Section 2.4—The Quotient Rule

In the previous section, we found that the derivative of the product of two functions is not the product of their derivatives. The quotient rule gives the derivative of a function that is the quotient of two functions. It is derived from the product rule.

The Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

Proof:

We want to find $h'(x)$, given that $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

We rewrite this as a product: $h(x)g(x) = f(x)$.

Using the product rule, $h'(x)g(x) + h(x)g'(x) = f'(x)$.

Solving for $h'(x)$, we get
$$h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$$
$$= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

The quotient rule provides us with an alternative approach to differentiate rational functions, in addition to what we learned last section.

Memory Aid for the Product and Quotient Rules

It is worth noting that the quotient rule is similar to the product rule in that both have $f'(x)g(x)$ and $f(x)g'(x)$. For the product rule, we put an addition sign between the terms. For the quotient rule, we put a subtraction sign between the terms and then divide the result by the square of the original denominator.

Take note that in the quotient rule the $f'(x)g(x)$ term must come first. This isn't the case with the product rule.

EXAMPLE 1

Applying the quotient rule

Determine the derivative of $h(x) = \frac{3x - 4}{x^2 + 5}$.

Solution

Since $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = 3x - 4$ and $g(x) = x^2 + 5$, use the quotient rule to find $h'(x)$.

Using the quotient rule, we get
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$
$$= \frac{(3)(x^2 + 5) - (3x - 4)(2x)}{(x^2 + 5)^2}$$
$$= \frac{3x^2 + 15 - 6x^2 + 8x}{(x^2 + 5)^2}$$
$$= \frac{-3x^2 + 8x + 15}{(x^2 + 5)^2}$$

EXAMPLE 2

Selecting a strategy to determine the equation of a line tangent to a rational function

Determine the equation of the tangent to $y = \frac{2x}{x^2 + 1}$ at $x = 0$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $\frac{dy}{dx}$.

By the quotient rule,
$$\frac{dy}{dx} = \frac{(2)(x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2}$$

At $x = 0$,
$$\frac{dy}{dx} = \frac{(2)(0 + 1) - (0)(0)}{(0 + 1)^2} = 2$$

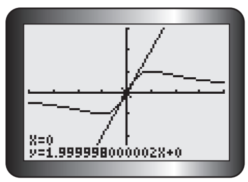
The slope of the tangent at $x = 0$ is 2 and the point of tangency is $(0, 0)$. The equation of the tangent is $y = 2x$.

Tech Support

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 0$. The calculator displays the equation of the tangent line.

The equation of the tangent line in this case is $y = 2x$.

EXAMPLE 3 Using the quotient rule to solve a problem

Determine the coordinates of each point on the graph of $f(x) = \dfrac{2x + 8}{\sqrt{x}}$ where the tangent is horizontal.

Solution

The slope of the tangent at any point on the graph is given by $f'(x)$. Using the quotient rule,

$$f'(x) = \frac{(2)(\sqrt{x}) - (2x + 8)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} - \dfrac{2x + 8}{2\sqrt{x}}}{x}$$

$$= \frac{\dfrac{2x}{\sqrt{x}} - \dfrac{x + 4}{\sqrt{x}}}{x}$$

$$= \frac{\dfrac{2x - x - 4}{\sqrt{x}}}{x}$$

$$= \frac{x - 4}{x\sqrt{x}}$$

The tangent will be horizontal when $f'(x) = 0$; that is, when $x = 4$. The point on the graph where the tangent is horizontal is $(4, 8)$.

IN SUMMARY

Key Ideas

- The derivative of a quotient of two differentiable functions is not the quotient of their derivatives.
- The **quotient rule** for differentiation:

  If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Need to Know

- To find the derivative of a rational function, you can use two methods:

Leave the function in fraction form, and use the quotient rule.	OR	Express the function as a product, and use the product and power of a function rules.
$f(x) = \dfrac{x - 2}{1 + x}$		$f(x) = (x - 2)(1 + x)^{-1}$

Exercise 2.4

PART A

1. What are the exponent rules? Give examples of each rule.

2. Copy the table, and complete it *without* using the quotient rule.

Function	Rewrite	Differentiate and Simplify, if Necessary
$f(x) = \frac{x^2 + 3x}{x}, x \neq 0$		
$g(x) = \frac{3x^{\frac{5}{3}}}{x}, x \neq 0$		
$h(x) = \frac{1}{10x^5}, x \neq 0$		
$y = \frac{8x^3 + 6x}{2x}, x \neq 0$		
$s = \frac{t^2 - 9}{t - 3}, t \neq 3$		

C 3. What are the different ways to find the derivative of a rational function? Give examples.

PART B

K 4. Use the quotient rule to differentiate each function. Simplify your answers.

a. $h(x) = \frac{x}{x + 1}$

b. $h(t) = \frac{2t - 3}{t + 5}$

c. $h(x) = \frac{x^3}{2x^2 - 1}$

d. $h(x) = \frac{1}{x^2 + 3}$

e. $y = \frac{x(3x + 5)}{1 - x^2}$

f. $y = \frac{x^2 - x + 1}{x^2 + 3}$

5. Determine $\frac{dy}{dx}$ at the given value of x.

a. $y = \frac{3x + 2}{x + 5}, x = -3$

b. $y = \frac{x^3}{x^2 + 9}, x = 1$

c. $y = \frac{x^2 - 25}{x^2 + 25}, x = 2$

d. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}, x = 4$

6. Determine the slope of the tangent to the curve $y = \frac{x^3}{x^2 - 6}$ at point $(3, 9)$.

7. Determine the points on the graph of $y = \frac{3x}{x - 4}$ where the slope of the tangent is $-\frac{12}{25}$.

T 8. Show that there are no tangents to the graph of $f(x) = \frac{5x + 2}{x + 2}$ that have a negative slope.

9. Find the point(s) at which the tangent to the curve is horizontal.

a. $y = \frac{2x^2}{x - 4}$

b. $y = \frac{x^2 - 1}{x^2 + x - 2}$

A 10. An initial population, p, of 1000 bacteria grows in number according to the equation $p(t) = 1000\left(1 + \frac{4t}{t^2 + 50}\right)$, where t is in hours. Find the rate at which the population is growing after 1 h and after 2 h.

11. Determine the equation of the tangent to the curve $y = \frac{x^2 - 1}{3x}$ at $x = 2$.

12. A motorboat coasts toward a dock with its engine off. Its distance s, in metres, from the dock t seconds after the engine is turned off is $s(t) = \frac{10(6 - t)}{t + 3}$ for $0 \leq t \leq 6$.

a. How far is the boat from the dock initially?

b. Find the velocity of the boat when it bumps into the dock.

13. a. The radius of a circular juice blot on a piece of paper towel t seconds after it was first seen is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where r is measured in centimetres. Calculate

i. the radius of the blot when it was first observed

ii. the time at which the radius of the blot was 1.5 cm

iii. the rate of increase of the area of the blot when the radius was 1.5 cm

b. According to this model, will the radius of the blot ever reach 2 cm? Explain your answer.

14. The graph of $f(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent line at $(2, -1)$. Find a and b. Check using a graphing calculator.

15. The concentration, c, of a drug in the blood t hours after the drug is taken orally is given by $c(t) = \frac{5t}{2t^2 + 7}$. When does the concentration reach its maximum value?

16. The position from its starting point, s, of an object that moves in a straight line at time t seconds is given by $s(t) = \frac{t}{t^2 + 8}$. Determine when the object changes direction.

PART C

17. Consider the function $f(x) = \frac{ax + b}{cx + d}$, $x \neq -\frac{d}{c}$, where a, b, c, and d are nonzero constants. What condition on a, b, c, and d ensures that each tangent to the graph of f has a positive slope?