Section 2.1—The Derivative Function

In this chapter, we will extend the concepts of the slope of a tangent and the rate of change to introduce the **derivative**. We will examine the methods of differentiation, which we can use to determine the derivatives of polynomial and rational functions. These methods include the use of the power rule, sum and difference rules, and product and quotient rules, as well as the chain rule for the composition of functions.

The Derivative at a Point

In the previous chapter, we encountered limits of the form $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$. This limit has two interpretations: the slope of the tangent to the graph $y = f(x)$ at the point $(a, f(a))$, and the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$. Since this limit plays a central role in calculus, it is given a name and a concise notation. It is called the **derivative of $f(x)$ at $x = a$**. It is denoted by $f'(a)$ and is read as "f prime of a."

The **derivative of f at the number a** is given by $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, provided that this limit exists.

EXAMPLE 1 **Selecting a limit strategy to determine the derivative at a number**

Determine the derivative of $f(x) = x^2$ at $x = -3$.

Solution

Using the definition, the derivative at $x = -3$ is given by

$$
\begin{aligned}
f'(-3) &= \lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h} \\
&= \lim_{h \to 0} \frac{(-3 + h)^2 - (-3)^2}{h} && \text{(Expand)} \\
&= \lim_{h \to 0} \frac{9 - 6h + h^2 - 9}{h} && \text{(Simplify and factor)} \\
&= \lim_{h \to 0} \frac{h(-6 + h)}{h} \\
&= \lim_{h \to 0} (-6 + h) \\
&= -6
\end{aligned}
$$

Therefore, the derivative of $f(x) = x^2$ at $x = -3$ is -6.

An alternative way of writing the derivative of f at the number a is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

In applications where we are required to find the value of the derivative for a number of particular values of x, using the definition repeatedly for each value is tedious.

The next example illustrates the efficiency of calculating the derivative of $f(x)$ at an arbitrary value of x and using the result to determine the derivatives at a number of particular x-values.

EXAMPLE 2

Connecting the derivative of a function to an arbitrary value

a. Determine the derivative of $f(x) = x^2$ at an arbitrary value of x.
b. Determine the slopes of the tangents to the parabola $y = x^2$ at $x = -2$, 0, and 1.

Solution

a. Using the definition,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} && \text{(Expand)} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} && \text{(Simplify and factor)} \\ &= \lim_{h \to 0} \frac{h(2x + h)}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

The derivative of $f(x) = x^2$ at an arbitrary value of x is $f'(x) = 2x$.

b. The required slopes of the tangents to $y = x^2$ are obtained by evaluating the derivative $f'(x) = 2x$ at the given x-values. We obtain the slopes by substituting for x:

$f'(-2) = -4$ $\qquad$ $f'(0) = 0$ $\qquad$ $f'(1) = 2$

The slopes are -4, 0, and 2, respectively.

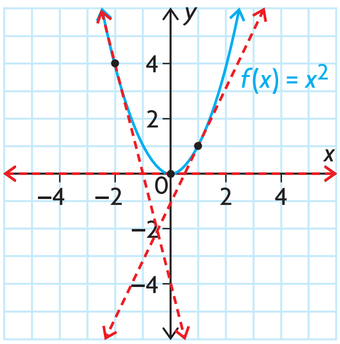
In fact, knowing the x-coordinate of a point on the parabola $y = x^2$, we can easily find the slope of the tangent at that point. For example, given the x-coordinates of points on the curve, we can produce the following table.

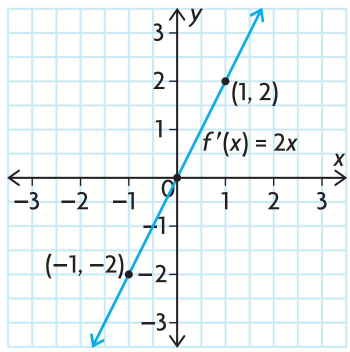
The slope of the tangent to the curve $f(x) = x^2$ at a point $P(x, y)$ is given by the derivative $f'(x) = 2x$. For each x-value, there is an associated value $2x$.

For the Parabola $f(x) = x^2$

$P(x, y)$	x-Coordinate of P	Slope of Tangent at P
$(-2, 4)$	-2	$2(-2) = -4$
$(-1, 1)$	-1	-2
$(0, 0)$	0	0
$(1, 1)$	1	2
$(2, 4)$	2	4
(a, a^2)	a	$2a$

The graphs of $f(x) = x^2$ and the derivative function $f'(x) = 2x$ are shown below. The tangents at $x = -2$, 0, and 1 are shown on the graph of $f(x) = x^2$.





Notice that the graph of the derivative function of the quadratic function (of degree two) is a linear function (of degree one).

INVESTIGATION

A. Determine the derivative with respect to x of each of the following functions:

a. $f(x) = x^3$ b. $f(x) = x^4$ c. $f(x) = x^5$

B. In Example 2, we showed that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Referring to step 1, what pattern do you see developing?

C. Use the pattern from step 2 to predict the derivative of $f(x) = x^{39}$.

D. What do you think $f'(x)$ would be for $f(x) = x^n$, where n is a positive integer?

The Derivative Function

The derivative of f at $x = a$ is a number $f'(a)$. If we let a be arbitrary and assume a general value in the domain of f, the derivative f' is a function. For example, if $f(x) = x^2$, $f'(x) = 2x$, which is itself a function.

The Definition of the Derivative Function

The derivative of $f(x)$ with respect to x is the function $f'(x)$, where

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that this limit exists.

The $f'(x)$ notation for this limit was developed by Joseph Louis Lagrange (1736–1813), a French mathematician. When you use this limit to determine the derivative of a function, it is called determining the derivative from first principles.

In Chapter 1, we discussed velocity at a point. We can now define (instantaneous) velocity as the derivative of position with respect to time. If the position of a body at time t is $s(t)$, then the velocity of the body at time t is

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}.$$

Likewise, the (instantaneous) rate of change of $f(x)$ with respect to x is the function $f'(x)$, whose value is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

EXAMPLE 3 Determining the derivative from first principles

Determine the derivative $f'(t)$ of the function $f(t) = \sqrt{t}$, $t \geq 0$.

Solution

Using the definition,

$$\begin{aligned}
f'(t) &= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h}\left(\frac{\sqrt{t + h} + \sqrt{t}}{\sqrt{t + h} + \sqrt{t}}\right) \quad \text{(Rationalize the numerator)} \\
&= \lim_{h \to 0} \frac{(t + h) - t}{h(\sqrt{t + h} + \sqrt{t})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{t + h} + \sqrt{t})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{t + h} + \sqrt{t}} \\
&= \frac{1}{2\sqrt{t}}, \text{ for } t > 0
\end{aligned}$$

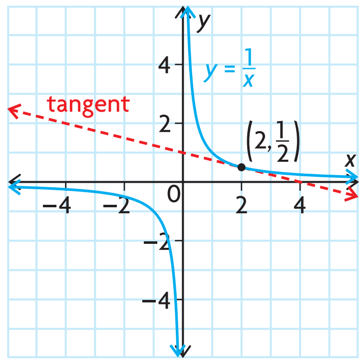
Note that $f(t) = \sqrt{t}$ is defined for all instances of $t \geq 0$, whereas its derivative $f'(t) = \frac{1}{2\sqrt{t}}$ is defined only for instances when $t > 0$. From this, we can see that a function need not have a derivative throughout its entire domain.

EXAMPLE 4 Selecting a strategy involving the derivative to determine the equation of a tangent

Determine an equation of the tangent to the graph of $f(x) = \frac{1}{x}$ at the point where $x = 2$.

Solution

When $x = 2$, $y = \frac{1}{2}$. The graph of $y = \frac{1}{x}$, the point $\left(2, \frac{1}{2}\right)$, and the tangent at the point are shown.



First find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

(Simplify the fraction)

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2}$$

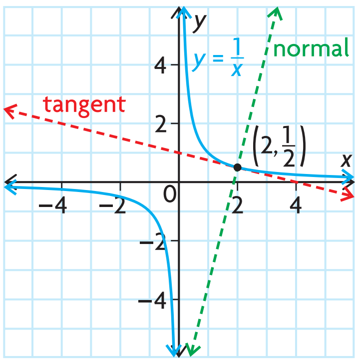
The slope of the tangent at $x = 2$ is $m = f'(2) = -\frac{1}{4}$. The equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$ or, in standard form, $x + 4y - 4 = 0$.

EXAMPLE 5 Selecting a strategy involving the derivative to solve a problem

Determine an equation of the line that is perpendicular to the tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$ and that intersects it at the point of tangency.

Solution

In Example 4, we found that the slope of the tangent at $x = 2$ is $f'(2) = -\frac{1}{4}$, and the point of tangency is $\left(2, \frac{1}{2}\right)$. The perpendicular line has slope 4, the negative reciprocal of $-\frac{1}{4}$. Therefore, the required equation is $y - \frac{1}{2} = 4(x - 2)$, or $8x - 2y - 15 = 0$.

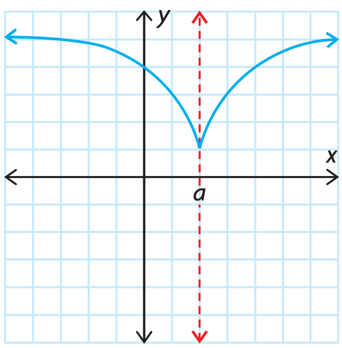


The line whose equation we found in Example 5 has a name.

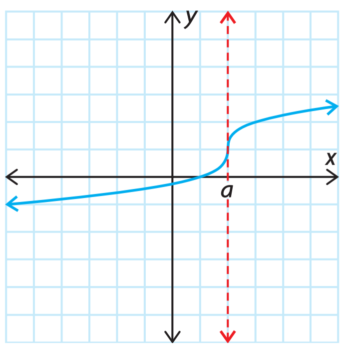
The **normal** to the graph of f at point P is the line that is perpendicular to the tangent at P.

The Existence of Derivatives

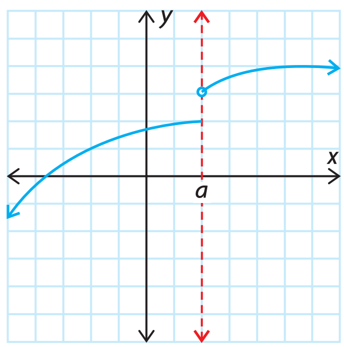
A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



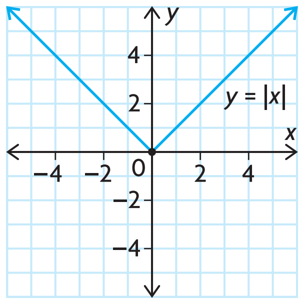
Discontinuity

EXAMPLE 6 Reasoning about differentiability at a point

Show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

The graph of $f(x) = |x|$ is shown. Because the slope for $x < 0$ is -1, whereas the slope for $x > 0$ is $+1$, the graph has a "corner" at $(0, 0)$, which prevents a unique tangent from being drawn there. We can show this using the definition of a derivative.



$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

Now, we will consider one-sided limits.

$|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$.

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$$

Since the left-hand limit and the right-hand limit are not the same, the derivative does not exist at $x = 0$.

From Example 6, we conclude that it is possible for a function to be **continuous** at a point and yet *not differentiable* at this point. However, if a function is differentiable at a point, then it is also continuous at this point.

Other Notation for Derivatives

Symbols other than $f'(x)$ are often used to denote the derivative. If $y = f(x)$, the symbols y' and $\frac{dy}{dx}$ are used instead of $f'(x)$. The notation $\frac{dy}{dx}$ was originally used by Leibniz and is read "dee y by dee x." For example, if $y = x^2$, the derivative is $y' = 2x$ or, in Leibniz notation, $\frac{dy}{dx} = 2x$. Similarly, in Example 4, we showed that if $y = \frac{1}{x}$, then $\frac{dy}{dx} = -\frac{1}{x^2}$. The Leibniz notation reminds us of the process by which the derivative is obtained—namely, as the limit of a difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

By omitting y and f altogether, we can combine these notations and write $\frac{d}{dx}(x^2) = 2x$, which is read "the derivative of x^2 with respect to x is $2x$." It is important to note that $\frac{dy}{dx}$ is *not a fraction*.

IN SUMMARY

Key Ideas

- The derivative of a function f at a point $(a, f(a))$ is $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if the limit exists.
- A function is said to be **differentiable** at a if $f'(a)$ exists. A function is differentiable on an interval if it is differentiable at every number in the interval.
- The derivative function for any function $f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, if the limit exists.

Need to Know

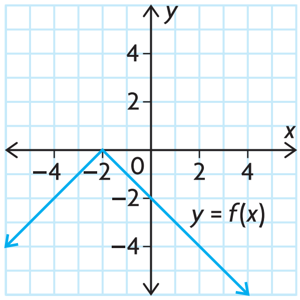
- To find the derivative at a point $x = a$, you can use $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.
- The derivative $f'(a)$ can be interpreted as either
  – the slope of the tangent at $(a, f(a))$, or
  – the instantaneous rate of change of $f(x)$ with respect to x when $x = a$.
- Other notations for the derivative of the function $y = f(x)$ are $f'(x)$, y', and $\frac{dy}{dx}$.
- The normal to the graph of a function at point P, is a line that is perpendicular to the tangent line that passes through point P.

Exercise 2.1

PART A

1. State the domain on which f is differentiable.

a.



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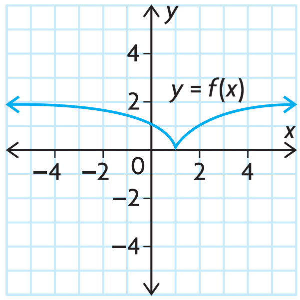

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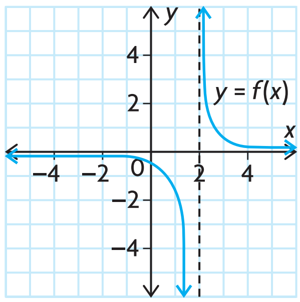

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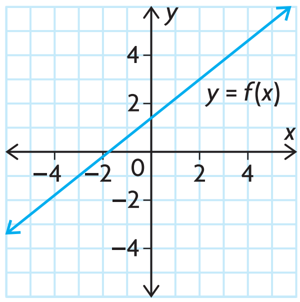

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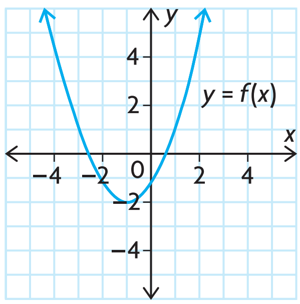

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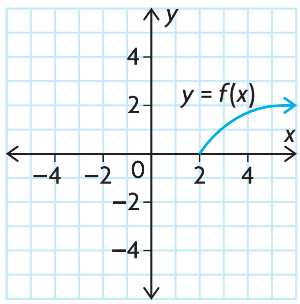


C 2. Explain what the derivative of a function represents.

3. Illustrate two situations in which a function does not have a derivative at $x = 1$.

4. For each function, find $f(a + h)$ and $f(a + h) - f(a)$.

a. $f(x) = 5x - 2$

b. $f(x) = x^2 + 3x - 1$

c. $f(x) = x^3 - 4x + 1$

d. $f(x) = x^2 + x - 6$

e. $f(x) = -7x + 4$

f. $f(x) = 4 - 2x - x^2$

PART B

K 5. For each function, find the value of the derivative $f'(a)$ for the given value of a.

a. $f(x) = x^2, a = 1$

b. $f(x) = x^2 + 3x + 1, a = 3$

c. $f(x) = \sqrt{x + 1}, a = 0$

d. $f(x) = \frac{5}{x}, a = -1$

6. Use the definition of the derivative to find $f'(x)$ for each function.

a. $f(x) = -5x - 8$

b. $f(x) = 2x^2 + 4x$

c. $f(x) = 6x^3 - 7x$

d. $f(x) = \sqrt{3x + 2}$

7. In each case, find the derivative $\frac{dy}{dx}$ from first principles.

a. $y = 6 - 7x$

b. $y = \frac{x + 1}{x - 1}$

c. $y = 3x^2$

8. Determine the slope of the tangents to $y = 2x^2 - 4x$ when $x = 0, x = 1$, and $x = 2$. Sketch the graph, showing these tangents.

9. a. Sketch the graph of $f(x) = x^3$.

b. Calculate the slopes of the tangents to $f(x) = x^3$ at points with x-coordinates $-2, -1, 0, 1, 2$.

c. Sketch the graph of the derivative function $f'(x)$.

d. Compare the graphs of $f(x)$ and $f'(x)$.

A 10. An object moves in a straight line with its position at time t seconds given by $s(t) = -t^2 + 8t$, where s is measured in metres. Find the velocity when $t = 0, t = 4$, and $t = 6$.

11. Determine an equation of the line that is tangent to the graph of $f(x) = \sqrt{x + 1}$ and parallel to $x - 6y + 4 = 0$.

12. For each function, use the definition of the derivative to determine $\frac{dy}{dx}$, where $a, b, c,$ and m are constants.

a. $y = c$

b. $y = x$

c. $y = mx + b$

d. $y = ax^2 + bx + c$

13. Does the function $f(x) = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

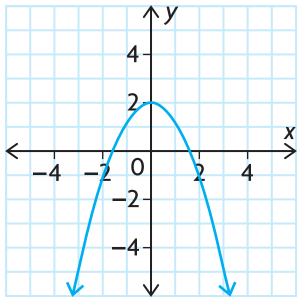
14. A football is kicked up into the air. Its height, h, above the ground, in metres, at t seconds can be modelled by $h(t) = 18t - 4.9t^2$.

a. Determine $h'(2)$.

b. What does $h'(2)$ represent?

15. Match each function in graphs **a**, **b**, and **c** with its corresponding derivative, graphed in **d**, **e**, and **f**.

a.



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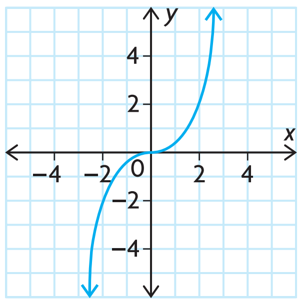

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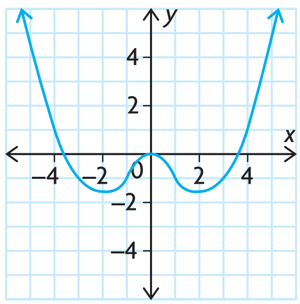

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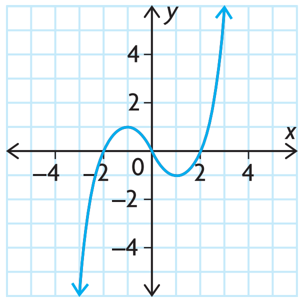

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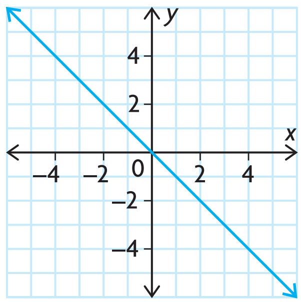

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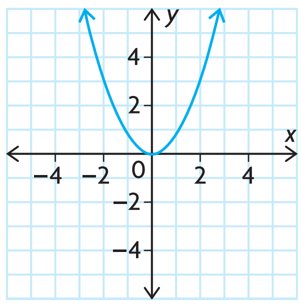

PART C

16. For the function $f(x) = x|x|$, show that $f'(0)$ exists. What is the value?

17. If $f(a) = 0$ and $f'(a) = 6$, find $\lim_{h \to 0} \frac{f(a + h)}{2h}$.

18. Give an example of a function that is continuous on $-\infty < x < \infty$ but is not differentiable at $x = 3$.

19. At what point on the graph of $y = x^2 - 4x - 5$ is the tangent parallel to $2x - y = 1$?

20. Determine the equations of both lines that are tangent to the graph of $f(x) = x^2$ and pass through point $(1, -3)$.