Chapter 2

DERIVATIVES

Imagine a driver speeding down a highway, at 140 km/h. He hears a police siren and is quickly pulled over. The police officer tells him that he was speeding, but the driver argues that because he has travelled 200 km from home in two hours, his average speed is within the 100 km/h limit. The driver's argument fails because police officers charge speeders based on their instantaneous speed, not their average speed.

There are many other situations in which the instantaneous rate of change is more important than the average rate of change. In calculus, the derivative is a tool for finding instantaneous rates of change. This chapter shows how the derivative can be determined and applied in a great variety of situations.

CHAPTER EXPECTATIONS

In this chapter, you will

- understand and determine derivatives of polynomial and simple rational functions from first principles, **Section 2.1**
- identify examples of functions that are not differentiable, **Section 2.1**
- justify and use the rules for determining derivatives, **Sections 2.2, 2.3, 2.4, 2.5**
- identify composition as two functions applied in succession, **Section 2.5**
- determine the composition of two functions expressed in notation, and decompose a given composite function into its parts, **Section 2.5**
- use the derivative to solve problems involving instantaneous rates of change, **Sections 2.2, 2.3, 2.4, 2.5**

Review of Prerequisite Skills

Before beginning your study of derivatives, it may be helpful to review the following concepts from previous courses and the previous chapter:

- Working with the properties of exponents
- Simplifying radical expressions
- Finding the slopes of parallel and perpendicular lines
- Simplifying rational expressions
- Expanding and factoring algebraic expressions
- Evaluating expressions
- Working with the difference quotient

Exercise

1. Use the exponent laws to simplify each of the following expressions. Express your answers with positive exponents.

a. $a^5 \times a^3$

b. $(-2a^2)^3$

c. $\frac{4p^7 \times 6p^9}{12p^{15}}$

d. $(a^4b^{-5})(a^{-6}b^{-2})$

e. $(3e^6)(2e^3)^4$

f. $\frac{(3a^{-4})[2a^3(-b)^3]}{12a^5b^2}$

2. Simplify and write each expression in exponential form.

a. $(x^{\frac{1}{2}})(x^{\frac{2}{3}})$

b. $(8x^6)^{\frac{2}{3}}$

c. $\frac{\sqrt{a}\sqrt[3]{a}}{\sqrt{a}}$

3. Determine the slope of a line that is perpendicular to a line with each given slope.

a. $\frac{2}{3}$

b. $-\frac{1}{2}$

c. $\frac{5}{3}$

d. -1

4. Determine the equation of each of the following lines:

a. passing through points $A(-3, -4)$ and $B(9, -2)$

b. passing through point $A(-2, -5)$ and parallel to the line $3x - 2y = 5$

c. perpendicular to the line $y = \frac{3}{4}x - 6$ and passing through point $A(4, -3)$

5. Expand, and collect like terms.

a. $(x - 3y)(2x + y)$
b. $(x - 2)(x^2 - 3x + 4)$
c. $(6x - 3)(2x + 7)$
d. $2(x + y) - 5(3x - 8y)$
e. $(2x - 3y)^2 + (5x + y)^2$
f. $3x(2x - y)^2 - x(5x - y)(5x + y)$

6. Simplify each expression.

a. $\frac{3x(x + 2)}{x^2} \times \frac{5x^3}{2x(x + 2)}$
b. $\frac{y}{(y + 2)(y - 5)} \times \frac{(y - 5)^2}{4y^3}$
c. $\frac{4}{(h + k)} \div \frac{9}{2(h + k)}$
d. $\frac{(x + y)(x - y)}{5(x - y)} \div \frac{(x + y)^3}{10}$
e. $\frac{x - 7}{2x} + \frac{5x}{x - 1}$
f. $\frac{x + 1}{x - 2} - \frac{x + 2}{x + 3}$

7. Factor each expression completely.

a. $4k^2 - 9$
b. $x^2 + 4x - 32$
c. $3a^2 - 4a - 7$
d. $x^4 - 1$
e. $x^3 - y^3$
f. $r^4 - 5r^2 + 4$

8. Use the factor theorem to factor the following expressions:

a. $a^3 - b^3$
b. $a^5 - b^5$
c. $a^7 - b^7$
d. $a^n - b^n$

9. If $f(x) = -2x^4 + 3x^2 + 7 - 2x$, evaluate

a. $f(2)$
b. $f(-1)$
c. $f\left(\frac{1}{2}\right)$
d. $f(-0.25)$

10. Rationalize the denominator in each of the following expressions:

a. $\frac{3}{\sqrt{2}}$
b. $\frac{4 - \sqrt{2}}{\sqrt{3}}$
c. $\frac{2 + 3\sqrt{2}}{3 - 4\sqrt{2}}$
d. $\frac{3\sqrt{2} - 4\sqrt{3}}{3\sqrt{2} + 4\sqrt{3}}$

11. a. If $f(x) = 3x^2 - 2x$, determine the expression for the difference quotient $\frac{f(a + h) - f(a)}{h}$ when $a = 2$. Explain what this expression can be used for.

b. Evaluate the expression you found in part a. for a small value of h where $h = 0.01$.

c. Explain what the value you determined in part b. represents.

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CHAPTER 2: THE ELASTICITY OF DEMAND

Have you ever wondered how businesses set prices for their goods and services? An important idea in marketing is *elasticity of demand*, or the response of consumers to a change in price. Consumers respond differently to a change in the price of a staple item, such as bread, than they do to a change in the price of a luxury item, such as jewellery. A family would probably still buy the same quantity of bread if the price increased by 20%. This is called *inelastic* demand. If the price of a gold chain, however, increased by 20%, sales would likely decrease 40% or more. This is called *elastic* demand. Mathematically, elasticity is defined as the negative of the relative (percent) change in the number demanded $\left(\frac{\Delta n}{n}\right)$ divided by the relative (percent) change in the price $\left(\frac{\Delta p}{p}\right)$:

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

For example, if a store increased the price of a CD from $17.99 to $19.99, and the number sold per week went from 120 to 80, the elasticity would be

$$E = -\left[\left(\frac{80 - 120}{120}\right) \div \left(\frac{19.99 - 17.99}{17.99}\right)\right] \doteq 3.00$$

An elasticity of about 3 means that the change in demand is three times as large, in percent terms, as the change in price. The CDs have an elastic demand because a small change in price can cause a large change in demand. In general, goods or services with elasticities greater than one ($E > 1$) are considered elastic (e.g., new cars), and those with elasticities less than one ($E < 1$) are considered inelastic (e.g., milk). In our example, we calculated the average elasticity between two price levels, but, in reality, businesses want to know the elasticity at a specific, or *instantaneous*, price level. In this chapter, you will develop the rules of differentiation that will enable you to calculate the instantaneous rate of change for several classes of functions.

Case Study—Marketer: Product Pricing

In addition to developing advertising strategies, marketing departments also conduct research into, and make decisions on, pricing. Suppose that the demand–price relationship for weekly movie rentals at a convenience store is $n(p) = \frac{500}{p}$, where $n(p)$ is demand and p is price.

DISCUSSION QUESTIONS

1. Generate two lists, each with at least five goods and services that you have purchased recently, classifying each of the goods and services as having elastic or inelastic demand.
2. Calculate and discuss the elasticity if a movie rental fee increases from $1.99 to $2.99.