } x \neq 1 \\ 0, \text{ if } x = 1 \end{cases}$$

a. Evaluate $\lim\limits_{x \to 1^+} g(x)$ and $\lim\limits_{x \to 1^-} g(x)$, and then determine whether $\lim\limits_{x \to 1} g(x)$ exists.

b. Sketch the graph of $g(x)$, and identify any points of discontinuity.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

An Olympic coach has developed a 6 min fitness test for her team members that sets target values for heart rates. The monitor they have available counts the total number of heartbeats, starting from a rest position at "time zero." The results for one of the team members are given in the table below.

Time (min)	Number of Heartbeats
0.0	0
1.0	55
2.0	120
3.0	195
4.0	280
5.0	375
6.0	480

a. The coach has established that each athlete's heart rate must not exceed 100 beats per minute at exactly 3 min. Using a graphical technique, determine if this athlete meets the coach's criterion.

b. The coach needs to know the instant in time when an athlete's heart rate actually exceeds 100 beats per minute. Explain how you would solve this problem graphically. Is a graphical solution an efficient method? Explain. How is this problem different from part a?

c. Build a mathematical model with the total number of heartbeats as a function of time ($n = f(t)$). First determine the degree of the polynomial, and then use a graphing calculator to obtain an algebraic model.

d. Solve part b algebraically by obtaining an expression for the instantaneous rate of change in the number of heartbeats (heart rate) as a function of time ($r = g(t)$) using the methods presented in the chapter. Compare the accuracy and efficiency of solving this problem graphically and algebraically.