Section 3.2—Maximum and Minimum on an Interval (Extreme Values)

INVESTIGATION The purpose of this investigation is to determine how the derivative can be used to determine the maximum (largest) value or the minimum (smallest) value of a function on a given interval. Together, these are called the **absolute extrema** on an interval.

A. For each of the following functions, determine, by completing the square, the value of x that produces a maximum or minimum function value on the given interval.

i. $f(x) = -x^2 + 6x - 3, 0 \leq x \leq 5$

ii. $f(x) = -x^2 - 2x + 11, -3 \leq x \leq 4$

iii. $f(x) = 4x^2 - 12x + 7, -1 \leq x \leq 4$

B. For each function in part A, determine the value of c such that $f'(c) = 0$.

C. Compare the values obtained in parts A and B for each function. Why does it make sense to say that the pattern you discovered is not merely a coincidence?

D. Using a graphing calculator, graph each of the following functions and determine all the values of x that produce a maximum or minimum value on the given interval.

i. $f(x) = x^3 - 3x^2 - 8x + 10, -2 \leq x \leq 4$

ii. $f(x) = x^3 - 12x + 5, -3 \leq x \leq 3$

iii. $f(x) = 3x^3 - 15x^2 + 9x + 23, 0 \leq x \leq 4$

iv. $f(x) = -2x^3 + 12x + 7, -2 \leq x \leq 2$

v. $f(x) = -x^3 - 2x^2 + 15x + 23, -4 \leq x \leq 3$

E. For each function in part D, determine all the values of c such that $f'(c) = 0$.

F. Compare the values obtained in parts D and E for each function. What do you notice?

G. From your comparisons in parts C and F, state a method for using the derivative of a function to determine values of the variable that give maximum or minimum values of the function.

H. Repeat part D for the following functions, using the given intervals.

i. $f(x) = -x^2 + 6x - 3, 4 \leq x \leq 8$

ii. $f(x) = 4x^2 - 12x + 7, 2 \leq x \leq 6$

iii. $f(x) = x^3 - 3x^2 - 9x + 10, -2 \leq x \leq 6$

iv. $f(x) = x^3 - 12x + 5, 0 \leq x \leq 5$

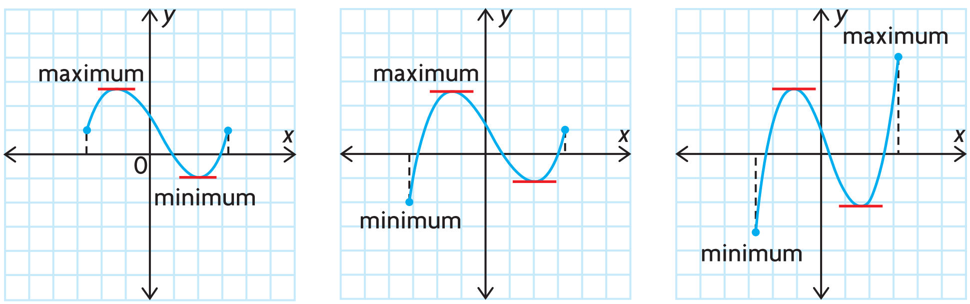
v. $f(x) = x^3 - 5x^2 + 3x + 7, -2 \leq x \leq 5$

I. In parts C and F, you saw that a maximum or minimum can occur at points $(c, f(c))$, where $f'(c) = 0$. From your observations in part H, state other values of the variable that can produce a maximum or minimum in a given interval.

Checkpoint: Check Your Understanding

The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" $(f'(c) = 0)$ or at an endpoint of the interval. The minimum value occurs at a "valley" $(f'(c) = 0)$ or at an endpoint. This is true no matter how many peaks and valleys the graph has in the interval.

In the following three graphs, the derivative equals zero at two points:



Algorithm for Finding Maximum or Minimum (Extreme) Values

If a function $f(x)$ has a derivative at every point in the interval $a \leq x \leq b$, calculate $f(x)$ at

- all points in the interval $a \leq x \leq b$, where $f'(x) = 0$
- the endpoints $x = a$ and $x = b$

The maximum value of $f(x)$ on the interval $a \leq x \leq b$ is the largest of these values, and the minimum value of $f(x)$ on the interval is the smallest of these values.

When using the algorithm above it is important to consider the function $f(x)$ on a finite interval—that is, an interval that includes its endpoints. Otherwise, the function may not attain a maximum or minimum value.

EXAMPLE 1 Selecting a strategy to determine absolute extrema

Find the extreme values of the function $f(x) = -2x^3 + 9x^2 + 4$ on the interval $x \in [-1, 5]$.

Solution

The derivative is $f'(x) = -6x^2 + 18x$.

If we set $f'(x) = 0$, we obtain $-6x(x - 3) = 0$, so $x = 0$ or $x = 3$.

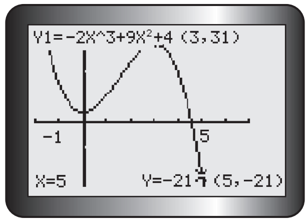
Both values lie in the given interval, $[-1, 5]$.

We can then evaluate $f(x)$ for these values and at the endpoints $x = -1$ and $x = 5$ to obtain

$$f(-1) = 15$$
$$f(0) = 4$$
$$f(3) = 31$$
$$f(5) = -21$$

Therefore, the maximum value of $f(x)$ on the interval $-1 \le x \le 5$ is $f(3) = 31$, and the minimum value is $f(5) = -21$.

Graphing the function on this interval verifies our analysis.



EXAMPLE 2 Solving a problem involving absolute extrema

The amount of current, in amperes (A), in an electrical system is given by the function $C(t) = -t^3 + t^2 + 21t$, where t is the time in seconds and $0 \le t \le 5$. Determine the times at which the current is at its maximum and minimum, and determine the amount of current in the system at these times.

Solution

The derivative is $\frac{dC}{dt} = -3t^2 + 2t + 21$.

If we set $\frac{dC}{dt} = 0$, we obtain

$$-3t^2 + 2t + 21 = 0 \quad \text{(Multiply by } -1\text{)}$$
$$3t^2 - 2t - 21 = 0 \quad \text{(Factor)}$$
$$(3t + 7)(t - 3) = 0 \quad \text{(Solve)}$$

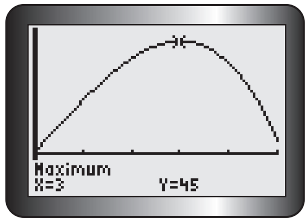
Therefore, $t = -\frac{7}{3}$ or $t = 3$.

Only $t = 3$ is in the given interval, so we evaluate $C(t)$ at $t = 0$, $t = 3$, and $t = 5$ as follows:

$$C(0) = 0$$
$$C(3) = -3^3 + 3^2 + 21(3) = 45$$
$$C(5) = -5^3 + 5^2 + 21(5) = 5$$

The maximum is 45 A at time $t = 3$ s, and the minimum is 0 A at time $t = 0$ s.

Graphing the function on this interval verifies our analysis.



EXAMPLE 3

Selecting a strategy to determine the absolute minimum

The amount of light intensity on a point is given by the function $I(t) = \frac{t^2 + 2t + 16}{t + 2}$, where t is the time in seconds and $t \in [0, 14]$. Determine the time of minimal intensity.

Solution

Note that the function is not defined for $t = -2$. Since this value is not in the given interval, we need not worry about it.

The derivative is

$$I'(t) = \frac{(2t + 2)(t + 2) - (t^2 + 2t + 16)(1)}{(t + 2)^2} \quad \text{(Quotient rule)}$$
$$= \frac{2t^2 + 6t + 4 - t^2 - 2t - 16}{(t + 2)^2} \quad \text{(Expand and simplify)}$$
$$= \frac{t^2 + 4t - 12}{(t + 2)^2}$$

If we set $I'(t) = 0$, we only need to consider when the numerator is 0.

$$t^2 + 4t - 12 = 0 \quad \text{(Factor)}$$
$$(t + 6)(t - 2) = 0 \quad \text{(Solve)}$$
$$t = -6 \text{ or } t = 2$$

Only $t = 2$ is in the given interval, so we evaluate $I(t)$ for $t = 0, 2,$ and 14.

$$I(0) = 8$$

$$I(2) = \frac{4 + 4 + 16}{4} = 6$$

$$I(14) = \frac{14^2 + 2(14) + 16}{16} = 15$$

Note that the calculation can be simplified by rewriting the intensity function as shown.

$$I(t) = \frac{t^2 + 2t}{t + 2} + \frac{16}{t + 2}$$
$$= t + 16(t + 2)^{-1}$$

Then $I'(t) = 1 - 16(t + 2)^{-2}$

$$= 1 - \frac{16}{(t + 2)^2}$$

Setting $I'(t) = 0$ gives

$$1 = \frac{16}{(t + 2)^2}$$

$$t^2 + 4t + 4 = 16$$

$$t^2 + 4t - 12 = 0$$

As before, $t = -6$ or $t = 2$.
The evaluations are also simplified.

$$I(0) = 0 + \frac{16}{2} = 8$$

$$I(2) = 2 + \frac{16}{4} = 6$$

$$I(14) = 14 + \frac{16}{16} = 15$$

Either way, the minimum amount of light intensity occurs at $t = 2$ s on the given time interval.

IN SUMMARY

Key Ideas

- The maximum and minimum values of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" ($f'(c) = 0$) or at an endpoint of the interval, [a, b],
- The minimum value occurs at a "valley" ($f'(c) = 0$) or at an endpoint of the interval, [a, b].

Need to Know

- **Algorithm for Finding Extreme Values:**

  For a function $f(x)$ that has a derivative at every point in an interval [a, b], the maximum or minimum values can be found by using the following procedure:

  1. Determine $f'(x)$. Find all points in the interval $a \le x \le b$, where $f'(x) = 0$.
  2. Evaluate $f(x)$ at the endpoints a and b, and at points where $f'(x) = 0$.
  3. Compare all the values found in step 2.
     - The largest of these values is the maximum value of $f(x)$ on the interval $a \le x \le b$.
     - The smallest of these values is the minimum value of $f(x)$ on the interval $a \le x \le b$.

Exercise 3.2

PART A

C 1. State, with reasons, why the maximum/minimum algorithm can or cannot be used to determine the maximum and minimum values of the following functions:

a. $y = x^3 - 5x^2 + 10, -5 \le x \le 5$

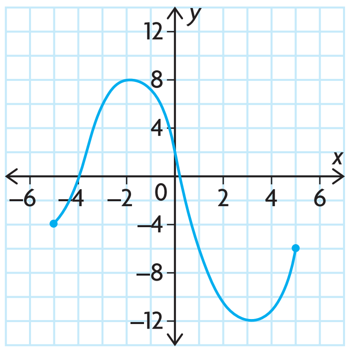
b. $y = \dfrac{3x}{x - 2}, -1 \le x \le 3$

c. $y = \dfrac{x}{x^2 - 4}, x \in [0, 5]$

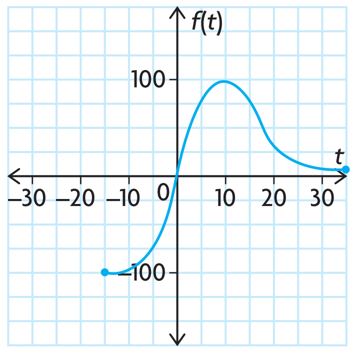
d. $y = \dfrac{x^2 - 1}{x + 3}, x \in [-2, 3]$

2. State the absolute maximum value and the absolute minimum value of each function, if the function is defined on the interval shown.

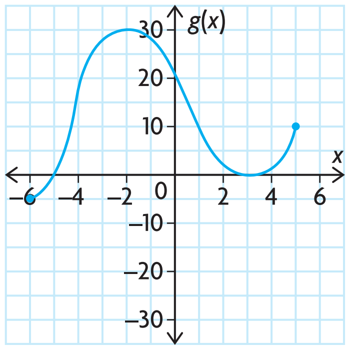
a.



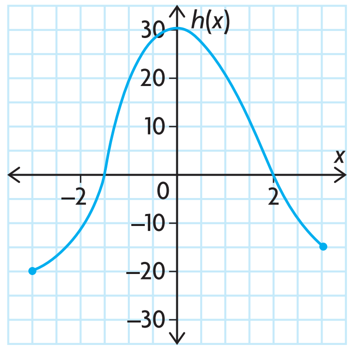
c.



b.



d.



K 3. Determine the absolute extrema of each function on the given interval. Illustrate your results by sketching the graph of each function.

a. $f(x) = x^2 - 4x + 3, 0 \leq x \leq 3$

b. $f(x) = (x - 2)^2, 0 \leq x \leq 2$

c. $f(x) = x^3 - 3x^2, -1 \leq x \leq 3$

d. $f(x) = x^3 - 3x^2, x \in [-2, 1]$

e. $f(x) = 2x^3 - 3x^2 - 12x + 1, x \in [-2, 0]$

f. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, x \in [0, 4]$

PART B

T 4. Using the algorithm for finding maximum or minimum values, determine the absolute extreme values of each function on the given interval.

a. $f(x) = x + \frac{4}{x}, 1 \leq x \leq 10$

b. $f(x) = 4\sqrt{x} - x, x \in [2, 9]$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20, x \in [-3, 4]$

e. $f(x) = \frac{4x}{x^2 + 1}, -2 \leq x \leq 4$

f. $f(x) = \frac{4x}{x^2 + 1}, x \in [2, 4]$

5. a. An object moves in a straight line. Its velocity, in m/s, at time t is $v(t) = \frac{4t^2}{4 + t^3}, t \geq 0$. Determine the maximum and minimum velocities over the time interval $1 \leq t \leq 4$.

b. Repeat part a., if $v(t) = \frac{4t^2}{1 + t^2}, t \geq 0$.

A 6. A swimming pool is treated periodically to control the growth of bacteria. Suppose that t days after a treatment, the number of bacteria per cubic centimetre is $N(t) = 30t^2 - 240t + 500$. Determine the lowest number of bacteria during the first week after the treatment.

7. The fuel efficiency, E, in litres per 100 kilometres, for a car driven at speed v, in km/h, is $E(v) = \frac{1600v}{v^2 + 6400}$.

a. If the speed limit is 100 km/h, determine the legal speed that will maximize the fuel efficiency.

b. Repeat part a., using a speed limit of 50 km/h.

c. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is increasing.

d. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is decreasing.

8. The concentration $C(t)$, in milligrams per cubic centimetre, of a certain medicine in a patient's bloodstream is given by $C(t) = \frac{0.1t}{(t + 3)^2}$, where t is the number of hours after the medicine is taken. Determine the maximum and minimum concentrations between the first and sixth hours after the medicine is taken.

9. Technicians working for the Ministry of Natural Resources found that the amount of a pollutant in a certain river can be represented by $P(t) = 2t + \frac{1}{(162t + 1)}$, $0 \leq t \leq 1$, where t is the time, in years, since a cleanup campaign started. At what time was the pollution at its lowest level?

10. A truck travelling at x km/h, where $30 \leq x \leq 120$, uses gasoline at the rate of $r(x)$ L/100 km, where $r(x) = \frac{1}{4}\left(\frac{4900}{x} + x\right)$. If fuel costs \$1.15/L, what speed will result in the lowest fuel cost for a trip of 200 km? What is the lowest total cost for the trip?

11. The polynomial function $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$, $0 \leq x \leq 75$, models the shape of a roller-coaster track, where f is the vertical displacement of the track and x is the horizontal displacement of the track. Both displacements are in metres. Determine the absolute maximum and minimum heights along this stretch of track.

12. a. Graph the cubic function with an absolute minimum at $(-2, -12)$, a local maximum at $(0, 3)$, a local minimum at $(2, -1)$, and an absolute maximum at $(4, 9)$. *Note*: local maximum and minimum values occur at peaks and valleys of a graph and do not have to be absolute extrema.

    b. What is the domain of this function?

    c. Where is the function increasing? Where is it decreasing?

13. What points on an interval must you consider to determine the absolute maximum or minimum value on the interval? Why?

PART C

T

14. In a certain manufacturing process, when the level of production is x units, the cost of production, in dollars, is $C(x) = 3000 + 9x + 0.05x^2$, $1 \leq x \leq 300$. What level of production, x, will minimize the unit cost, $U(x) = \frac{C(x)}{x}$? Keep in mind that the production level must be an integer.

15. Repeat question 14. If the cost of production is $C(x) = 6000 + 9x + 0.05x^2$, $1 \leq x \leq 300$.