Section 2.5—The Derivatives of Composite Functions

Recall that one way of combining functions is through a process called **composition**. We start with a number x in the domain of g, find its image $g(x)$, and then take the value of f at $g(x)$, provided that $g(x)$ is in the domain of f. The result is the new function $h(x) = f(g(x))$, which is called the **composite function** of f and g, and is denoted $(f \circ g)$.

Definition of a composite function

Given two functions f and g, the **composite function** $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$.

EXAMPLE 1

Reflecting on the process of composition

If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following values:

a. $f(g(4))$ b. $g(f(4))$ c. $f(g(x))$ d. $g(f(x))$

Solution

a. Since $g(4) = 9$, we have $f(g(4)) = f(9) = 3$.

b. Since $f(4) = 2$, we have $g(f(4)) = g(2) = 7$. *Note:* $f(g(4)) \neq g(f(4))$.

c. $f(g(x)) = f(x + 5) = \sqrt{x + 5}$

d. $g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 5$ *Note:* $f(g(x)) \neq g(f(x))$.

The chain rule states how to compute the derivative of the composite function $h(x) = f(g(x))$ in terms of the derivatives of f and g.

The Chain Rule

If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.

In words, the chain rule says, "the derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function."

Proof:

By the definition of the derivative, $[f(g(x))]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$.

Assuming that $g(x+h) - g(x) \neq 0$, we can write

$$[f(g(x))]' = \lim_{h \to 0}\left[\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right)\right]$$

$$= \lim_{h \to 0}\left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right]\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

(Property of limits)

Since $\lim_{h \to 0}[g(x+h) - g(x)] = 0$, let $g(x+h) - g(x) = k$ and $k \to 0$ as $h \to 0$. We obtain

$$[f(g(x))]' = \lim_{k \to 0}\left[\frac{f(g(x) + k) - f(g(x))}{k}\right]\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

Therefore, $[f(g(x))]' = f'(g(x))g'(x)$.

This proof is not valid for all circumstances. When dividing by $g(x+h) - g(x)$, we assume that $g(x+h) - g(x) \neq 0$. A proof that covers all cases can be found in advanced calculus textbooks.

EXAMPLE 2 Applying the chain rule

Differentiate $h(x) = (x^2 + x)^{\frac{3}{2}}$.

Solution

The inner function is $g(x) = x^2 + x$, and the outer function is $f(x) = x^{\frac{3}{2}}$.

The derivative of the inner function is $g'(x) = 2x + 1$.

The derivative of the outer function is $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$.

The derivative of the outer function evaluated at the inner function $g(x)$ is $f'(x^2 + x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}$.

By the chain rule, $h'(x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}(2x + 1)$.

The Chain Rule in Leibniz Notation

If y is a function of u and u is a function of x (so that y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

If we interpret derivatives as rates of change, the chain rule states that if y is a function of x through the intermediate variable u, then the rate of change of y

with respect to x is equal to the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

EXAMPLE 3

Applying the chain rule using Leibniz notation

If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at $x = 4$.

Solution

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2 - 2)\left[2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right]$$

$$= (3u^2 - 2)\left(\frac{1}{\sqrt{x}}\right)$$

It is not necessary to write the derivative entirely in terms of x.

When $x = 4$, $u = 2\sqrt{4} = 4$ and $\frac{dy}{dx} = [3(4)^2 - 2]\left(\frac{1}{\sqrt{4}}\right) = (46)\left(\frac{1}{2}\right) = 23$.

EXAMPLE 4

Selecting a strategy involving the chain rule to solve a problem

An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air can be modelled by the function $C(p) = \sqrt{0.5p^2 + 17}$, where $C(p)$ is in parts per million and population p is expressed in thousands. It is estimated that t years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand. At what rate will the carbon monoxide level be changing with respect to time three years from now?

Solution

We are asked to find the value of $\frac{dC}{dt}$, when $t = 3$.

We can find the rate of change by using the chain rule.

Therefore, $\frac{dC}{dt} = \frac{dC}{dp}\frac{dp}{dt}$

$$= \frac{d(0.5p^2 + 17)^{\frac{1}{2}}}{dp}\frac{d(3.1 + 0.1t^2)}{dt}$$

$$= \left[\frac{1}{2}(0.5p^2 + 17)^{-\frac{1}{2}}(0.5)(2p)\right](0.2t)$$

When $t = 3$, $p(3) = 3.1 + 0.1(3)^2 = 4$.

So, $\frac{dC}{dt} = \left[\frac{1}{2}(0.5(4)^2 + 17)^{-\frac{1}{2}}(0.5)(2(4))\right](0.2(3))$

$= 0.24$

Since the sign of $\frac{dC}{dt}$ is positive, the carbon monoxide level will be increasing at the rate of 0.24 parts per million per year three years from now.

EXAMPLE 5 Using the chain rule to differentiate a power of a function

If $y = (x^2 - 5)^7$, find $\frac{dy}{dx}$.

Solution

The inner function is $g(x) = x^2 - 5$, and the outer function is $f(x) = x^7$.
By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= 7(x^2 - 5)^6(2x)\\ &= 14x(x^2 - 5)^6\end{aligned}$$

Example 5 is a special case of the chain rule in which the outer function is a power function of the form $y = [g(x)]^n$. This leads to a generalization of the power rule seen earlier.

Power of a Function Rule

If n is a real number and $u = g(x)$, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$,

or $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$.

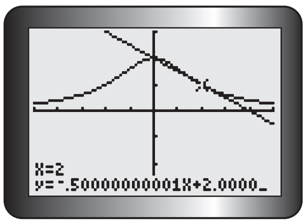
EXAMPLE 6 Connecting the derivative to the slope of a tangent

Using a graphing calculator, sketch the graph of the function $f(x) = \frac{8}{x^2 + 4}$.

Find the equation of the tangent at the point $(2, 1)$ on the graph.

Solution

Using a graphing calculator, the graph is



Tech Support

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

The slope of the tangent at point $(2, 1)$ is given by $f'(2)$.
We first write the function as $f(x) = 8(x^2 + 4)^{-1}$.

By the power of a function rule, $f'(x) = -8(x^2 + 4)^{-2}(2x)$.

The slope at $(2, 1)$ is

$$\begin{aligned}f'(2) &= -8(4 + 4)^{-2}(4)\\ &= -\frac{32}{(8)^2}\\ &= -0.5\end{aligned}$$

The equation of the tangent is $y - 1 = -\frac{1}{2}(x - 2)$, or $x + 2y - 4 = 0$.

EXAMPLE 7 Combining derivative rules to differentiate a complex product

Differentiate $h(x) = (x^2 + 3)^4(4x - 5)^3$. Express your answer in a simplified factored form.

Solution

Here we use the product rule and the chain rule.

$$h'(x) = \frac{d}{dx}[(x^2 + 3)^4] \cdot (4x - 5)^3 + \frac{d}{dx}[(4x - 5)^3] \cdot (x^2 + 3)^4 \quad \text{(Product rule)}$$

$$= [4(x^2 + 3)^3(2x)] \cdot (4x - 5)^3 + [3(4x - 5)^2(4)] \cdot (x^2 + 3)^4 \quad \text{(Chain rule)}$$

$$= 8x(x^2 + 3)^3(4x - 5)^3 + 12(4x - 5)^2(x^2 + 3)^4 \quad \text{(Simplify)}$$

$$= 4(x^2 + 3)^3(4x - 5)^2[2x(4x - 5) + 3(x^2 + 3)] \quad \text{(Factor)}$$

$$= 4(x^2 + 3)^3(4x - 5)^2(11x^2 - 10x + 9)$$

EXAMPLE 8 Combining derivative rules to differentiate a complex quotient

Determine the derivative of $g(x) = \left(\frac{1 + x^2}{1 - x^2}\right)^{10}$.

Solution A – Using the product and chain rule

There are several approaches to this problem. You could keep the function as it is and use the chain rule and the quotient rule. You could also decompose the function and express it as $g(x) = \frac{(1 + x^2)^{10}}{(1 - x^2)^{10}}$, and then apply the quotient rule and the chain rule. Here we will express the function as the product $g(x) = (1 + x^2)^{10}(1 - x^2)^{-10}$ and apply the product rule and the chain rule.

$$g'(x) = \frac{d}{dx}\left[(1 + x^2)^{10}\right](1 - x^2)^{-10} + (1 + x^2)^{10}\frac{d}{dx}\left[(1 - x^2)^{-10}\right]$$

$$= 10(1 + x^2)^9(2x)(1 - x^2)^{-10} + (1 + x^2)^{10}(-10)(1 - x^2)^{-11}(-2x)$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-10} + (20x)(1 + x^2)^{10}(1 - x^2)^{-11} \quad \text{(Simplify)}$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-11}[(1 - x^2) + (1 + x^2)] \quad \text{(Factor)}$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-11}(2)$$

$$= \frac{40x(1 + x^2)^9}{(1 - x^2)^{11}} \quad \text{(Rewrite using positive exponents)}$$

Solution B – Using the chain and quotient rule

In this solution, we will use the chain rule and the quotient rule, where $u = \frac{1 + x^2}{1 - x^2}$ is the inner function and u^{10} is the outer function.

$$g'(x) = \frac{dg}{du}\frac{du}{dx}$$

$$g'(x) = \frac{d\left[\left(\frac{1 + x^2}{1 - x^2}\right)^{10}\right]}{d\left(\frac{1 + x^2}{1 - x^2}\right)} \frac{d}{dx}\left(\frac{1 + x^2}{1 - x^2}\right) \quad \text{(Chain rule and quotient rule)}$$

$$= 10\left(\frac{1 + x^2}{1 - x^2}\right)^9 \frac{d}{dx}\left(\frac{1 + x^2}{1 - x^2}\right)$$

$$= 10\left(\frac{1 + x^2}{1 - x^2}\right)^9 \left[\frac{2x(1 - x^2) - (-2x)(1 + x^2)}{(1 - x^2)^2}\right] \quad \text{(Expand)}$$

$$= 10\left(\frac{1 + x^2}{1 - x^2}\right)^9 \left[\frac{2x - 2x^3 + 2x + 2x^3}{(1 - x^2)^2}\right] \quad \text{(Simplify)}$$

$$= 10\left(\frac{1 + x^2}{1 - x^2}\right)^9 \left[\frac{4x}{(1 - x^2)^2}\right]$$

$$= \frac{10(1 + x^2)^9}{(1 - x^2)^9} \frac{4x}{(1 - x^2)^2}$$

$$= \frac{40x(1 + x^2)^9}{(1 - x^2)^{11}}$$

IN SUMMARY

Key Idea

- The **chain rule**:
  If y is a function of u, and u is a function of x (i.e., y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.
  Therefore, if $h(x) = (f \circ g)(x)$, then
  $$h'(x) = f'(g(x)) \cdot g'(x) \quad \text{(Function notation)}$$
  or
  $$\frac{d[h(x)]}{dx} = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx} \quad \text{(Leibniz notation)}$$

Need to Know

- When the outer function is a power function of the form $y = [g(x)]^n$, we have a special case of the chain rule, called the **power of a function rule**:
  $$\frac{d}{dx}[g(x)]^n = \frac{d[g(x)]^n}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$
  $$= n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 2.5

PART A

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following value:

   a. $f(g(1))$
   b. $g(f(1))$
   c. $g(f(0))$
   d. $f(g(-4))$
   e. $f(g(x))$
   f. $g(f(x))$

2. For each of the following pairs of functions, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the composite functions equal?

   a. $f(x) = x^2$
      $g(x) = \sqrt{x}$
   b. $f(x) = \frac{1}{x}$
      $g(x) = x^2 + 1$
   c. $f(x) = \frac{1}{x}$
      $g(x) = \sqrt{x + 2}$

C 3. What is the rule for calculating the derivative of the composition of two differentiable functions? Give examples, and show how the derivative is determined.

4. Differentiate each function. Do not expand any expression before differentiating.

   a. $f(x) = (2x + 3)^4$
   b. $g(x) = (x^2 - 4)^3$
   c. $h(x) = (2x^2 + 3x - 5)^4$
   d. $f(x) = (\pi^2 - x^2)^3$
   e. $y = \sqrt{x^2 - 3}$
   f. $f(x) = \frac{1}{(x^2 - 16)^5}$

PART B

K 5. Rewrite each of the following in the form $y = u^n$ or $y = ku^n$, and then differentiate.

   a. $y = -\frac{2}{x^3}$
   b. $y = \frac{1}{x + 1}$
   c. $y = \frac{1}{x^2 - 4}$
   d. $y = \frac{3}{9 - x^2}$
   e. $y = \frac{1}{5x^2 + x}$
   f. $y = \frac{1}{(x^2 + x + 1)^4}$

6. Given $h = g \circ f$, where f and g are continuous functions, use the information in the table to evaluate $h(-1)$ and $h'(-1)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	1	18	−5	−15
0	−2	5	−1	−11
1	−1	−4	3	−7
2	4	−9	7	−3
3	13	−10	11	1

7. Given $f(x) = (x - 3)^2$, $g(x) = \frac{1}{x}$, and $h(x) = f(g(x))$, determine $h'(x)$.

8. Differentiate each function. Express your answer in a simplified factored form.
   a. $f(x) = (x + 4)^3(x - 3)^6$
   b. $y = (x^2 + 3)^3(x^3 + 3)^2$
   c. $y = \frac{3x^2 + 2x}{x^2 + 1}$
   d. $h(x) = x^3(3x - 5)^2$
   e. $y = x^4(1 - 4x^2)^3$
   f. $y = \left(\frac{x^2 - 3}{x^2 + 3}\right)^4$
9. Find the rate of change of each function at the given value of t. Leave your answers as rational numbers, or in terms of roots and the number π.
   a. $s(t) = t^{\frac{1}{3}}(4t - 5)^{\frac{2}{3}}, t = 8$
   b. $s(t) = \left(\frac{t - \pi}{t - 6\pi}\right)^{\frac{1}{3}}, t = 2\pi$
10. For what values of x do the curves $y = (1 + x^3)^2$ and $y = 2x^6$ have the same slope?
11. Find the slope of the tangent to the curve $y = (3x - x^2)^{-2}$ at $\left(2, \frac{1}{4}\right)$.
12. Find the equation of the tangent to the curve $y = (x^3 - 7)^5$ at $x = 2$.
13. Use the chain rule, in Leibniz notation, to find $\frac{dy}{dx}$ at the given value of x.
   a. $y = 3u^2 - 5u + 2, u = x^2 - 1, x = 2$
   b. $y = 2u^3 + 3u^2, u = x + x^{\frac{1}{2}}, x = 1$
   c. $y = u(u^2 + 3)^3, u = (x + 3)^2, x = -2$
   d. $y = u^3 - 5(u^3 - 7u)^2, u = \sqrt{x}, x = 4$
14. Find $h'(2)$, given $h(x) = f(g(x)), f(u) = u^2 - 1, g(2) = 3$, and $g'(2) = -1$.

A 15. A 50 000 L tank can be drained in 30 min. The volume of water remaining in the tank after t minutes is $V(t) = 50\,000\left(1 - \frac{t}{30}\right)^2, 0 \le t \le 30$. At what rate, to the nearest whole number, is the water flowing out of the tank when $t = 10$?

16. The function $s(t) = (t^3 + t^2)^{\frac{1}{2}}, t \ge 0$, represents the displacement s, in metres, of a particle moving along a straight line after t seconds. Determine the velocity when $t = 3$.

PART C

17. a. Write an expression for $h'(x)$ if $h(x) = p(x)q(x)r(x)$.
    b. If $h(x) = x(2x + 7)^4(x - 1)^2$, find $h'(-3)$.

T 18. Show that the tangent to the curve $y = (x^2 + x - 2)^3 + 3$ at the point $(1, 3)$ is also the tangent to the curve at another point.

19. Differentiate $y = \frac{x^2(1 - x^3)}{(1 + x)^3}$.

Technology Extension: Derivatives on Graphing Calculators

Numerical derivatives can be approximated on a TI-83/84 Plus using **nDeriv(**.

To approximate $f'(0)$ for $f(x) = \frac{2x}{x^2 + 1}$ follow these steps:

Press MATH, and scroll down to **8:nDeriv(** under the MATH menu.

Press ENTER, and the display on the screen will be **nDeriv(**.

To find the derivative, key in the *expression*, the *variable*, the *value* at which we want the derivative, and a value for ε.

For this example, the display will be **nDeriv** (2X/(X² + 1), X, 0, 0.01).

Press ENTER, and the value **1.99980002** will be returned.

Therefore, $f'(0)$is approximately 1.999 800 02.

A better approximation can be found by using a smaller value for ε, such as $\varepsilon = 0.0001$. The default value for ε is 0.001.

Try These:

a. Use the **nDeriv(** function on a graphing calculator to determine the value of the derivative of each of the following functions at the given point.

i. $f(x) = x^3, x = -1$

ii. $f(x) = x^4, x = 2$

iii. $f(x) = x^3 - 6x, x = -2$

iv. $f(x) = (x^2 + 1)(2x - 1)^4, x = 0$

v. $f(x) = x^2 + \frac{16}{x} - 4\sqrt{x}, x = 4$

vi. $f(x) = \frac{x^2 - 1}{x^2 + x - 2}, x = -1$

b. Determine the actual value of each derivative at the given point using the rules of differentiation.

The TI-89, TI-92, an TI-Nspire can find exact symbolic and numerical derivatives. If you have access to either model, try some of the functions above and compare your answers with those found using a TI-83/84 Plus. For example, on the TI-89 press DIFFERENTIATE under the CALCULATE menu, key $d(2x/(x^2 + 1), x)|x = 0$ and press ENTER.

CHAPTER 2: THE ELASTICITY OF DEMAND

An electronics retailing chain has established the monthly price (p)–demand (n) relationship for an electronic game as

$$n(p) = 1000 - 10\,\frac{(p-1)^{\frac{4}{3}}}{\sqrt[3]{p}}$$

They are trying to set a price level that will provide maximum revenue (R). They know that when demand is *elastic* ($E > 1$), a drop in price will result in higher overall revenues ($R = np$), and that when demand is *inelastic* ($E < 1$), an increase in price will result in higher overall revenues. To complete the questions in this task, you will have to use the elasticity definition

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

converted into differential $\left(\frac{\Delta n}{\Delta p} = \frac{dn}{dp}\right)$ notation.

a. Determine the elasticity of demand at \$20 and \$80, classifying these price points as having elastic or inelastic demand. What does this say about where the optimum price is in terms of generating the maximum revenue? Explain. Also calculate the revenue at the \$20 and \$80 price points.

b. Approximate the demand curve as a linear function (tangent) at a price point of \$50. Plot the demand function and its linear approximation on the graphing calculator. What do you notice? Explain this by looking at the demand function.

c. Use your linear approximation to determine the price point that will generate the maximum revenue. (*Hint:* Think about the specific value of E where you will not want to increase or decrease the price to generate higher revenues.) What revenue is generated at this price point?

d. A second game has a price–demand relationship of

$$n(p) = \frac{12\,500}{p - 25}$$

The price is currently set at \$50. Should the company increase or decrease the price? Explain.